THE CONDITIONAL DISTRIBUTION OF SETS OF TESTS ON A SYSTEM SIMULATED FROM TESTS ON ITS COMPONENTS[1]

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1. Introduction. This note is concerned with a system which is made up of components in such a way that failure of any component causes the failure of the system, and the system cannot fail unless some component fails. A number of units of each component are tested and the result of the test for each unit is a success or a failure. Tests on units of the system may be simulated by randomly selecting one result for each component and regarding the collection of results as comprising a trial for the system. If there are no failures among the component results, then the system trial is a success; otherwise, it is a failure.

This method of simulating system trials from component results is appealing when only a few systems are to be constructed, but components are more plentiful. Testing only a few units of the system will not yield very precise estimates of the probability of success of the system (or briefly, its reliability), but testing larger numbers of units of the components and simulating tests on units of the system will provide more precise estimates. If the tests are destructive, the appeal of simulation is even greater, for it may be unreasonable to destroy units of the system merely for the purpose of estimating their reliability. But, in view of the smaller loss involved in the destruction of components, it may be quite reasonable to subject some units to destructive testing.

The number of system units which can be simulated without repeating the use of a unit of any component is the least number of units tested for any component. The trials for these simulated system units will be called a set of system trials. If the tests on the units of the components are statistically independent, then the simulated system trials are independent. Further, if the probability that a unit will successfully pass the test is a constant for all units of the same component, then the probability that a simulated system trial will be a success is a constant. The estimate of the reliability of a component is the proportion of successes, and of the system is the product of the reliabilities of the components. Exact confidence limits for the system may be obtained by use of any of the methods developed for the binomial distribution.

It seems intuitively reasonable that more information can be obtained about the unknown system reliability by considering all sets of system trials which can be generated by the simulation process, than by considering only one such set. This approach has been taken in [1], where two methods of determining confidence intervals have been suggested. However, the authors were unable to associate exact confidence statements with the methods, and the present note does not develop such statements.

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The purpose of this note is as follows. There are n_i trials for component i, $i = 1, \cdots, k$, of which n_{i1} are successes and the rest are failures. Let

$$n^* = \min_i n_i, \qquad n_1^* = \min_i n_{i1}. \tag{1.1}$$

A trial for the system is simulated by drawing at random a result for each component. If all component results are successes, the system trial is a success. If not, the system trial is a failure. The process is continued, without replacement of component results, until n^* system trials have been generated. The principal purpose of this note is to derive the conditional distribution of S, the number of successful system trials, $0 \leqq S \leqq n_1^*$, given $n_{11}, \cdots, n_{k1}$. It also will be shown that the conditional expectation of S, $E(S)$, is the product of the estimated component reliabilities, i.e.,

$$E(S) = \prod_i (n_{i1}/n_i). \tag{1.2}$$

2. Derivation of the conditional distribution. Let A_j be the event that the jth system trial is a success, $j = 1, \cdots, n^*$. Then for any $j_1, \cdots, j_m$, $1 \leqq m \leqq n_1^*$,

$$P(A_{j_1} \cdots A_{j_m}) = \prod_{i=1}^{k} \left[\binom{n_{i1}}{m} m!\right]\left[\binom{n_i - m}{n^* - m}(n^* - m)!\right] \Big/ \left[\binom{n_i}{n^*} n^*!\right]. \tag{2.1}$$

Let

$$S_m = \sum_{j_1, \cdots, j_m} P(A_{j_1} \cdots A_{j_m}).$$

Then

$$S_m = \binom{n^*}{m} P(A_1 \cdots A_m) = \binom{n^*}{m}^{-(k-1)} \prod_{i=1}^{k} \binom{n_{i1}}{m}\binom{n_i - m}{n^* - m} \Big/ \binom{n_i}{n^*}, \tag{2.2}$$

and by [2], Section IV.3, p. 96, Formula (3.1), it follows that the probability that $S = s$ is

$$p(s) = \sum_{j=0}^{n_1^* - s} (-1)^j \binom{s+j}{s} S_{s+j}. \tag{2.3}$$

This is summarized in the following theorem.

THEOREM 1. *The conditional probability of exactly s successful simulated system trials is given by Formula* (2.3).

3. Average of the conditional distribution. It will now be demonstrated that the average simulated system trial is equal to the product of the $\hat{p}_i$'s, where $\hat{p}_i = n_{i1}/n_i$.

The expected value of the conditional probability distribution of the number of successes in a set of simulated system trials is

$$E(S) = \sum_{y=0}^{n_1^*} p(y) y / n^*. \tag{3.1}$$

The coefficient of S_m/n^* in (3.1) is

$$\sum_{z=1}^{m} (-1)^{m+z} z \binom{m}{z}, \tag{3.2}$$

which is zero for $m \geqq 2$. This can be demonstrated by setting $p = q = 1$ in

$$m(p-q)^{m-1} = mp^{m-1} - m\binom{m-1}{1} p^{m-2} q + \cdots . \tag{3.3}$$

(See [2], p. 61, (12.1), third formula.) Thus, the only non-zero term in $E(S)$ occurs for $m = 1$, and is

$$p(1)/n^* = \prod_{i=1}^{k} n_{i1} \Big/ \prod_{i=1}^{k} n_i , \tag{3.4}$$

which completes the proof of the following theorem:

THEOREM 2. *The expected value of the conditional probability distribution of the number of successes in a set of simulated system trials is* $E(S) = \prod_{i=1}^{k} \hat{p}_i$, *where* $\hat{p}_i = n_{i1}/n_i$, *the ratio of the number of successes to the number of trials for the* ith *component.*

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REFERENCES

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[2] FELLER, WILLIAM (1957). *An Introduction to Probability Theory and Its Applications*, **1** 2nd ed. Wiley, New York.