A RENEWAL THEOREM FOR RANDOM VARIABLES WHICH ARE DEPENDENT OR NON-IDENTICALLY DISTRIBUTED[1]

By Y. S. Chow and Herbert Robbins

Columbia University

1. Introduction. Let $y_1, y_2, \cdots$ be a sequence of not necessarily independent r.v.'s, let $x_n = y_1 + \cdots + y_n$, and for $T > 0$ set $M = M(T) =$ first $n \geqq 1$ such that $x_n \geqq T$. We shall be concerned with finding conditions on the joint distribution of the sequence (y_n) which will ensure that

$$\lim_{T\to\infty} EM/T = 1/\mu, \tag{1}$$

where μ is some positive constant, thus generalizing the theorem in renewal theory which asserts that (1) holds if the y_n are independent and identically distributed with $Ey_n = \mu$, $0 < \mu < \infty$. Some results in the independent but non-identically distributed case may be found in [2], [3].

(a) *The dependent case.* Denoting by $E(\cdot \mid \mathfrak{F}_{n-1})$ the conditional expectation of $\cdot$ given $y_1, \cdots, y_{n-1}$ for $n > 1$, with $E(\cdot \mid \mathfrak{F}_0) = E(\cdot)$, our assumptions are

$$E(y_n \mid \mathfrak{F}_{n-1}) = Ey_n = \mu_n \qquad \text{(constants)}, \tag{2}$$

$$\lim_{n\to\infty} (\mu_1 + \cdots + \mu_n)/n = \mu, \qquad 0 < \mu < \infty, \tag{3}$$

and

$$E(|y_n - \mu_n|^\alpha \mid \mathfrak{F}_{n-1}) \leqq K < \infty \qquad \text{for some } \alpha > 1. \tag{4}$$

THEOREM 1. *If* (2), (3), *and* (4) *hold, then* (1) *holds.*

(b) *The independent case.* Here we shall assume that the y_n are independent, with means $\mu_n = Ey_n$ for which (3) holds, but we shall replace (4) by the assumption that

$$\lim_{n\to\infty} \int_{\{y_n-\mu_n>n\epsilon\}} (y_n - \mu_n) = 0 \qquad \text{for every } \epsilon > 0, \tag{5}$$

and that either

$$\int_{\{y_n-\mu_n<0\}} (y_n - \mu_n) \geqq -K > -\infty, \tag{6}$$

or that

$$y_n \geqq 0. \tag{7}$$

We remark that (5) and (6) hold if $E(|y_n - \mu_n|^\alpha) \leqq K < \infty$ for some $\alpha > 1$, or if the y_n are uniformly integrable about their means, i.e. if

$$\lim_{k\to\infty} \left[\sup_{n\geqq1} \int_{\{|y_n-\mu_n|\geqq k\}} |y_n - \mu_n| \right] = 0.$$

Received December 19, 1962.

[1] Research supported in part by National Science Foundation Grant NSF-G14146 at Columbia University.

THEOREM 2. *If the* y_n *are independent, and if* (3) *and* (5) *and either* (6) *or* (7) *hold, then* (1) *holds.*

2. Proof of Theorem 1. Assume (2), (3), and (4) and set $\rho_n = \mu_1 + \cdots + \mu_n$; then $\lim_{n\to\infty} \rho_n/n = \mu$, $0 < \mu < \infty$. By [4], p. 286, $\lim_{n\to\infty} x_n/n = \mu$ a.s., so that $P(M < \infty) = 1$.

For any $0 < \delta < \mu/3$ define $y_n' = \min(y_n, \mu_n + n\delta)$. Then

$$\sum_1^\infty P(y_n' \neq y_n) = \sum_1^\infty P(y_n - \mu_n > n\delta)$$
$$\leqq \sum_1^\infty P(|y_n - \mu_n|^\alpha > n^\alpha \delta^\alpha) \leqq \sum_1^\infty K(n\delta)^{-\alpha} < \infty,$$

so that $P(y_n' \neq y_n \text{ i.o.}) = 0$. Hence, setting $x_n' = y_1' + \cdots + y_n'$, $\mu_n' = Ey_n'$, $\rho_n' = \mu_1' + \cdots + \mu_n'$, and $M' = M'(T) =$ first $n \geqq 1$ such that $x_n' \geqq T$, we have $\lim_{n\to\infty} x_n'/n = \mu$ a.s., and therefore $P(M' < \infty) = 1$ also. Moreover, $M \leqq M'$. We note also that by (4),

$$0 \leqq \mu_n - \mu_n' = \int_{\{y_n - \mu_n > n\delta\}} (y_n - \mu_n - n\delta)$$
$$\leqq \int_{\{y_n - \mu_n > n\delta\}} (y_n - \mu_n)$$
$$\leqq \left(\frac{1}{n\delta}\right)^{\alpha-1} \cdot \int_{\{y_n - \mu_n > n\delta\}} (y_n - \mu_n)^\alpha$$
$$\leqq K/(n\delta)^{\alpha-1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\lim_{n\to\infty} \rho_n'/n = \mu$.

Now define for any $k = 1, 2, \cdots$, $M^* = \min(k, M')$. Then by a martingale system theorem ([1], p. 302), $E\{\sum_1^{M^*} [y_i' - E(y_i' \mid \mathfrak{F}_{i-1})]\} = 0$, so that

$$Ex_{M^*}' = E\left[\sum_1^{M^*} E(y_i' \mid \mathfrak{F}_{i-1})\right]$$
$$= \sum_{i=1}^k \int_{\{M^* = i\}} [E(y_1' \mid \mathfrak{F}_0) + \cdots + E(y_i' \mid \mathfrak{F}_{i-1})]$$
$$= \int_{\{M' \geqq 1\}} E(y_1' \mid \mathfrak{F}_0) + \int_{\{M' \geqq 2\}} E(y_2' \mid \mathfrak{F}_1) + \cdots + \int_{\{M' \geqq k\}} E(y_k' \mid \mathfrak{F}_{k-1})$$
$$= \int_{\{M' \geqq 1\}} \mu_1 + \int_{\{M' \geqq 2\}} \mu_2 + \cdots + \int_{\{M' \geqq k\}} \mu_k$$
$$- \int_{\{M' \geqq 1\}} E[(y_1 - \mu_1 - \delta)\chi_{\{y_1 - \mu_1 > \delta\}} \mid \mathfrak{F}_0]$$
$$- \cdots - \int_{\{M' \geqq k\}} E[(y_k - \mu_k - k\delta)\chi_{\{y_k - \mu_k > k\delta\}} \mid \mathfrak{F}_{k-1}],$$

where χ_A denotes the characteristic function of the set A. And by the Hölder inequality for conditional expectations,

$$\begin{aligned}\int_{\{M'\geqq i\}} & E[(y_i - \mu_i - i\delta)\chi_{\{y_i-\mu_i>i\delta\}} \mid \mathfrak{F}_{i-1}] \\ &\leqq \int_{\{M'\geqq i\}} E[(y_i - \mu_i)\chi_{\{y_i-\mu_i>i\delta\}} \mid \mathfrak{F}_{i-1})] \\ &\leqq \int_{\{M'\geqq i\}} E^{1/\alpha}[|y_i - \mu_i|^\alpha \mid \mathfrak{F}_{i-1}]\cdot P^{1/\alpha'}(y_i - \mu_i > i\delta \mid \mathfrak{F}_{i-1}) \quad (\alpha + \alpha' = \alpha\alpha') \\ &\leqq K\cdot(i\delta)^{\alpha/\alpha'}\cdot P(M' \geqq i) = \epsilon_i P(M' \geqq i), \quad \text{say,}\end{aligned}$$

where $(\epsilon_1 + \cdots + \epsilon_n)/n \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}E(x'_{M^*}) &\geqq \int_{\{M'\geqq 1\}} (\mu_1 - \epsilon_1) + \cdots + \int_{\{M'\geqq k\}} (\mu_k - \epsilon_k) \\ &= (\mu_1 - \epsilon_1)[P(M' = 1) + \cdots + P(M' = k) + P(M' > k)] \\ &\qquad + \cdots + (\mu_k - \epsilon_k)[P(M' = k) + P(M' > k)] \\ &= P(M' = 1)(\mu_1 - \epsilon_1) + P(M' = 2)(\mu_1 + \mu_2 - \epsilon_1 - \epsilon_2) \\ &\qquad + \cdots + P(M' = k)(\mu_1 + \cdots + \mu_k - \epsilon_1 - \cdots - \epsilon_k) \\ &\qquad + P(M' > k)(\mu_1 + \cdots + \mu_k - \epsilon_1 - \cdots - \epsilon_k) \\ &\geqq \textstyle\sum_{i=1}^k P(M' = i)(\rho_i - \epsilon_1 - \cdots - \epsilon_i) \qquad \text{if } k \text{ is large.}\end{aligned}$$

But $E(x'_{M^*}) \leqq T + \int_{\{M'\leqq k\}} y'_{M'} \leqq T + \int_{\{M \leqq k\}} (\mu_{M'} + \delta M')$. Hence

$$\begin{aligned}T &\geqq -\int_{\{M'\leqq k\}} M'\left(\delta + \frac{|\mu_{M'}|}{M'}\right) + \sum_{i=1}^k iP(M' = i)\left(\frac{\rho_i - \epsilon_1 - \cdots - \epsilon_i}{i}\right) \\ &\geqq A - 2\delta \int_{\{M'\leqq k\}} M' + \sum_{i=1}^k iP(M' = i)(\mu - \delta),\end{aligned}$$

where A is uniformly bounded below in k and T. Thus

$$T \geqq A + (\mu - 3\delta)\int_{\{M'\leqq k\}} M',$$

and letting $k \to \infty$ it follows that $T \geqq A + (\mu - 3\delta)EM'$, and hence that

$$\limsup_{T\to\infty} (EM/T) \leqq \limsup_{T\to\infty} (EM'/T) \leqq 1/\mu. \tag{8}$$

To establish an inequality in the opposite direction we observe that for the martingale $(x_n - \rho_n, \mathfrak{F}_n, n \geqq 1)$,

$$E(|x_{n+1} - \rho_{n+1} - x_n + \rho_n| \mid \mathfrak{F}_n) = E(|y_{n+1} - \mu_{n+1}| \mid \mathfrak{F}_n) \leqq K^{1/\alpha}$$

and that $EM < \infty$. Hence ([1], p. 302) $E(x_M - \rho_M) = 0$, so that for $i \geqq i_0$,

$$T \leqq E(x_M) = E(\rho_M) \leqq \int_{\{M \leqq i\}} (|\mu_1| + \cdots + |\mu_i|) + \int_{\{M > i\}} M(\mu + \epsilon) \tag{9}$$
$$\leqq |\mu_1| + \cdots + |\mu_i| + (\mu + \epsilon)EM.$$

Hence

$$\liminf_{T\to\infty} EM/T \geqq 1/\mu, \tag{10}$$

which, with (8), proves (1).

We remark that from the fact that $x_n/n \to \mu$ a.s. it follows easily that $\lim_{T\to\infty} (M/T) = 1/\mu$ a.s. The obvious way to prove (1) would be to show that the random variables M/T are uniformly integrable in T, but we have not attempted to do this.

3. Proof of Theorem 2. Assume that the y_n are independent and that (3) and (5) hold. Given any $0 < \epsilon < \mu/3$, choose $i = i(\epsilon)$ by (5) so that

$$\int_{\{y_n - \mu_n > n\epsilon\}} (y_n - \mu_n) < \epsilon \qquad \text{for all } n \geqq i.$$

Define $y_n' = \min(y_n, \mu_n + n\epsilon) \leqq y_n$ and $\mu_n' = Ey_n' \leqq \mu_n = Ey_n$. Then for all $n \geqq i$ we have

$$0 \leqq \mu_n - \mu_n' = \int_{\{y_n - \mu_n > n\epsilon\}} (y_n - \mu_n - n\epsilon) \leqq \int_{\{y_n - \mu_n > n\epsilon\}} (y_n - \mu_n) < \epsilon. \tag{11}$$

Define x_n', ρ_n, ρ_n', M' as in Section 2. It follows from (11) that

$$\liminf_{n\to\infty} \rho_n'/n \geqq \mu - \epsilon, \tag{12}$$

and we may therefore assume that i has been chosen so large that for all $n \geqq i$ we have in addition to (11) the inequalities

$$\mu - \epsilon \leqq \rho_n/n \leqq \mu + \epsilon, \qquad \mu - 2\epsilon \leqq \rho_n'/n \leqq \mu + \epsilon. \tag{13}$$

Then for $n > i$,

$$\sum_{j=1}^{n} \frac{\mu_j'}{j} = \sum_{j=1}^{i-1} \frac{\rho_j'}{j(j+1)} + \sum_{j=i}^{n-1} \frac{\rho_j'}{j(j+1)} + \frac{\rho_n'}{n}$$
$$\geqq \sum_{j=1}^{i-1} \frac{\rho_j'}{j(j+1)} + (\mu - 2\epsilon) \sum_{j=i}^{n} \frac{1}{j+1}.$$

Hence

$$\sum_{j=1}^{\infty} (\mu_j'/j) = \infty. \tag{14}$$

We shall now prove that $\limsup_{n\to\infty} x_n' = \infty$ a.s. Suppose in fact that

$$P(\limsup_{n\to\infty} x_n' < \infty) > 0. \tag{15}$$

Since

$$\sum_{j=1}^{n} \frac{y_j'}{j} = \sum_{j=1}^{n} \frac{(x_j' - x_{j-1}')}{j} = \sum_{j=1}^{n-1} \frac{x_j'}{j(j+1)} + \frac{x_n'}{n}$$

it follows from (15) that $P(\limsup_{n\to\infty} \sum_{j=1}^{n} (y_j'/j) < \infty) > 0$, and hence from (14) that

$$P\left(\lim_{n\to\infty} \sum_{j=1}^{n} \frac{(y_j' - \mu_j')}{j} = -\infty\right) > 0. \tag{16}$$

But for $n \geqq i$ we have from (11) that

$$(y_n' - \mu_n')/n \leqq (\mu_n + n\epsilon - \mu_n + \epsilon)/n < 2\epsilon,$$

so that

$$E[\sup_{n\geqq 1} (y_n' - \mu_n')/n] < \infty. \tag{17}$$

It follows ([2], p. 319) from (16) and (17) that $P\{\sum_{n=1}^{\infty} [(y_n' - \mu_n')/n]$ converges$\} = 1$, which contradicts (16). Hence (15) cannot be true, and therefore $P(\limsup_{n\to\infty} x_n' = \infty) = 1$, so that

$$1 \leqq M \leqq M' < \infty \qquad \text{a.s.} \tag{18}$$

Defining M^* as in Section 2, we have

$$x_{M^*}' = y_1' + \cdots + y_{M^*}' = \sum_{n=1}^{k} y_n'\varphi_{n-1}(M^*),$$

where by definition

$$\begin{aligned} \varphi_{n-1}(i) &= 1 \text{ if } i \geqq n, \\ &= 0 \text{ if } i < n. \end{aligned}$$

Since the event $M^* < n$ is independent of x_n it follows that

$$\begin{aligned} E(x_{M^*}') &= \sum_{n=1}^{k} \mu_n' P(M^* \geqq n) \\ &= \sum_{n=1}^{k} \mu_n'[P(M' = n) + P(M' = n+1) + \cdots + P(M' = k) \\ &\qquad\qquad + P(M' > k)] \\ &= \sum_{n=1}^{k} \rho_n' P(M' = n) + \rho_k' P(M' > k) \geqq \sum_{n=1}^{k} \rho_n' P(M' = n) \quad \text{for} \quad k > i. \end{aligned}$$

But $E(x_{M^*}') \leqq T + \sum_{n=1}^{k} (\mu_n + \epsilon n)P(M' = n)$. Hence

$$T \geqq \sum_{n=1}^{k} (\rho_n' - \mu_n - \epsilon n)P(M' = n)$$

$$\geqq O(1) + \sum_{n=i}^{k} (\rho'_{n-1} - (n+1)\epsilon)P(M' = n)$$

$$\geqq O(1) + \sum_{n=i}^{k} [(n-1)(\mu - 2\epsilon) - (n+1)\epsilon]P(M' = n)$$

$$= O(1) + (\mu - 3\epsilon) \sum_{n=i}^{k} (n-1)P(M' = n)$$

$$= O(1) + (\mu - 3\epsilon) \sum_{n=1}^{k} (n-1)P(M' = n),$$

where $O(1)$ is uniformly bounded in k and T. Hence $T \geqq O(1) + (\mu - 3\epsilon)E(M' - 1)$, and therefore (8) holds.

So far we have not used (6) or (7). Suppose now that (7) holds. Then $x_M = \sum_{n=1}^{\infty} y_n \varphi_{n-1}(M)$, and since all the terms of the series are non-negative,

$$T \leqq E(x_M) = \sum_{n=1}^{\infty} \mu_n P(M \geqq n) = \sum_{n=1}^{\infty} (\rho_n/n) \cdot n \cdot P(M = n)$$

$$\leqq \rho_{i-1} + (\mu + \epsilon) \cdot EM - (\mu + \epsilon) \sum_{n=1}^{i-1} nP(M = n).$$

It follows that (10) holds.

To prove (10) under the Assumption (6), set $z_n = x_n - \rho_n$. Then $(z_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and by (7),

$$E[(z_{n+1} - z_n)^- \mid \mathfrak{F}_n] = E[(y_{n+1} - \mu_{n+1})^-] \leqq K,$$

where $a^- = \max(0, -a)$. Since $EM < \infty$ it follows ([1], p. 303) that $Ez_M \leqq Ez_1 = 0$. Hence $T \leqq E(x_M) = E(z_M + \rho_M) \leqq E(\rho_M)$, which takes us back to (9), and (10) follows. This completes the proof.

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