Then for fixed ω not in one of the exceptional sets and large enough k,

$$\sum_{i>k} M_{n_i}(\omega)2^{-n_i/6} \leqq \sum_{i>k} 2n_i \log 2v(n_i)2^{-n_i/6} \leqq 2 \log 2 \sum_{n>n_k} n^2 2^{-n/6} < \infty.$$

We apply the Borel-Cantelli lemma with respect to the measure $\mu[E, \omega]$ to see that $\mu[T(\omega), \omega] = 1$, where $T(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{i\geqq k} \bigcup_j A(j, n_i)$. Since $A(j, n_i)$ is part of the boundary of the convex set $J(\omega) \cap C(j, n_i)$, which is a subset of $C(j, n_i)$, the length, $|A(j, n_i)|$, of $A(j, n_i)$ is less than $2\pi \cdot 2^{-n_i/6}$. Take $\epsilon_k = 2\pi \cdot 2^{-n_k/6}$. We have:

$$h_{\epsilon_k}^*(T(\omega)) \leqq \sum_{i\geqq k} \sum_j h(|A(j, n_i)|) \leqq \sum_{i\geqq k} M_{n_i}(\omega)h(2\pi \cdot 2^{-n_i/6}).$$

From (3) and the properties of $v(n)$ and $a(n)$, we obtain

$$h_{\epsilon_k}^*(T(\omega)) \leqq \sum_{i>k} 2n_i v(n_i)h(2\pi \cdot 2^{-n_i/6}) \log 2$$

$$= 2 \log 2 \sum_{i>k} n_i a(n_i)h(2\pi \cdot 2^{-n_i/6})/h(2\pi \cdot 2^{-n_i/6}) \log 2^{n_i} = 2\sum_{i>k} a(n_i).$$

Since $\sum a(n_i) < \infty$, $\lim_{k\to\infty} h_{\epsilon_k}^*(T(\omega)) = 0$, so $h^*(T(\omega)) = 0$.

REMARK. From the uniformity of the Brownian motion, it would be surprising if $K(\omega)$ had actual corners. One might even suspect that if $k(t)$ satisfies (A) and $\lim_{t\to\infty} k(t) \log 1/t = \infty$, one would have $k^*(E) = \infty$ for any E such that $\mu[T(\omega) \cap E, \omega] > 0$.

REFERENCES

[1] BAXTER, G. (1961). A combinatorial lemma for complex numbers. *Ann. Math. Statist.* **32** 901–909.

[2] CARLESON, L. (1950). *On a class of meromorphic functions and its associated exceptional sets.* Thesis, University of Uppsala.

[3] LÉVY, P. (1948). *Processus Stochastiques et Mouvement Brownien.* Gautier-Villars, Paris.

ON THE SAMPLE FUNCTIONS OF PROCESSES WHICH CAN BE ADDED TO A GAUSSIAN PROCESS

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Let $x(t)$ be a real measurable Gaussian process on an interval T with mean 0 and correlation function $R(s, t)$. We assume $\int_T \int_T R^2(s, t)\,ds\,dt < \infty$ so that $R(s, t)$ has an L_2 expansion $\sum \lambda_i \varphi_i(s)\varphi_i(t)$ with $\sum \lambda_i^2 < \infty$. We will write R for the compact integral operator gotten from $R(s, t)$. For any f satisfying $\int_T [R(t, t)]^{\frac{1}{2}} |f(t)|\,dt < \infty$ we can form the random variables $\theta(f, x) =$

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$\int_T x(t)f(t)\,dt$ ([1], Theorem 2.7, pg. 62). $\theta(f, x)$ will be Gaussian ([1], Theorem 2.8, pg. 64) and, writing P_x for the measure associated with $x(t)$, $\int \theta(f, x)\,dP_x = 0$, $\int \theta(f, x)\theta(g, x)\,dP_x = (Rf, g)$. Hence, if f_n converges to f in L_2, $\theta(f_n, x)$ converges in mean to a Gaussian random variable which we write $\theta(f, x)$ and which satisfies the above equations. Moreover, if f_n converges to f and the f_n are in the range of $R^{\frac{1}{2}}$, $\theta(R^{-\frac{1}{2}}f_n, x)$ converges to a Gaussian random variable which we write $\theta(R^{-\frac{1}{2}}f, x)$. It is easily verified that this notation is consistent and that the above equations continue to hold. In particular, $\theta_i = \lambda_i^{-\frac{1}{2}}\theta(\varphi_i, x)$ are independent normalized Gaussian random variables.

We wish to investigate the behavior of sample functions of processes $y(t)$ such that $P_{x+y} < P_x$ (P_{x+y} is absolutely continuous with respect to P_x) where $(x + y)(t)$ is the process gotten by adding independent versions of $x(t)$ and $y(t)$. If we write η_i for $\lambda_i^{-\frac{1}{2}}\theta(\varphi_i, y)$, then clearly $P_{x+y} < P_x$ implies that the sequence $(\theta_i + \eta_i)$ is absolutely continuous with respect to the sequence (θ_i) in the sense that every property possessed by the sequence (θ_i) with probability one also holds for the sequence $(\theta_i + \eta_i)$ with probability one. Conversely, if (θ_i) is any sequence of independent normalized Gaussian random variables and (η_i) is any other sequence of random variables such that $(\theta_i + \eta_i)$ is absolutely continuous with respect to (θ_i), then $P_{x+y} < P_x$ where $x(t) = \sum \lambda_i^{\frac{1}{2}}\theta_i\varphi_i(t)$ and $y(t) = \sum \lambda_i^{\frac{1}{2}}\eta_i\varphi_i(t)$, if λ_i decreases rapidly enough and φ_i is an orthonormal set of sufficiently smooth functions on an interval T. The results of this paper can thus be interpreted in either context and both points of view will be used in the proofs.

If $y(t)$ is a "sure" function, then $P_{x+y} < P_x$ if and only if $y = R^{\frac{1}{2}}f$ for some square integrable f [4], and it is not hard to show that any process $y(t)$ whose sample functions are almost all drawn from the range of $R^{\frac{1}{2}}$ satisfies $P_{x+y} < P_x$. The following example shows however that there are y's satisfying $P_{x+y} < P_x$ whose sample functions almost all lie outside the range of $R^{\frac{1}{2}}$.

Example: Let $y(t)$ be the Gaussian process on T with mean 0 and correlation function $S(s, t) = \sum \lambda_i\mu_i\varphi_i(s)\varphi_i(t)$ where $\sum \mu_i = \infty$ and $\sum \mu_i^2 < \infty$. Then $P_{x+y} < P_x$ since $R^{-\frac{1}{2}}(R - R - S)R^{-\frac{1}{2}} = -R^{-\frac{1}{2}}SR^{-\frac{1}{2}}$ is a Hilbert-Schmidt operator [2]. The norm of $R^{-\frac{1}{2}}y$ if it existed would be $\sum \mu_i[\theta(\varphi_i, y)/(\lambda_i\mu_i)^{\frac{1}{2}}]^2$, but this is infinite with probability one under the assumptions since the $\theta(\varphi_i, y)/(\mu_i\lambda_i)^{\frac{1}{2}}$ are (with respect to P_y) independent normalized Gaussian random variables.

This example shows that no necessary and sufficient conditions on the sample functions of $y(t)$ can be found to guarantee $P_{x+y} < P_x$. The situation cannot be saved by making special assumptions on the $x(t)$ process since it didn't really enter into the construction. Of course, the vector process $(x(t) + y(t), y(t))$ is singular with respect to $(x(t), y(t))$ since, once one knows $y(t)$, the usual perfect test can be applied to separate $x(t)$ and $x(t) + y(t)$. The following theorem clarifies the situation somewhat in the case of Gaussian $y(t)$.

THEOREM 1. *If $y(t)$ is Gaussian, has mean 0, is independent of $x(t)$ and $P_{x+y} < P_x$, then the following are equivalent*:

(1) *The sample functions of $y(t)$ are in the range of $R^{\frac{1}{2}}$ with probability* 1.

(2) *The vector process* $(x(t) + y(t), y(t))$ *is absolutely continuous with respect to* $(x(t), y(t))$.

(3) $x(t) + y(t)$ *is strongly continuous with respect to* $x(t)$ *in the sense of Hajek* [5].

PROOF. If $S(s, t)$ is the correlation function of $y(t)$ and S is the associated integral operator, then by hypothesis $R^{-\frac{1}{2}}SR^{-\frac{1}{2}} = \sum \mu_i P_i$, ($P_i$ being the projection on the subspace spanned by the normalized function ψ_i) is Hilbert-Schmidt, i.e., $\sum \mu_i^2 < \infty$. The random variables $\theta(R^{-\frac{1}{2}}\psi_i, x)$ are independent and Gaussian, and have mean 0 and variances 1, μ_i, and $1 + \mu_i$ with respect to P_x, P_y, and P_{x+y} respectively. We shall show that each of the conditions is equivalent to $\sum \mu_i < \infty$.

Condition 1. The norm of $R^{-\frac{1}{2}}y(t)$ if it exists is given by $\sum \theta(R^{-\frac{1}{2}}\psi_i, y)^2 = \sum \mu_i\eta_i^2$ where η_i are independent normalized Gaussian random variables and this series converges or diverges with probability one depending on whether $\sum \mu_i$ converges or diverges. (This shows also that the sample functions are either all in the range of $R^{\frac{1}{2}}$ or all outside it.)

Condition 2. According to Gel'fand [3] this is equivalent to

$$\int \log [dP_{(x+y,\, y)}/dP_x\, dP_y]\, dP_{(x+y,\, y)} < \infty$$

and an easy calculation shows that the integral is $\frac{1}{2}\sum (2\mu_i + \mu_i^2)/(1 + \mu_i) - \log(1 + \mu_i)$ whose convergence is equivalent to that of $\sum \mu_i$.

Condition 3. Strong equivalence means the P_x convergence of

$$\sum \theta(R^{-\frac{1}{2}}\psi_j, x)^2[1 - 1/(1 + \mu_i)]$$

which is equivalent to the convergence of $\sum [1 - 1/(1 + \mu_i)]$ which is equivalent to the convergence of $\sum \mu_i$.

We now turn to the problem of finding necessary conditions on the sample functions of $y(t)$ without assuming that it is Gaussian. We will need the following lemma.

LEMMA 1. *Let* $y(t)$ *be a process independent of* $x(t)$ *with* $P_{x+y} < P_x$. *Let* A *be a measurable subset of sample space with* $P_y(A) > 0$ *and* ν *the measure defined by* $\nu(B) = P_y(A \cap B)/P_y(A)$. Then there is a process $z(t)$ with $P_z = \nu$ and $P_{x+z} < P_x$.

PROOF. The existence of z follows from Kolmogoroff's theorem. For any set B, $P_{x+z}(B) = \int P_x(B + f)P_z(df) \leqq 1/P_y(A) \int P_x(B + f)P_y(df) = P_{x+y}(B)/P_y(A)$ which implies $P_{x+z} < P_x$.

THEOREM 2. *If* $P_{x+y} < P_x$, *then* $\theta_i = \theta(R^{-\frac{1}{2}}\varphi_i, x)$ *is defined almost everywhere* P_y *and for any numbers* ϵ_i *with* $0 \leqq \epsilon_i < 1$ *and* $\sum \epsilon_i^2 < \infty$, θ_i *must satisfy* $\sum \epsilon_i\theta_i^2 < \infty$ *with* P_y *probability one.*

This leaves open the question of whether $\sum \theta_i^4$ must converge.

PROOF. The existence of θ_i with respect P_x and hence P_{x+y} implies its existence with respect to P_y. Let D_n be the derivative of P_{x+y} with respect to P_x over the field generated by $\theta_1, \cdots, \theta_n$; then for $0 < \alpha_i < 1$

$$\left(\int D_n^{\frac{1}{2}}\, dP_x\right)^2 = \left(\int \cdots \int [\exp(-\tfrac{1}{2}\textstyle\sum t_i^2)/(2\pi)^{n/2}]\right.$$

$$\left.\cdot\left[\int \exp\left(\textstyle\sum t_i\,\theta_i(y) - \tfrac{1}{2}\,\theta_i^2(y)\right) dP_y\right]^{\frac{1}{2}} dt_1 \cdots dt_n\right)^2$$

$$= (2\pi)^{-n}\left(\int \cdots \int \exp(-\tfrac{1}{2}\textstyle\sum \alpha_i\, t_i^2)\right.$$

$$\left.\cdot\left[\int \exp\left(-\textstyle\sum (1-\alpha_i)\, t_i^2 - t_i\,\theta_i(y) + \tfrac{1}{2}\,\theta_i^2(y)\right) dP_y\right]^{\frac{1}{2}} dt_1 \cdots dt_n\right)$$

$$\leqq (2\pi)^{-n}\int \cdots \int \exp(-\textstyle\sum \alpha_i\, t_i^2)\, dt_1 \cdots dt_n$$

$$\cdot\int \cdots \int \exp\left(-\textstyle\sum (1-\alpha_i) t_i^2 - t_i\,\theta_i(y) + \tfrac{1}{2}\,\theta_i^2(y)\right) dP_y\, dt_1 \cdots dt_n$$

The first integral equals $(2\pi)^{n/2}\prod (2\alpha_i)^{-\frac{1}{2}}$ and the second is

$$(2\pi)^{n/2}\prod (2(1-\alpha_i))^{-\frac{1}{2}}\int \exp\{-\textstyle\sum [(1-2\alpha_i)/4(1-\alpha_i)]\,\theta_i^2(y)\}\, dP_y$$

so, setting $\alpha_i = (1+\epsilon_i)/2$, we have

$$\left(\int D_n^{\frac{1}{2}}\, dP_x\right)^2 \leqq \int \exp\{-\tfrac{1}{2}\textstyle\sum [\epsilon_i/(1-\epsilon_i)]\,\theta_i^2(y) + \log(1-\epsilon_i^2)\}\, dP_y\,.$$

If $\sum_1^\infty [\epsilon_i/(1-\epsilon_i)]\theta_i^2(y) + \log(1-\epsilon_i^2) = \infty$ almost everywhere, then

$$\int [\lim D_n(x)]^{\frac{1}{2}}\, dP_x = 0$$

and P_{x+y} is singular with respect to P_x contrary to hypothesis. If the serie diverges on a set A of positive measure and y' is the process gotten by restrictin y to A as in the lemma, then the same contradiction can be gotten by using y' i place of y. Hence, the series cannot diverge to $+\infty$ except on a set of measure for any set of ϵ_i with $|\epsilon_i| < 1$. If $\sum \epsilon_i^2 < \infty$, then $\sum \log(1-\epsilon_i^2) > -\infty$ s that the series $\sum[\epsilon_i/(1-\epsilon_i)]\theta_i^2(y)$ and hence, also the series $\sum \epsilon_i\theta_i^2(y)$ mus converge almost everywhere.

Corollary. *If $\sum \lambda_i^2 < \infty$ (which does not imply that the sample functions c $x(t)$ are in L_2), then the sample functions of $y(t)$ are in L_2 with probability one.*

Proof. This follows from the above theorem since $\int_T |y(t)|^2\, dt = \sum \lambda_i\theta_i^2(y)$

The next theorem makes more sense in connection with sequences so we stat it that way. The following lemma will be needed in the proof.

Lemma 2. *If a_i is a sequence of positive numbers with $\sum a_i = \infty$ and χ_i is an sequence of random variables taking the values 1 and 0 with probabilities $p_i \geqq \epsilon$ an $1 - p_i$, then $\sum \chi_i a_i$ diverges with probability at least ϵ.*

Proof. Let $f_n = \sum_1^n \chi_i a_i/\sum_1^n a_i$. Then $0 \leqq f_n \leqq 1$ and $E(f_n) \geqq \epsilon$. Let $p(\alpha$

be the probability that $f_n \leqq \alpha$. Then

$$\epsilon \leqq E(f_n) \leqq \alpha p(\alpha) + 1 - p(\alpha)$$

so $p(\alpha) \leqq (1 - \epsilon)/(1 - \alpha)$. Thus,

$$\text{prob } (\textstyle\sum_1^\infty \chi_i a_i \geqq \alpha \sum_1^n a_i) \geqq \text{prob } (f_n \geqq \alpha)$$
$$\geqq 1 - p(\alpha) \geqq 1 - [(1 - \epsilon)/(1 - \alpha)].$$

Letting n go to ∞ and then α go to zero completes the proof.

THEOREM 3. *Let (θ_i) be a sequence of independent normalized Gaussian random variables and let the η_i's be independent of them. If $(\theta_i + \eta_i)$ is equivalent to (θ_i) and if m_i is any set of numbers with* $\text{prob}(\eta_i^2 \geqq m_i) \geqq \epsilon > 0$, *then* $\sum m_i^2 < \infty$ *and* $\sum m_i \eta_i^2 < \infty$ *with probability one.*

PROOF. The second assertion will follow from the first by Theorem 2. If $\sum m_i^2 = \infty$, we can choose a set of numbers β_i to satisfy $|\beta_i m_i| < 1$, $\sum \beta_i^2 m_i^2 < \infty$ and $\sum \beta_i m_i^2 = \infty$. From Theorem 2 we have $\sum \beta_i m_i \eta_i^2 < \infty$ with probability one. If χ_i is the characteristic function of $\eta_i^2 \geqq m_i$, then $\sum \beta_i m_i \eta_i^2 \geqq \sum \beta_i m_i^2 \chi_i$ and the previous lemma gives a contradiction, completing the proof.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[2] FELDMAN, J. (1958). Equivalence and perpendicularity of Gaussian processes. *Pacific J. Math.* **8** 699–708.

[3] GEL'FAND, I. M. (1957). On the amount of information concerning one random function which is contained in another random function. *Uspehi Mat. Nauk.* **1** No. 1 (73), 3–52.

[4] GRENANDER, U. (1950). Stochastic processes and statistical inference. *Ark. Mat.*, Band 1 195–277.

[5] HÁJEK, J. (1962). On linear statistical problems in stochastic processes. *Czechoslovak Math. J.* **12** (87).

NOTE ON TWO BINOMIAL COEFFICIENT SUMS FOUND BY RIORDAN[1]

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In a recent paper on enumeration of graphs Riordan [7] has noted the following two combinatorial identities:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k}(k+1)! = n^n \tag{1}$$

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