SOME CONSEQUENCES OF RANDOMIZATION IN A GENERALIZATION OF THE BALANCED INCOMPLETE BLOCK DESIGN[1]

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1. Summary. The present paper envisages a generalized situation of the balanced incomplete block design in the sense of allowing for the sampling of sources of experimental material, of blocks within sources, of experimental units within blocks, and of treatments under consideration. A model for an arbitrary observation of a generalized balanced incomplete block design is derived explicitly from the physical way in which the experiment is performed, i.e., from the way in which the sampling and randomization procedures are carried out. The correlational structure of the observations is therefore implicit in the model. The model initially uses no assumptions of additivity of treatments with experimental material. It is shown that expected values of squares of partial observational means, as well as the expected values of products of individual observations, admit simple and easily specifiable expressions in terms of quantities called cap sigmas and denoted by Σ's. The expected values of mean squares in the analysis of variance table are then derived. Consequences of the presence of various types of nonadditivity on the usual test of no treatment effects are discussed for fixed, mixed and random situations. For example, when the blocks actually used in the experiment form a random sample from an infinite population of blocks then the presence of interactions of blocks with treatments produces no bias in the comparison of the error and adjusted treatment mean squares. The correlational structure of the observations under the simplifying additivity assumption is examined for the standard balanced incomplete block design. It is shown that the usual forms of estimators of treatment comparisons are appropriate and that the Σ's play the roles which the block and plot variances have in the corresponding assumed infinite model. In the presence of nonadditivities of treatments with the experimental material the usual forms of the linear estimators are no longer best.

2. Introduction. The balanced incomplete block design was first introduced by Yates (1936a). Although it has always been vaguely recognized that the validity of the suggested analysis of the design has its roots in the randomization procedures followed in performing the experiment, the analyses actually given have claimed their justification from some *a priori* assumed error structure.

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Thus, there exists the intrablock analysis which purports to assume only one kind of error, and there is also the analysis, first exhibited by Yates (1940), which combines intra and interblock information, based on the assumptions of two kinds of error. Equivalently to the last procedure both intra and interblock information can be utilized by considering in the assumed model blocks as random uncorrelated variables and by carrying out the consequent generalized least squares analysis. Rao (1959) obtained the expectations under randomization of mean squares with non-additivity, for the standard case of the incomplete block design; that is, no sampling was considered of any of the entities, i.e. blocks, units in blocks, treatments.

In recent years the models for a large number of experimental designs have been expressed in a form exhibiting explicitly a one-one correspondence with the way the experiment has been carried. Many of the models have been studied under general conditions allowing for the sampling of the entities under consideration and involving the presence of all possible interactions and not involving any specialized assumptions. For a large class of balanced situations, e.g. completely randomized, randomized block, Latin square, split-plot, and a number of cases in which treatments are subject to error, it has been found that the introduction of population quantities termed Σ's allows considerable simplification and unification in obtaining second moment results, and particularly in obtaining expected mean squares in the analysis of variance. Some references on these matters are Kempthorne (1955), Wilk (1955a, 1955b), Wilk and Kempthorne (1957), Cox (1958), Zyskind and Kempthorne (1960), Folks and Kempthorne (1960), Kempthorne et al (1961), Zyskind (1962). The balanced incomplete block design structure is not balanced in the sense that there is no unique analysis of variance which gives all of the relevant information. The question arises therefore as to the utility and applicability of the above general approach to the case of the balanced incomplete block design. In addition to the question of the possible simplicity of the Σ expressions for the expectations of squares of sample means there is also the novel issue as to whether the expectations of the products of two different sample observations admit simple and easily specifiable Σ expressions. This issue arises in the balanced incomplete block design because here some of the mean squares, such as that for treatments eliminating blocks, are not expressible as linear functions of squares of partial observational means. It seemed appropriate therefore to consider the case of the balanced incomplete block design in as broad a way as possible from the point of view of sampling of the entities such as sources of experimental material, blocks, units within blocks and treatments, which are involved. The results of the present paper were obtained independently of those of Rao (1959). They are less general in the sense that they are limited to the case of the balanced incomplete block design only; however, they are more general in the sense that they allow for the possibility of sampling of all of the entities under consideration and hence for the exploration of the mixed, fixed and random situations. In addition the present paper explores the role played by the Σ quantities. A preliminary report of the present results was presented at the annual statistical meetings in Washington in 1959, and subsequently an abstract was published by Zyskind (1960).

3. Considerations under the general case. We consider that the experimental material has a hierarchical structure as follows. The experimental units are arranged in sets of size K, each set being called a block. Also, the blocks are arranged into sets of size B, and there are S such sets, called sources, of experimental material altogether.

The treatments whose effects are to be investigated are T in number and are identified by the subscript w, $w = 1, 2, \cdots, T$. There is also an *a priori* arrangement of the first t integers into b groups each of size k, where each one of the integers occurs altogether in r of the groups, any two integers appear together in λ of the groups, and

$$tr = bk. \tag{1}$$

The different groups are denoted by the subscript u, $u = 1, 2, \cdots, b$; and the elements in the groups by the subscript v, $v = 1, 2, \cdots, k$; so that every uv combination denotes a particular integer z, $z = 1, 2, \cdots, t$.

The above situation is adapted to the standard case of the balanced incomplete block design by putting the number of treatments equal to t and having the t treatments associated through their subscripts with the corresponding t integers, thus obtaining b distinct treatment groups. Further, the number of sources of experimental material is taken equal to one, the number of blocks within a source equal to b, and the number of experimental units within a block equal to k. The experimental procedure consists then of associating randomly the treatment groups with blocks, and then within the achieved combinations of treatment group and block of applying the individual treatments of the group to the individual units of the block at random. If the block and unit entities are denoted by the indices j and m respectively, $j = 1, 2, \cdots, b$; $m = 1, 2, \cdots, k$; then the experimental procedure is fully characterized mathematically by the introduction of two sets of random variables as follows:

$B_j^u = 1$ if treatment group u is associated with block j in the outcome of the experiment and

$B_j^u = 0$ otherwise;

$P_{um}^{uv} = 1$ if in the block with which treatment group u is associated the treatment uv falls on plot m and

$P_{um}^{uv} = 0$ otherwise.

The distributions of the random variables so defined follow entirely from the randomization scheme employed in carrying out the experiment. Thus, $P\{B_j^u = 1\} = b^{-1}$, $P\{B_j^u = 0\} = 1 - b^{-1}$, $P\{P_{um}^{uv} = 1\} = k^{-1}$, $P\{P_{um}^{uv} = 0\} = 1 - k^{-1}$. Also, the variables B_j^u are uncorrelated with the variables P_{um}^{uv} and the moments of these variables can be easily obtained. To develop a model for the simple balanced incomplete block case denote the conceptual response due to treatment

$w = uv$ falling on plot m of block j by $Y_{jmuv} = Y_{jmw}$. Then there are bkt such conceptual responses and the performance of the experiment yields a subset of bk of these. If the observed response on the (uv)th treatment is denoted by X_{uv}, then the relation of X_{uv} to the set of observed responses is given by

$$X_{uv} = \sum_{jm} B_j^u P_{um}^{uv} Y_{jmuv}. \tag{2}$$

Because the properties of the variables B_u^j, P_{um}^{uv} can be easily obtained, the properties of the X_{uv}'s can also be worked out.

For the general case described at the beginning of the present section, a random sample of t treatments is taken from the T treatments under consideration. The chosen treatments are then associated with the integers 1, 2, 3, $\cdots$, t according to the order of their choice, so that in this manner b chosen treatment groups each of size k are obtained. Independently for each chosen source the b treatment groups are randomly associated with b out of the B existing blocks of that source, within each chosen block treatment group combination the k treatments of the group are randomly associated with k out of the K experimental units of the block, and altogether s sources out of S are randomly selected for the performance of the experiment.

Among the random variables describing the experimental procedure we now have

$S_i^{i^*} = 1$ if the i^*th chosen source in the sample is the ith source of the population and

$S_i^{i^*} = 0$ otherwise; $i^* = 1, 2, \cdots, s$, and $i = 1, 2, \cdots, S$.

Other sets of variables $B_{i^*j}^{i^*u}$, $P_{i^*um}^{i^*uv}$, T_w^{uv} are defined analogously to the standard case already described. Thus, for example, $B_{i^*j}^{i^*u} = 1$ if in the i^*th selected source integer group u is associated with the jth block of the source and is equal to zero otherwise.

In this general situation denote the generic conceptual response by Y_{ijmw} and the observed value on the treatment associated with uv in the i^*th selected source by X_{i^*uv}. Then

$$X_{i^*uv} = \sum_{ijmw} S_i^{i^*} B_{i^*j}^{i^*u} P_{i^*um}^{i^*uv} T_w^{uv} Y_{ijmw}. \tag{3}$$

The expression for X_{i^*uv} can be put into a more familiar form when use is made of the following identical decomposition for Y_{ijmw}, where the absence of subscripts denotes the averaging over the population ranges of the indices omitted. Now

$$\begin{aligned} Y_{ijmw} &= Y + (Y_i - Y) + (Y_{ij} - Y_i) + (Y_{ijm} - Y_{ij}) + (Y_w - Y) \\ &\quad + (Y_{iw} - Y_i - Y_w + Y) + (Y_{ijw} - Y_{ij} - Y_{iw} + Y_i) \\ &\quad + (Y_{ijmw} - Y_{ijm} - Y_{ijw} + Y_{ij}) \\ &= \mu + S_i + B_{i(j)} + P_{ij(m)} + T_w + (ST)_{iw} + (BT)_{i(jw)} \\ &\quad + (PT)_{ij(mw)}. \end{aligned} \tag{4}$$

The correspondence between the combinations of Y's within the brackets and the briefer symbolic expressions for these population components should be evident.

Substituting the identical expansion for Y_{ijmw} in the expression for X_{i^*uv} we obtain

$$
\begin{aligned}
X_{i^*uv} = \mu &+ \sum_{i} S_i^{i^*} S_i + \sum_{ij} S_i^{i^*} B_{i^*j}^{i^*u} B_{i(j)} \\
&+ \sum_{ijm} S_i^{i^*} B_{i^*j}^{i^*u} P_{i^*um}^{i^*uv} P_{ij(m)} + \sum_{w} T_w^{uv} T_w + \sum_{iw} S_i^{i^*} T_w^{uv} (ST)_{iw} \\
&+ \sum_{ijw} S_i^{i^*} B_{i^*j}^{i^*u} T_w^{uv} (BT)_{iw} + \sum_{ijmw} S_i^{i^*} B_{i^*j}^{i^*u} P_{i^*um}^{i^*uv} T_w^{uv} (PT)_{ij(mw)} .
\end{aligned} \tag{5}
$$

In order to relate the considerations given so far to a more general framework which has been found useful in the treatment of many randomized designs a brief digression on experimental structures follows.

Identities like the one for the conceptual response Y_{ijmw} above can in general be arrived at by the following considerations. Partial population means can be obtained by averaging over the entire range of values of particular sets of subscripts. Partial means are here denoted by the usual symbol for a response but with omission of subscripts over which the average has been taken. An admissible mean is defined as one in which whenever a nested index appears then all the indices which nest it appear also. Our considerations are restricted to admissible means only. The indices of an admissible partial mean which nest no other indices of that mean are said to constitute the set of indices belonging to the rightmost bracket. It is convenient to indicate the grouping of the indices of the rightmost bracket by using parentheses, ( ), and also to group in this way other sets of indices when we wish to emphasize that for some structural reason they belong to the same category. In the present case the admissible partial means are eight in number and may be denoted by Y, Y_i, $Y_{i(j)}$, $Y_{ij(m)}$, Y_w, Y_{iw}, $Y_{i(jw)}$, $Y_{ij(mw)}$.

From every partial mean linear combinations of means can be formed which are of special physical and formal significance. These linear combinations, henceforth called components, are obtained by selecting all those partial means which are yielded by the mean in question when some, all, or none of its rightmost bracket subscripts are omitted in all possible ways. Whenever an odd number of indices is omitted the mean is to be preceded by a negative sign, whenever an even number is omitted the mean is to be preceded by a positive sign. The number zero is considered even. For example, in the present case the partial mean $Y_{i(j)}$ leads to the component $(Y_{i(j)} - Y_i)$, the mean $Y_{i(jw)}$ to $(Y_{i(jw)} - Y_{i(j)} - Y_{iw} + Y_i)$, and the mean Y to (Y). The components thus constructed have a correspondence with the effects and interactions of the usual assumed linear models. It has been shown, Zyskind (1962), that for any given population structure the typical response can be expressed identically as a sum of all its corresponding components. The above relation is called the population identity.

The component of variation corresponding to a given type of population component is defined to be the sum of squares of all the different values of the type divided by the number of linearly independent values of the component. Thus, some examples in the present case are

$$\sigma^2_{(\phi)} = \mu^2 = Y^2, \qquad \sigma^2_S = \frac{\sum_{i=1}^S S_i^2}{S-1},$$

$$\sigma^2_T = \frac{\sum_{w=1}^T (Y_w - Y)^2}{(T-1)}, \qquad \sigma^2_{SB(PT)} = \frac{\sum_{ijmw} (PT)^2_{ij(mw)}}{SB(K-1)(T-1)}.$$

The introduction of quantities called cap sigmas and denoted by Σ's with appropriate sets of subscripts, has been found very useful in obtaining general expressions for the expected values of mean squares in a large class of experimental situations. Included in this class are the cases of complex pure sampling schemes, and also those of multifactor completely randomized, randomized block, generalizations of the split-plot, and the Latin square designs—as well as various modifications of these standard designs when in addition the applications of intended treatment amounts are subject to error. The use of Σ's will also prove to be of value in the present paper. The Σ's are defined for the general situation as follows.

Definition: Consider a particular type of component and all σ^2's of the following form:

(i) the set of subscripts of σ^2 includes the set of subscripts corresponding to the leading term of the component as a subset,

(ii) the excess subscripts belong exclusively to the rightmost bracket of σ^2.

The linear combination of all such σ^2's, where the coefficient of a particular σ^2 with c excess subscripts is $(-1)^c$/(Product of population ranges of the excess indices), is defined as the Σ corresponding to the type of component under consideration. The subscript notation for the Σ is to be the same as for the type of component.

It should be pointed out that the component of variation corresponding to the null set is $Y^2 = \sigma^2_{(\phi)}$, and that the corresponding $\Sigma_{(\phi)}$ is uniquely defined. The introduction of $\Sigma_{(\phi)}$ is of importance to the development of the present approach. In the case of the balanced to incomplete block the experimental units are nested in blocks which in turn are nested in sources. This structure leads to the following expressions for the Σ's:

$$\Sigma_\phi = \sigma^2_\phi - S^{-1}\sigma^2_S - T^{-1}\sigma^2_T + (ST)^{-1}\sigma^2_{ST},$$

$$\Sigma_S = \sigma^2_S - B^{-1}\sigma^2_{S(B)} - T^{-1}\sigma^2_{ST} + (BT)^{-1}\sigma^2_{S(BT)}, \qquad \Sigma_T = \sigma^2_T - S^{-1}\sigma^2_{ST},$$

$$\Sigma_{ST} = \sigma^2_{ST} - B^{-1}\sigma^2_{S(BT)}, \qquad \Sigma_{S(B)} = \sigma^2_{S(B)} - T^{-1}\sigma^2_{S(BT)} - K^{-1}\sigma^2_{SB(P)} + (TK)^{-1}\sigma^2_{SB(PT)}, \tag{6}$$

$$\Sigma_{S(BT)} = \sigma^2_{S(BT)} - K^{-1}\sigma^2_{SB(PT)},$$

$$\Sigma_{SB(P)} = \sigma^2_{SB(P)} - T^{-1}\sigma^2_{SB(PT)} \qquad \Sigma_{SB(PT)} = \sigma^2_{SB(PT)}.$$

By making use of the properties of the sampling and design random variables various quantities of interest in the balanced incomplete block situation can now be calculated. In particular the expected value of the square of the overall experimental mean, $E(X^2)$, is found to have in terms of Σ's the very simple form

$$E(X^2) = E(\sum_{i^*uv} X_{i^*uv}/sbk)^2 = \Sigma_{(\phi)} + s^{-1}\Sigma_S + (sb)^{-1}\Sigma_{S(B)} + (sbk)^{-1}\Sigma_{SB(P)} + t^{-1}\Sigma_T + (st)^{-1}\Sigma_{ST} + (sbk)^{-1}\Sigma_{S(BT)} + (sbk)^{-1}\Sigma_{SB(PT)} . \tag{7}$$

The coefficient of each Σ_i in the above expansion is unity divided by the number of possibly different values of the component of type i entering into the formation of the overall experimental mean.

Since X_{i^*} is the observed mean for the observations in the i^*th chosen source it should be clear that the moments of X_{i^*}, $E(X_{i^*}^p)$, $i^* = 1, 2, \cdots, s$, are all equal for the different values of i^*. $E(X_{i^*}^2)$ can therefore be regarded as the expected value of the square of the overall experimental mean when the number of sources in the sample is unity. Hence the form for $E(X_{i^*}^2)$ is identical with the form for $E(X^2)$ when the substitution $s = 1$ is made in it. Thus

$$E(X_{i^*}^2) = \Sigma_\phi + \Sigma_S + b^{-1}\Sigma_{S(B)} + (bk)^{-1}\Sigma_{SB(P)} + t^{-1}\Sigma_T + t^{-1}\Sigma_{ST} + (bk)^{-1}\Sigma_{S(BT)} + (bk)^{-1}\Sigma_{SB(PT)} . \tag{8}$$

The relevance of the above expressions to the computation of expected value of mean squares in the analysis of variance table becomes clear when one notes that many analysis of variance quantities of interest are exemplified by the expected value of the between sources mean square, which can be written in two forms

$$E\{(s-1)^{-1}\textstyle\sum_{i^*uv} (X_{i^*} - X)^2\} = sbk(s-1)^{-1}E(X_{i^*}^2 - X^2). \tag{9}$$

Thus, in many cases the expected value of a mean square of a line in the analysis of variance table is expressible as a constant times a known linear combination of expected values of squares of partial experimental means.

If X_{i^*u} denotes the observed mean of the treatment group u in the i^*th chosen source, then

$$E(X_{i^*u}^2) = \Sigma_\phi + \Sigma_S + \Sigma_{S(B)} + k^{-1}\Sigma_{SB(I)} + k^{-1}\Sigma_T + k^{-1}\Sigma_{ST} + k^{-1}\Sigma_{S(BT)} + k^{-1}\Sigma_{SB(PT)} . \tag{10}$$

It will be noted that here again the coefficient of each Σ_i quantity is unity divided by the number of different values of component i entering into the formation of X_{i^*u}. A similar expression applies also to $E(X_{w^*}^2)$, where X_{w^*} is the observed mean of the treatment to which w^*, one of the numbers 1, 2, $\cdots$, t, has been randomly assigned. We define analogously the symbols $X_{i^*w^*}$ and $X_{i^*w^*f}$, where for example $X_{i^*w^*f}$ stands for the fth observed value on the w^*th chosen treatment in the i^*th chosen replicate.

Because blocks and treatments are not orthogonal some of the values of mean

TABLE 1

Due to	D.F.	E.M.S.
Sources	$s-1$	$b k\Sigma_S + k\Sigma_{S(B)} + \Sigma_{SB(P)} + r\Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)}$ $= bk\sigma_S^2 + k\left(1-\frac{b}{B}\right)\sigma_{S(B)}^2 + \left(1-\frac{k}{K}\right)\sigma_{SB(P)}^2 + 0\sigma_T^2 + \left(-\frac{bk}{T}+r\right)\sigma_{ST}^2 + \left(\frac{bk}{BT}-\frac{k}{T}-\frac{r}{B}+1\right)\sigma_{S(BT)}^2$ $+\left(1-\frac{1}{K}-\frac{1}{T}+\frac{k}{KT}\right)\sigma_{SB(PT)}^2$
Blocks ignoring Treatments	$s(b-1)$	$k\Sigma_{S(B)} + \Sigma_{SB(P)} + \frac{b(t-k)}{t(b-1)}\Sigma_T + \frac{b(t-k)}{t(b-1)}\Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)}$ $= k\sigma_{S(B)}^2 + \left(1-\frac{k}{K}\right)\sigma_{SB(P)}^2 + \frac{b(t-k)}{t(b-1)}\sigma_T^2 + \frac{b(t-k)}{t(b-1)}\left(1-\frac{1}{S}\right)\sigma_{ST}^2 + \left[1-\frac{k}{T}-\frac{b(t-k)}{t(b-1)K}\right]\sigma_{S(BT)}^2$ $+\left(1-\frac{1}{K}-\frac{1}{T}+\frac{k}{KT}\right)\sigma_{SB(PT)}^2$
Treatments adjusted	$t-1$	$\Sigma_{SB(P)} + sb\frac{(k-1)}{(t-1)}\Sigma_T + b\frac{(k-1)}{(t-1)}\Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)}$ $= sb\frac{(k-1)}{(t-1)}\sigma_T^2 + b\frac{(k-1)}{(t-1)}\left(1-\frac{s}{S}\right)\sigma_{ST}^2 + \left[1-\frac{b(k-1)}{B(t-1)}\right]\sigma_{S(BT)}^2 + \sigma_{SB(P)}^2$ $+\left(1-\frac{1}{K}-\frac{1}{T}\right)\sigma_{SB(PT)}^2$
Error	$sbk-sb-t+1$	$\Sigma_{SB(P)} + 0\Sigma_T + \frac{rts-rt-sb+b}{sbk-sb-t+1}\Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)}$ $= \frac{rts-rt-sb+b}{sbk-sb-t+1}\sigma_{ST}^2 + \left[1-\frac{1}{B}\frac{rts-rt-sb+b}{sbk-sb-t+1}\right]\sigma_{S(BT)}^2 + \sigma_{SB(P)}^2$ $+\left(1-\frac{1}{K}-\frac{1}{T}\right)\sigma_{SB(PT)}^2$

squares, as for example that for treatments eliminating blocks, cannot be expressed as simple known linear functions of values of squares of partial observational sample means. Fortunately however, a few other simple and useful results can be derived and used for such cases. These are

$$E(X_{i^*uv}X_{i^*uv'}) = \Sigma_\phi + \Sigma_S + \Sigma_{S(B)}, \qquad v \neq v';$$

$$E(X_{i^*w^*f}X_{i^*w^*f'}) = \Sigma_\phi + \Sigma_S + \Sigma_{ST} + \Sigma_T, \qquad f \neq f';$$

$$E(X_{i^*w^*f}X_{i^{*\prime}w^*f'}) = \Sigma_\phi + \Sigma_T, \qquad i^* \neq i^{*\prime};$$

and

$$E(X_{i^*w^*f}X_{i^{*\prime}w^{*\prime}f'}) = \Sigma_\phi, \qquad i^* \neq i^{*\prime}, w^* \neq w^{*\prime}.$$

By making use of expressions like the above we can obtain, for instance, the expected value of the mean square for treatments eliminating blocks. The expected value of the error mean square can then be obtained by subtraction. After some computations, then, we obtain the analysis of variance exhibited in Table 1.

Inspection of the table reveals the following facts. If the treatments are all identical then $\sigma_T^2 = \sigma_{ST}^2 = \sigma_{S(BT)}^2 = \sigma_{SB(PT)}^2 = 0$, and the expectation of the adjusted treatment mean square equals the expectation of the error mean square. Thus, for the test of Fisher's hypothesis that the effects of all treatments are identical the design is an unbiased one in the sense of Yates (1936b). Further, in the absence of interactions of treatments with experimental material the expectations of the two mean squares in question are equal when $\sigma_T^2 = 0$. It is informative to note that the presence of interactions of experimental units within block with treatments does not introduce any comparative bias into the two mean squares under consideration, no matter what the relation between k and K, and also between t and T. Other interactions however do introduce biases as follows. When $S = s = 1$ then the entire bias is due to the block-treatment interaction and is of the amount $-[b(k - 1)/B(t - 1)]\sigma_{S(BT)}^2$. Further when $b = B$ then the above expression becomes $-(k - 1)(t - 1)^{-1}\sigma_{S(BT)}^2$, which after comparison of terms can be checked to be also the value of the negative bias obtained by Rao (1959). On the other hand when the chosen blocks form a sample from an infinite population of blocks, i.e. when blocks are random, then the bias due to $\sigma_{S(BT)}^2$ becomes zero. This fact is worth observing because when, for example, the blocks are litters of mice then under many circumstances it would be appropriate to treat the sample of chosen blocks as drawn from a large population of litters.

If the number of chosen sources, s exceeds unity and if also treatments interact with sources then one of the following cases obtains. If the sources are fixed, i.e. if $s = S$, then the contribution of σ_{ST}^2 to the expected value of the adjusted treatment mean square is zero and its contribution to the expected value of the error mean square is a positive quantity. Thus, in this case σ_{ST}^2 introduces a negative bias. If however the number of sources is "large," i.e. if the sources are random, then the bias introduced by σ_{ST}^2 is

$$\left[\frac{b(k-1)}{t-1} - \frac{(s-1)b(k-1)}{sbk - sb - t + 1}\right]\sigma^2_{ST}$$

which is a positive quantity.

As an example consider the case where $k = 3$, $r = 5$, $t = 6$, $b = 10$, $s = 5$. Then the coefficient of σ^2_{ST} in the expected value of the mean square for treatments eliminating blocks is 4 while it is 80/95 in the expected value of the mean square for error.

Thus, the bias introduced by σ^2_{ST} is positive when the sources are "random" while it is negative when they are "fixed." This fact should be taken into account because the interaction of treatments with sources is likely to be larger than that of treatments with blocks.

It should be noted that the analysis of variance exhibited in Table 1 is the customary one and would normally be recommended if treatments and the experimental material were expected to be additive. When $s > 1$ the bias in the test for treatments, computed in terms of Σ's, can be removed by partitioning the error sum of squares into the sum of two parts: the interaction of treatments eliminating blocks with sources, which involves $(t - 1)(s - 1)$ degrees of freedom, and the pooled error within sources sum of squares with $s(bk - b - t + 1)$ degrees of freedom. The expected values of the mean squares appropriate to the quantities are respectively:

$$\Sigma_{SB(P)} + \frac{b(k-1)}{(t-1)}\Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)} = \frac{b(k-1)}{(t-1)}\sigma^2_{ST} + \left[1 - \frac{b(k-1)}{B(t-1)}\right]\sigma^2_{S(BT)} + \sigma^2_{SB(P)} + \left(1 - \frac{1}{K} - \frac{1}{T}\right)\sigma^2_{SB(PT)},$$

and

$$\Sigma_{SB(P)} + \Sigma_{S(BT)} + \Sigma_{SB(PT)} = \sigma^2_{SB(P)} + \sigma^2_{S(BT)} + \left(1 - \frac{1}{K} - \frac{1}{T}\right)\sigma^2_{SB(PT)}.$$

It is evident that for S "large" the mean square for the interaction of sources with treatments eliminating blocks provides a "proper" error mean for the test that $\sigma^2_T = 0$. Under all other circumstances the numerator mean square has a bias of the amount $-(s/S)[b(k-1)/(t-1)]\sigma^2_{ST}$. The test has no bias originating from the interaction of treatments with blocks within groups.

On occasion one may desire to partition the total of the sum of squares for blocks ignoring treatments and for treatments eliminating blocks into the sum of squares for treatments ignoring blocks plus the sum of squares for blocks eliminating treatments. Now the sum of squares for treatments ignoring blocks is expressible as a linear function of squares of partial observational means so that the Σ form of its expected value can be obtained at once. The Σ form of the expected value for blocks eliminating treatments is obtained as a difference and yields for the expected value of the mean square

$$\left(k - \frac{t-k}{s(b-1)}\right) \Sigma_{S(B)} + \Sigma_{SB(P)} + 0\, \Sigma_T$$
$$+ \frac{(s-1)(b-r)}{s(b-1)} \Sigma_{ST} + \Sigma_{S(BT)} + \Sigma_{SB(PT)}$$

If uncorrelated measurement errors are also operative, then their presence does not complicate the discussion in any essential way for their contribution adds the variance of measurement errors with coefficient one to each line of expected mean squares of the analysis of variance table.

If in addition, the realizations of intended treatment amounts are also non-reproducible, i.e. treatments are subject to error, then the tables of expected mean squares have to be expanded to include contributions due to the treatment errors and to their interactions with the experimental material. The expectations differ according as the ways of performing the experiment with respect to the application of these treatment errors are varied. The expectations can be obtained and their consequences assessed along the lines developed in Zyskind and Kempthorne (1960).

4. Considerations under usual conditions of design and under additivity assumptions. In this section we take $S = s = 1$, $B = b$, $K = k$, and $T = t$, which corresponds to the common use of the balanced incomplete block design. Further, we assume that the interactions of treatments with experimental material are all zero.

If the treatment w, $w = 1, 2, \cdots, t$, occurs in the treatment set u we denote its observed value by X_{uw}. Using Equation (2) under the present assumptions, and denoting the variance of quantities by the symbol V, we obtain easily

$$V(X_{uw}) = V(B) + V(P);$$
$$(12) \qquad \mathrm{Cov}\,(X_{uw}, X_{uw'}) = V(B) - (k-1)^{-1}V(P), \qquad w \neq w';$$
$$\mathrm{Cov}\,(X_{uw}, X_{u'w'}) = -(b-1)^{-1}V(B), \qquad u \neq u'.$$

Denote the variance and covariances on the left-hand sides above by α, β, and γ respectively.

Thus, the variance of any observation is the sum of the block and plot variances, the covariance of any two different observations in the same block is equal to the block variance minus $(k-1)^{-1}$ times the plot variance, and the covariance of any two observations in different blocks is $-(b-1)^{-1}$ times the block variance.

If we consider the unbiased estimation of a treatment contrast $\sum_w \delta_w t_w$, where $\sum_w \delta_w = 0$, by a linear function of the observations $\sum_{uw} c_{uw} X_{uw}$, then $\sum_w \delta_w t_w = (\sum_{uw} c_{uw})\mu + \sum_w (\sum_u c_{uw}) t_w$, whatever the values of μ and t_w's. Hence $\sum_{uw} c_{uw} = C_{..} = 0$, $\sum_u c_{uw} = C_{.w} = \delta_w$. Denote also $\sum_w c_{uw}$ by $C_{u.}$. The variance of $\sum_{uw} c_{uw} X_{uw}$ is then given by

$$V(\sum_{uw} c_{uw}X_{uw}) = (\alpha - \beta)\sum_{uw} c_{uw}^2 + (\beta - \gamma)(\sum_u C_{u\cdot}^2) \tag{13}$$
$$+ C_{\cdot\cdot}^2\gamma = (\alpha - \beta)\sum_{uw} c_{uw}^2 + (\beta - \gamma)(\sum_u C_{u\cdot}^2).$$

Now $\alpha - \beta = V(P) + (k-1)^{-1}V(P) = \sigma_{B(P)}^2 = \Sigma_{B(P)}$, and $\beta - \gamma = V(B) + (b-1)^{-1}V(B) - (k-1)^{-1}V(P) = \sigma_B^2 - k^{-1}\sigma_{B(P)}^2 = \Sigma_B$.

The mathematical problem of finding the minimum variance unbiased linear estimator of $\sum_w \delta_w t_w$ amounts therefore to finding the minimum with respect to the c_{uw}'s of

$$(\Sigma_{B(P)})\sum_{uw} c_{uw}^2 + (\Sigma_B)\sum_u C_{u\cdot}^2, \tag{14}$$

subject to $\sum_u c_{uw} = C_{\cdot w} = \delta_w$, $C_{\cdot\cdot} = 0$.

Under the usual infinite model, i.e. where $V(X_{uw}) = \sigma^2 + \sigma_b^2$; Cov $(X_{uw}, X_{uw'}(= \sigma_b^2, w \neq w'$; Cov $(X_{uw}, X_{u'w'}) = 0$, $u \neq u'$; the analogous problem of estimation leads to finding the minimum with respect to the c_{uw}'s of the expression

$$\sigma^2\sum_{uw} c_{uw}^2 + \sigma_b^2\sum_u C_{u\cdot}^2, \tag{15}$$

subject to $\sum_u c_{uw} = C_{\cdot w} = \delta_w$, $C_{\cdot\cdot} = 0$.

The two minimization problems are therefore identical in form. The forms of the best linear estimator and its variance in the case of the finite randomization model can therefore be obtained from those holding in the infinite case, which are known and are given for example in Kempthorne (1952) on page 535, by replacing σ^2 with $\Sigma_{B(P)}$ and σ_b^2 with Σ_B. In actual applications estimators of Σ_B and $\Sigma_{B(P)}$ will have to be used. Now, the relevant lines for the estimation of the σ^2's and Σ's, of the analysis of variance tables in the infinite and finite cases are:

	E.M.S Finite Model	E.M.S Infinite Model
Blocks eliminating treatments	$\Sigma_{B(P)} + \frac{bk-t}{b-1}\Sigma_B$	$\sigma^2 + \frac{bk-t}{b-1}\sigma_b^2$
Error	$\Sigma_{B(P)}$	σ^2.

It follows that the A.O.V. estimators of σ^2 and σ_b^2 from the infinite model are identical with those of $\Sigma_{B(P)}$ and Σ_B respectively obtained with the finite model, and hence that the actually employed estimators of treatment contrasts will be the same regardless of which model is taken as the basis for the analysis.

Considerations along the lines of the present section indicate that when treatments interact with experimental material then unless strong additional assumptions hold, the usual estimators of treatment contrasts are no longer best.

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