APPLICATION OF METHODS IN SEQUENTIAL ANALYSIS TO DAM THEORY

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1. Summary and introduction. Wald's Fundamental Identity in sequential analysis [15] has been widely used for various applications apart from sequential sampling. Bartlett [1] used it for the insurance risk problem and also for the random walk and gambler's ruin problem. It is the purpose of this paper to show how the Fundamental Identity can be used to derive certain results in Dam Theory. The paper is purely of an expository nature and considers only simple examples; more difficult problems will be dealt with in a future paper.

We consider here a continuous time dam model due to Gani and Prabhu ([5], [6], [7]), based on Moran's ([10], [11]) discrete time model. Briefly it is as follows:

(a) The dam has finite capacity K.

(b) Let $X(T)$ represent the input (including the amount overflowed, if any) during a time interval of length T; we assume that $X(T)$ is an additive process with stationary increments. It is known that for such processes the m.g.f. is given by $E\{e^{\theta X(T)}\} = e^{-T\xi(\theta)}$ where $\xi(\theta)$ is a function of a specified type (Lévy [9]).

(c) The release is continuous and occurs at a unit rate except when the dam is empty.

It then follows that the net input (including the amount overflowed, if any) in the dam during the time interval $(0, T)$ is $Y(T) = X(T) - T$ whose m.g.f. is $M_T(\theta) = e^{-\theta T - T\xi(\theta)}$. For $T = 1$ this gives $M_1(\theta) = e^{-\theta-\xi(\theta)}$.

Let us denote the dam content at time t by $Z(t)$, with the initial content $Z(0) = u$ $(0 < u < K)$; we have, then $Z(t) = u + X(t) - t$. It follows from the above assumptions that the stochastic process $Z(t)$ is a temporally homogeneous Markov process. Following Moran's first paper, attention was concentrated mainly on investigating the stationary properties of the system. Later, Kendall [8], Gani [4], Prabhu [13] and Weesakul [16] considered the problem of emptiness of the dam; it is with this second problem that the present paper is concerned.

Considering the dam process described above as a random walk with barriers at $Z = 0$ and $Z = K$, the process starting at $Z(0) = u$, we obtain the probability that the dam becomes empty (i.e. the process terminates at 0) before it overflows (i.e. the process terminates at K) and the probability of the reverse situation; further, we derive the probability distribution of the time at which the dam becomes empty. We do this by making use of the following extension of Wald's Fundamental Identity to continuous time parameter by Dvoretzky, Kiefer and Wolfowitz [3].

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If $\{Y(T); T \geqq 0\}$, $Y(0) = 0$, is a process with stationary independent increments and if $M_1(\theta) = E[e^{\theta Y(1)}]$ exists for all real θ, then

$$E[e^{\theta Y(t+)} M_1(\theta)^{-t}] = 1, \tag{1.1}$$

where t is such that $Y(t) \geqq b$ or $Y(t) \leqq a$, while $a < Y(\tau) < b$ for all $\tau < t$. (Here a and b are constants, $a < 0$, $b > 0$.)

We also require a lemma due to Bartlett [1].

LEMMA. *Let Y be a random variable such that:* (a) *$E(Y)$ exists and is not equal to zero;* (b) *$M(\theta) = E[e^{Y\theta}]$ exists for θ in a finite interval;* (c) *$M(\theta) \to \infty$ as $\theta \to \pm\infty$. Then there exists one and only one real $\theta_0 \neq 0$ such that $M(\theta_0) = 1$.*

Since the Identity as stated above is applicable only when the process starts from zero, we put $b = K - u$, and $a = -u$. Applying the above lemma to the input process of the dam model, we find that there exists a nonzero real solution θ_0 of the equation

$$\theta + \xi(\theta) = 0; \tag{1.2}$$

we then have, putting $\theta = \theta_0$ in (1.1)

$$E[e^{\theta_0 Y(t+)}] = 1. \tag{1.3}$$

If the boundaries a and b are *exactly* reached at stage "t" we put $Y(t) = a$ with probability P_a and $Y(t) = b$ with probability $P_b = 1 - P_a$; then (1.3) gives $P_a e^{a\theta_0} + (1 - P_a)e^{b\theta_0} = 1$, so that

$$P_a = (1 - e^{b\theta_0})/(e^{a\theta_0} - e^{b\theta_0}). \tag{1.4}$$

Using (1.3) the characteristic function (c.f.) of the time "t" at which the dam becomes empty for the first time before touching the barrier K can also be determined. It is known that the equation $\theta + \xi(\theta) = i\phi$ has two roots $\theta_1(\phi)$ and $\theta_2(\phi)$ such that $\theta_1(\phi) \to 0$ and $\theta_2(\phi) \to \theta_0$ as $\phi \to 0$. Hence, in the case where the boundaries "a" and "b" are exactly reached, we obtain from (1.1)

$$\begin{aligned} P_a e^{a\theta_1(\phi)} C_a(\phi) + P_b e^{b\theta_1(\phi)} C_b(\phi) &= 1, \\ P_a e^{a\theta_2(\phi)} C_a(\phi) + P_b e^{b\theta_2(\phi)} C_b(\phi) &= 1, \end{aligned} \tag{1.5}$$

where $C_a(\phi)$ is c.f. of t conditional on $Y(t) = a$, and $C_b(\phi)$ is defined similarly, so that $E[e^{i\phi t}] = P_a C_a(\phi) + P_b C_b(\phi)$. Equations (1.4) and (1.5) give $P_a C_a(\phi)$, the c.f. of "t" the time at which the dam becomes empty for the first time without overflowing, (cf. Bartlett [1] pp. 18–19).

Before we use the Fundamental Identity in any problem let us consider the advantages of this method. The main advantage is its simplicity and easy applicability. Moreover, by a slight modification we can derive the more important probability distribution of the times of emptiness, no matter how often the dam has overflowed in the meantime. In random walk terminology this means that we have an absorbing barrier at 0 and a reflecting barrier at K. Another advantage of this method is that it can be used even when the inputs are not independent, as assumed in the above model, but Markovian. In this case the extension of Wald's Fundamental Identity to Markov Processes (Bellman [2], Tweedle [14], Phatarfod [11]) has to be applied; this problem will be considered in a future paper.

2. A compound Poisson input. We consider first the case where the inputs arrive at random at an average rate λ, and the amount of each input has a negative exponential distribution with parameter μ, while the release is assumed to be continuous at a uniform unit rate. The total input in the dam during the time interval $(0, T)$ is then given by a compound Poisson distribution whose m.g.f. is $\exp[\lambda T\{(1 - \theta/\mu)^{-1} - 1\}]$; the net input is $Y(T) = X(T) - T$, with the m.g.f. $M_T(\theta) = \exp[\lambda T\{(1 - \theta/\mu)^{-1} - 1\} - \theta T]$. If $\mu \neq \lambda$, the nonzero real solution of $M_1(\theta) = 1$ is easily seen to be $\theta_0 = \mu - \lambda$.

We shall derive our results by first considering the process to start from zero with barriers at b and a and then putting $b = K - u$ and $a = -u$. At the termination of the process, $Y(t)$ takes the value "a" or lies in the internal (b, ∞). Let the $\Pr\{Y(t) = a\} = P_a$, and $\Pr\{Y(t) \geqq b\} = P_b = 1 - P_a$. Also let the conditional p.d.f. of $Y(t) - b$ conditional on $Y(t) \geqq b$ be $p(x)$; because of the "forgetful property" of negative exponential distribution, $p(x) = \mu e^{-\mu x}$. Hence, we have, from (1.3)

$$1 = P_a e^{a\theta_0} + (1 - P_a) \int_0^\infty e^{(b+x)\theta_0} \mu e^{-\mu x}\, dx = P_a e^{a\theta_0} + (1 - P_a) \frac{e^{b\theta_0}}{1 - \theta_0/\mu}.$$

Putting $\theta_0 = \mu - \lambda = \nu$, $a = -u$, $b = K - u$ and denoting the probability by P_u, we obtain,

$$P_u = (\lambda e^{\nu u} - \mu e^{\nu K})/(\lambda - \mu e^{\nu K}), \qquad (\mu \neq \lambda). \tag{2.1}$$

If, however, $\mu = \lambda$ (i.e. mean input λ/μ = output 1) then, by a limiting process we obtain,

$$P_u = 1 - u/(K + 1/\lambda). \tag{2.2}$$

We will now proceed to derive the probability distribution of t, the time at which the dam becomes empty for the first time, before it overflows. The roots $\theta_1(\phi)$, $\theta_2(\phi)$ of $\log M_1(\theta) = -i\phi$ are found to be

$$\theta_1(\phi), \theta_2(\phi) = \tfrac{1}{2}[\nu + i\phi \pm \{(\nu + i\phi)^2 - 4\, i\phi\mu\}^{\frac{1}{2}}]. \tag{2.3}$$

Putting $\theta = \theta_1(\phi)$ in (1.1) we obtain,

$$\begin{aligned} 1 &= P_a e^{a\theta_1(\phi)} C_a(\phi) + P_b C_b(\phi) \int_0^\infty e^{(b+x)\theta_1} \mu e^{-\mu x}\, dx \\ &= P_a e^{a\theta_1(\phi)} C_a(\phi) + \frac{P_b C_b(\phi) e^{b\theta_1(\phi)}}{1 - \theta_1/\mu} \end{aligned} \tag{2.4}$$

where $C_a(\phi)$ is c.f. of t, conditional on $Y(t) = a$, and $C_b(\phi)$ is c.f. of t, conditional on $Y(t) \geqq b$.

Similarly putting $\theta = \theta_2(\phi)$ in (1.1), we obtain

$$P_a e^{a\theta_2(\phi)}\, C_a(\phi) + P_b\, C_b(\phi) e^{b\theta_2(\phi)}/(1 - \theta_2/\mu) = 1. \tag{2.5}$$

Substituting $b = K - u$, $a = -u$ and writing $P_a C_a(\phi)$ as $P_u C_u(\phi)$ in (2.4) and (2.5) we obtain

$$P_u C_u(\phi) = e^{u(i\phi+\nu)} \frac{(\mu - \theta_1)e^{(K-u)\theta_2} - (\mu - \theta_2)e^{(K-u)\theta_1}}{(\mu - \theta_1)e^{K\theta_2} - (\mu - \theta_2)e^{K\theta_1}} \tag{2.6}$$

as the c.f. of "t" the time at which the dam becomes empty for the first time, without overflowing.

Using the above result, we shall now derive the c.f. $C(\phi)$ of the time of first emptiness, regardless of how many times the dam overflows in the meantime. The barrier zero can be reached from u after 0, 1, 2, $\cdots$ number of passages through K. We derive the conditional characteristic functions of t for all these cases; their sum will give the required result.

The c.f. of t, conditional on not touching K is obviously $C_0'(\phi) = P_u C_u(\phi)$. The c.f. of t conditional on only one passage through K is

$$C_1'(\phi) = Q_u D_u(\phi) P_K C_K(\phi),$$

where $Q_u D_u(\phi)$ is the c.f. of "t", of reaching K before reaching zero and $P_K C_K(\phi)$ the c.f. of "t" of reaching zero from K before passing through K. The first expression can be obtained from (2.4) and (2.5) in a manner similar to that used for $P_u C_u(\phi)$, while the second can be obtained from (2.6) by putting $u = K$. Thus

$$Q_u D_u(\phi) = \lambda\{e^{u\theta_1} - e^{u\theta_2}\}/[(\mu - \theta_2)e^{K\theta_1} - (\mu - \theta_1)e^{K\theta_2}] \tag{2.7}$$

$$P_K C_K(\phi) = e^{K(i\phi+\nu)}(\theta_2 - \theta_1)/[(\mu - \theta_1)e^{K\theta_2} - (\mu - \theta_2)e^{K\theta_1}]. \tag{2.8}$$

Similarly the c.f. of t conditional on two passages through K is $C_2'(\phi) = Q_u D_u(\phi) Q_K D_K(\phi) P_K C_K(\phi)$. Hence

$$\begin{aligned} C(\phi) &= C_0'(\phi) + C_1'(\phi) + C_2'(\phi) + \cdots \\ &= P_u C_u(\phi) + Q_u D_u(\phi) P_K C_K(\phi)/(1 - Q_K D_K(\phi)), \end{aligned} \tag{2.9}$$

for $|Q_K D_K(\phi)| < 1$.

Substituting the values of P_u, Q_u, $C_u(\phi)$, $D_u(\phi)$ etc., in (2.9), we obtain

$$C(\phi) = e^{u(i\phi+\nu)} \frac{\{(\mu - \theta_1)e^{(K-u)\theta_2} - (\mu - \theta_2)e^{(K-u)\theta_1} - \lambda[e^{(K-u)\theta_2} - e^{(K-u)\theta_1}]\}}{(\mu - \theta_1)e^{K\theta_2} - (\mu - \theta_2)e^{K\theta_1} - \lambda(e^{K\theta_2} - e^{K\theta_1}]} \tag{2.10}$$

as the c.f. of the time at which the dam becomes empty, for the first time regardless of overflow in the meantime.

3. A further application: Discrete time model with geometric input. Here we consider a discrete time model in which the dam is fed during time intervals $(t, t + 1)$ $(t = 0, 1, \cdots)$ by independent inputs with $\Pr\{X(t) = r\} = pq^r$ $(r = 0, 1, \cdots, 0 < p < 1, q = 1 - p)$, and there is a release of one unit at the end of every such interval, except when the dam is empty. The net input $Y(t) = X(t) - 1$ has the m.g.f. $M(\theta) = pe^{-\theta}(1 - qe^{\theta})^{-1}$. The nonzero solution of $M(\theta) = 1$ is $\theta_0 = \log p/q$, $(p \neq q)$. For the sake of simplicity we assume that u and K are integers.

We proceed as in Sections 1 and 2, using Wald's Fundamental Identity instead of its extension, and noting that, since the geometric distribution has the "forgetful property", the excess over the upper boundary "b" has the same distribution as $X(t)$. Thus

$$P_u G_u(s) = s^u(p/q)^u \frac{\lambda_1^{K-u+1}(s) - \lambda_2^{K-u+1}(s)}{\lambda_1^{K+1}(s) - \lambda_2^{K+1}(s)}, \tag{3.1}$$

$$G(s) = s^u(p/q)^u \frac{\lambda_1^{K-u+1} - \lambda_2^{K-u+1} - s(\lambda_1^{K-u-1} - \lambda_2^{K-u-1})}{\lambda_1^{K+1} - \lambda_2^{K+1} - s(\lambda_1^{K-1} - \lambda_2^{K-1})} \tag{3.2}$$

where $G_u(s)$ is the conditional probability generating function (p.g.f.) of "t", the time at which the dam becomes empty for the first time, without overflowing, $G(s)$ is the p.g.f. of the time of first emptiness regardless of overflow and $\lambda_1(s)$, $\lambda_2(s) = (1 \pm \{1 - 4pqs\}^{\frac{1}{2}})/2q$.

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REFERENCES

[1] Bartlett, M. S. (1955). *An Introduction to Stochastic Processes*. Cambridge Univ. Press (1955).

[2] Bellman, R. (1957). On a generalization of the fundamental identity of Wald. *Proc. Cambridge Philos. Soc.* **53** 257–259.

[3] Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1953). Sequential decision problems for processes with continuous time parameter testing hypotheses. *Ann. Math. Statist.* **24** 254–264.

[4] Gani, J. (1958). Elementary methods in an occupancy problem of storage. *Math. Ann.* **136** 454–465.

[5] Gani, J. and Prabhu, N. U. (1958). Continuous time treatment of a storage problem. *Nature* **182** 39–40.

[6] Gani, J. and Prabhu, N. U. (1959). Remarks on the dam with Poisson type inputs. *Austral. J. Appl. Sci.* **10** 113–122.

[7] Gani, J. and Prabhu, N. U. (1959). The time-dependent solution for a storage model with Poisson input. *J. Math. Mech.* **8** 653–664.

[8] Kendall, D. G. (1957). Some problems in the theory of dams. *J. Roy. Statist. Soc. Ser. B* **19** 207–212.

[9] Lévy, P. (1948). *Processus stochastiques et Mouvement Brownien*. Gauthier-Villars, Paris.

[10] Moran, P. A. P. (1954). A probability theory of dams and storage systems. *Austral. J. Appl. Sci.* **5** 116–124.

[11] Moran, P. A. P. (1955). A probability theory of dams and storage systems: modification of the release rules. *Austral. J. Appl. Sci.* **6** 117–130.

[12] Phatarfod, R. M. (1959). Further problems in the theory of sequential analysis. Ph.D. Thesis, Manchester Univ.

[13] Prabhu, N. U. (1958). Some exact results for the finite dam. *Ann. Math. Statist.* **29** 1234–1243.

[14] Tweedie, M. C. K. (1960). Generalization of Wald's fundamental identity of sequential analysis to Markov chains. *Proc. Cambridge Philos. Soc.* **56** 205–214.

[15] Wald, A. (1947). *Sequential Analysis*. Wiley, New York.

[16] Weesakul, B. (1961). First emptiness in a finite dam. *J. Roy. Statist. Soc. Ser. B* **23** 343–351.