ALMOST PERIODIC VARIANCES[1]

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0. Summary. Second-order stationary random sequences, especially if Gaussian, may be of statistical interest because their covariances may, under certain conditions, be consistently estimated with finite realizations of the sequences. It is shown that there is a class of non-stationary random sequences, namely sequences of orthogonal random variables with zero means and variances f_t which form a uniformly almost periodic sequence, which are of statistical interest, at least in the Gaussian case, in the following sense. f_t admits a generalized Fourier series expansion, and the Fourier coefficients γ_s of this expansion can be consistently estimated with finite realizations of the sequences. In certain situations, the nonconstant variance sequence f_t may be directly estimated. The sequences of interest may be converted, via Fourier transformations, into second-order stationary random functions, and the Fourier coefficients γ_s, of the expansion of f_t, are shown to form a sequence of stationary covariances. A multiplicative representation is given for the nonstationary sequences considered.

I. PRELIMINARIES

I.1. Introduction. A second-order stationary random sequence X_t $(t = 0, \pm 1, \pm 2, \cdots)$ is a sequence of random variables (assumed real-valued and with zero first moments, unless otherwise stated) such that for all integers t, t',

$$\text{ave}\,\{X_t X_{t'}\} = \gamma_{t'-t} < +\infty \tag{1.1}$$

"ave" denoting the usual mathematical expectation. Thus the (finite) covariances of pairs of random elements of such sequences depend only on the differences between the indices of the elements. A *Gaussian* or *normal random sequence* is one such that any finite subset of random elements of the sequence has a multivariate Gaussian or normal joint probability distribution. From the viewpoint of one interested in statistical inference concerning the parameters $\gamma_{t'-t}$ figuring above, an important property of second-order stationary random sequences whose covariances satisfy certain restrictions, is that, for each integer $s (s = 0, \pm 1, \pm 2, \cdots)$,

$$\lim_{N\to\infty} (\text{q.m.}) N^{-1} \sum_{t=1}^{N} X_t X_{t+s} = \gamma_s, \tag{1.2}$$

where "lim q.m." denotes quadratic-mean convergence.

The present study begins with the question: Are there nonstationary (i.e., not second-order-stationary) random sequences whose covariances are specified

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by a (countable) set of parameters γ_s, say ($s = 0, \pm 1, \pm 2, \cdots$), such that the γ_s may be consistently estimated from finite stretches of data, in the spirit (though not the letter) of (1.2)? In other words, can one find nonstationary random sequences which give rise to finite series of data amenable to inference methods whose precision "improves" as the available length of series increases?

The answer to this question is affirmative, at least in the Gaussian case. There are Gaussian nonstationary random sequences X_t of potential statistical interest in the sense foregoing. These are sequences of statistically independent real-valued Gaussian random variables X_t ($t = 0, \pm 1, \pm 2, \cdots$), assumed to have zero first moments for all integers t, whose associated sequence of variances $f_t \equiv \text{var}(X_t) = \text{ave}\{X_t^2\}$ ($t = 0, \pm 1, \pm 2, \cdots$) is assumed to be *uniformly almost periodic* in the sense of H. Bohr (1951, transl.). Such uniformly almost periodic sequences f_t have formal *generalized Fourier Series* representations

$$f_t = \sum_{s=-\infty}^{\infty} \gamma_s z^{-t\lambda_s}, \tag{1.3}$$

where for real finite λ, $z^\lambda \equiv e^{2\pi i\lambda}$, with

$$\sum_{s=-\infty}^{\infty} |\gamma_s|^2 < +\infty. \tag{1.4}$$

The complex generalized Fourier coefficients γ_s figuring in (1.3) and (1.4) offer reasonable targets (given a sequence of the above-described type) at which to direct inference techniques, at least in the following sense. If $X_1, X_2, \cdots, X_N$ denote (for $N = 1, 2, \cdots$) finite consecutive subsets of random elements of such a sequence, then for each integer s,

$$\lim_{N\to\infty} (\text{q.m.}) N^{-1} \sum_{t=1}^{N} X_t^2 z^{t\lambda_s} = \gamma_s, \tag{1.5}$$

that is,

$$\begin{aligned} &\lim_{N\to\infty} (\text{q.m.}) N^{-1} \sum_{t=1}^{N} X_t^2 \cos 2\pi t\lambda_s = \text{re}\,\gamma_s, \\ &\lim_{N\to\infty} (\text{q.m.}) N^{-1} \sum_{t=1}^{N} X_t^2 \sin 2\pi t\lambda_s = \text{im}\,\gamma_s, \end{aligned} \tag{1.6}$$

where "re γ_s", "im γ_s" denote, respectively, the real and imaginary parts of γ_s. Part III of this paper is devoted to proofs of (1.5) and its variants.

I.2. Almost periodic sequences. Almost periodic function theory is due to Bohr (op. cit.), Besicovitch (reprinted 1954), Bochner, Weyl, and others. We now lead up to a definition of uniformly almost periodic sequences. The preliminary steps, as well as the final definition, are specializations to the case of sequences, of those of Besicovitch (op. cit.), the original forms being Bohr's.

DEFINITION I.2.1. A set E of integers is called *relatively dense* if there is an integer $l > 0$ such that any interval of the real line, of length l, contains at least one integer of E.

DEFINITION I.2.2. Let f_t $(t = 0, \pm 1, \pm 2, \cdots)$ be a real or complex sequence. An integer T is called a translation number of f_t belonging to $\epsilon \geqq 0$ if

$$\text{upper bound}_{-\infty < t < +\infty} |f_{t+T} - f_t| \leqq \epsilon. \tag{2.2}$$

We denote the set of all translation numbers of a sequence f_t, belonging to ϵ, by $E\{\epsilon, f_t\}$.

The precise definition of uniformly almost periodic sequences follows.

DEFINITION I.2.3. A sequence $f_t(t = 0, \pm 1, \pm 2, \cdots)$ is called *uniformly almost periodic* if for any $\epsilon > 0$, the set $E\{\epsilon, f_t\}$ is relatively dense.

For any u.a.p. (uniformly almost periodic) sequence, it follows from arguments of Besicovitch that f_t has formal expansion (1.3), provided we interpret the equality either in the sense of uniform convergence of the series to f_t, or in the sense that there exists a sequence of polynomials

$$\sum_{s=-\infty}^{\infty} p_s^{(k)} \gamma_s z^{-t\lambda_s} \qquad (k = 1, 2, \cdots) \tag{2.3}$$

(where $0 \leqq p_s^{(k)} \leqq 1$ and where for each k only a finite number of the factors p differ from zero) which converge to f_t uniformly in t, and converge formally to (1.3), in the sense that for each s

$$\lim_{k\to\infty} p_s^{(k)} = 1. \tag{2.4}$$

For such sequences, the inequality (1.4) holds.

I.3. Cyclical random sequences, harmonizable covariances. The consistency theorem expressed by (1.5), and its variants, use normality assumptions in their proofs. The balance of the theorems which follow do not require this assumption. It will therefore be convenient to redefine those nonstationary random sequences alluded to previously, now without the normality assumption.

DEFINITION I.3.1. A *cyclical random sequence* $X_t(t = 0, \pm 1, \pm 2, \cdots)$ will denote a sequence X_t of real-valued random variables with zero first moments for all integers t, whose covariances, for all integers t, t', have form

$$\text{ave}\,\{X_t X_{t'}\} = f_t \delta_{t'-t}, \tag{3.1}$$

where f_t is a u.a.p. sequence, and $\delta_{t'-t} = 1$ if $t' = t$, otherwise $\delta_{t'-t} = 0$. Part II of this paper develops some properties of cyclical random sequences.

The covariances of form (3.1) may be expressed in form close to that of the harmonizable covariances of Loève (1948). These are of form

$$\gamma_{t,t'} = \iint z^{t\lambda} z^{-t'\lambda'} \, dd'\Gamma(\lambda, \lambda'), \tag{3.2}$$

where $\Gamma(\lambda, \lambda')$ is a covariance of bounded variation on its domain of definition.

Since

$$\delta_{t'-t} = \int_{-\frac{1}{2}}^{\frac{1}{2}} z^{(t'-t)\lambda}\, d\lambda, \tag{3.3}$$

one may define

$$\Gamma(\lambda') = \sum_{\lambda_s < \lambda'} \gamma_s\,, \tag{3.4}$$

and write covariances satisfying (3.1) in the form

$$\gamma_{t,t'} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} z^{t'(\lambda-\lambda')} z^{-t\lambda}\, d\lambda\, d'\Gamma(\lambda'). \tag{3.5}$$

II. CYCLICAL RANDOM SEQUENCES

II.1. Properties of u.a.p. sequences. In what follows we specialize results from the monograph of A. S. Besicovitch (op. cit.). He considers only continuous u.a.p. functions defined on the whole real line, and consequently uses integrals rather than sums in defining the important mean limit (II.1.1). The proofs of specializations given here of Besicovitch's theorems all follow Besicovitch's proofs nearly word-for-word, changes being required only in the domains of definition and in the replacement of integrals with sums in the appropriate places. We omit these proofs.

A few elementary theorems about u.a.p. sequences follow.

THEOREM II.1.1. *Any* u.a.p. *sequence is bounded.*

THEOREM II.1.2. *If* f_t *is* u.a.p., *then so is any uniformly continuous function of* f_t. *Thus* cf_t (constant), f_t^2, $|f_t|^2$, *are* u.a.p. *as is* f_t^{-1}, *provided* $|f_t| > 0$ (*all* t).

THEOREM II.1.3. *A uniformly convergent series* $\sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}$ (*for real* λ_s), *is* u.a.p.

The crucial mean-value existence theorem for u.a.p. sequences follows.

THEOREM II.1.4. *For any* u.a.p. *sequence* f_t, *the mean value*

$$M\{f_t\} = \lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} f_{t+T} \tag{1.1}$$

exists, independently of $T (T = 0, \pm 1, \pm 2, \cdots)$. *If* $f_t \geqq 0 (t = 0, \pm 1, \pm 2, \cdots)$ *and for some* $t = t^*$, *say*, $f_{t^*} > 0$, *then* $M\{f_t\} > 0$.

The following theorems develop the Fourier series approach to u.a.p. sequences.

THEOREM II.1.5. *If* f_t *is* u.a.p., *then*

$$M\{f_t z^{t\lambda}\} = \gamma(\lambda) \tag{1.2}$$

is defined for all real λ, *but nonzero for an at-most-countable* λ*-set.*

If f_t is real, *then* $\overline{\gamma(\lambda)} = \gamma(-\lambda)$ (the overbar denoting complex conjugation). Hence if $\gamma(\lambda)$ is nonzero, then $\gamma(-\lambda)$ is nonzero, and we can write the λ-set for which $\gamma(\lambda)$ is nonzero as λ_s, $s = 0, \pm 1, \pm 2, \cdots$, where $\lambda_{-s} = -\lambda_s$. Since $\gamma(0)$ is real, we take $\lambda_0 = 0$. The formal expression

$$f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}, \qquad \gamma_s = \gamma(\lambda_s), \tag{1.3}$$

is called the generalized Fourier series (for short, Fourier series) of f_t, and we write (1.3) whether or not the series converges.

THEOREM II.1.6. *The Fourier series of a* u.a.p. *sequence represented by the sum of a uniformly convergent trigonometric series* $f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}$ *coincides with this series.*

THEOREM II.1.7. *If a sequence*

$$f_t^{(k)} = \sum_{-\infty}^{\infty} \gamma_s^{(k)} z^{-t\lambda_s} \qquad (k = 1, 2, \cdots)$$

of u.a.p. *sequences converges uniformly to a sequence* f_t, *then the Fourier series of* f_t *is given by*

$$f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}, \tag{1.4}$$

where $\gamma_s = \lim_{N\to\infty} \gamma_s^{(k)}$ *for all* s.

THEOREM II.1.8. *If* 2 u.a.p. *sequences have the same Fourier series, then they are identical.*

THEOREM II.1.9. *Let* f_t *be a* u.a.p. *sequence with generalized Fourier coefficients* γ_s. *Assume* f_t *is real. Then*

$$M\{f_t^2\} = \sum_{-\infty}^{\infty} |\gamma_s|^2 < +\infty. \tag{1.5}$$

II.2. Properties of cyclical random sequences and u.a.p. variances. In this section we study properties of cyclical sequences and their associated variance sequences.

THEOREM II.2.1. *If* X_t $(t = 0, \pm 1, \pm 2, \cdots)$ *is a cyclical random sequence, with* u.a.p. *variance sequence* f_t *possessing a formal expansion*

$$f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s} \qquad (t = 0, \pm 1, \pm 2, \cdots), \tag{2.1}$$

then $\gamma_s (s = 0, \pm 1, \pm 2, \cdots)$ *is a complex valued stationary covariance sequence.*

PROOF. Define

$$J_N(\lambda) = N^{-\frac{1}{2}} \sum_{t=1}^{N} X_t z^{-t\lambda} \qquad (\lambda \text{ real, finite}). \tag{2.2}$$

Then

$$\begin{aligned} \lim_{N\to\infty} \text{ave}\,\{J_N(\lambda)\overline{J_N(\lambda')}\} &= \lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} f_t z^{t(\lambda'-\lambda)} \\ &= M\{f_t z^{t(\lambda'-\lambda)}\} = \gamma(\lambda' - \lambda) < +\infty. \end{aligned} \tag{2.3}$$

Hence $\gamma(\lambda' - \lambda)$ is the limit of a sequence of complex stationary covariances and the theorem follows by the specialization of Theorem 34.1.B (Loève, 1960) to stationary covariances.

II.3. Multiplicative form of cyclical random sequences. We give a simple product form for cyclical random sequences, based upon the following lemma of Bohr (specialized to sequences).

LEMMA II.3.1. *If f_t and g_t are* u.a.p. *sequences with Fourier series*

$$f_t = \sum_{-\infty}^{\infty} \alpha_s z^{-t\mu_s}, \qquad g_t = \sum_{-\infty}^{\infty} \beta_s z^{-t\nu_s}, \tag{3.1}$$

then the product $f_t g_t$ is u.a.p. *with Fourier series*

$$f_t g_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}, \tag{3.2}$$

where

$$\gamma_s = \sum_{\mu_p + \nu_q = \lambda_s} \alpha_p \beta_q , \tag{3.3}$$

meaning that the exponent λ_s runs through all numbers of form $\mu_p + \nu_q$, and that the corresponding coefficient γ_s is given by the indicated sum, where the series, if it has an infinite number of terms, is absolutely convergent.

The desired theorem follows.

THEOREM II.3.1. *Let X_t be a cyclical random sequence, with*

$$\text{var}\,(X_t) = f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s} \quad (t = 0, \pm 1, \pm 2, \cdots). \tag{3.4}$$

There is at least one other cyclical random sequence X'_t whose associated variance sequence f'_t is identically equal to f_t, $f'_t = f_t$ $(t = 0, \pm 1, \pm 2, \cdots)$ and such that X'_t has form

$$X'_t = \sigma_t \eta_t \qquad (t = 0, \pm 1, \pm 2, \cdots), \tag{3.5}$$

where the η_t form of a sequence of uncorrelated random variables with zero means and unit variances and σ_t forms a u.a.p. sequence, with Fourier series

$$\sigma_t = \sum_{-\infty}^{\infty} \alpha_s z^{-t\mu_s}, \tag{3.6}$$

say. Correspondingly, there is at least one representation of γ_s (figuring in (3.4)) of form

$$\gamma_s = \sum_{\mu_p - \mu_q = \lambda_s} \alpha_p \overline{\alpha_q} . \tag{3.7}$$

PROOF. Set $\sigma_t = f_t^{\frac{1}{2}}$, where $f_t^{\frac{1}{2}}$ denotes the (real) non-negative square root sequence associated with the non-negative sequence f_t. Then σ_t is u.a.p. since the non-negative square root of f is a uniformly continuous function of f. Let σ_t have Fourier series (3.6). Since $f_t = \sigma_t \overline{\sigma_t}$, we have, by the preceding lemma, that

$$f_t = \sum_{-\infty}^{\infty} \Big(\sum_{\mu_p - \mu_q = \lambda'_s} \alpha_p \overline{\alpha_q} \Big) z^{-t\lambda'_s} \tag{3.8}$$

say, and the uniqueness of the Fourier series representation for u.a.p. sequences implies that $\lambda_s' = \lambda_s$ and that (3.7) holds. If a random sequence X_t' is defined by (3.5), then for all integers t, t'.

(3.9) $$\text{ave}\,\{X_t'X_{t'}'\} = f_t\delta_{t'-t}\,.$$

III. CONSISTENCY THEOREMS

III.1. Consistent estimation of Fourier coefficients. We prove the limit relation I:(1.5) for cyclical Gaussian sequences.

THEOREM III.1.1. *Let X_t be a cyclical Gaussian sequence whose variance sequence*

(1.1) $$f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-t\lambda_s}.$$

Then for each integer s,

(1.2) $$\lim_{N\to\infty}\,(\text{q.m.})N^{-1}\sum_{t=1}^{N} X_t^2 z^{t\lambda_s} = \gamma_s\,.$$

PROOF. We have

(1.3) $$\lim_{N\to\infty} \text{ave}\,\{N^{-1}\sum_{t=1}^{N} X_t^2 z^{t\lambda_s}\} = \lim_{N\to\infty} N^{-1}\sum_{t=1}^{N} f_t z^{t\lambda_s} = M\{f_t z^{t\lambda_s}\} = \gamma_s\,,$$

and

(1.4) $$\lim_{N\to\infty} N \,\text{var}\,(N^{-1}\sum_{t=1}^{N} X_t^2 z^{t\lambda_s}) = 2\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N} f_t^2 = 2M\{f_t^2\} < +\infty,$$

since f_t^2 is u.a.p. whenever f_t is u.a.p. Thus

(1.5) $$\lim_{N\to\infty} \text{var}\, N^{-1}\sum_{t=1}^{N} X_t^2 z^{t\lambda_s} = 0,$$

completing the proof.

III.2. Continuous-parameter extension. We sketch the setting for a continuous-parameter formal extension of Theorem II.1.1. The theorem is stated without proof, since the proof proceeds in the same manner as that of Theorem II.1.1.

DEFINITION III.2.1. A function $f(x)$ defined on $-\infty < x < \infty$ is called u.a.p. if it is continuous and its set of translation numbers $E\{\epsilon, f(x)\}$ is relatively dense for any $\epsilon > 0$. Here we modify the definition of translation numbers to include any x-values, and we modify similarly the definition of relative density.

DEFINITION III.2.2. A real-valued random function $X(x)$ defined on $-\infty < x < \infty$, with zero first moments for all x, is called *cyclical* if for any x, x', $x \neq x'$, $X(x)$ and $X(x')$ are uncorrelated and var $(X(x))$, the variance function associated with $X(x)$, is u.a.p. We consider in particular Gaussian cyclical random functions, i.e., cyclical random functions $X(x)$ such that for each fixed x, the random variable $X(x)$ has a marginal probability distribution which is Gaussian.

We require a generalization of the arithmetic means appearing in the proof of Theorem II.1.1, namely, the integral means with respect to a random function $V(x)$:

$$T^{-1} \int_0^T V(x)\, dx, \tag{2.1}$$

(T a finite real number). The integral figuring above will be interpreted as a quadratic-mean Riemann integral as defined in Loève (1948). This integral will exist wherever $f_V(x) = \operatorname{var}(V(x))$ is continuous and bounded on the domain of integration. Thus the integrals will be defined if $f_V(x)$ is a u.a.p. function.

Using the preceding ideas, the following formal extension of Theorem II.1.1 can be readily proved.

THEOREM III.2.1. *Let $X(x)$ be a Gaussian cyclical random function defined on $-\infty < x < \infty$. Let $f(x) = \operatorname{var}(X(x))$ have expansion*

$$f(x) = \sum_{-\infty}^{\infty} \gamma_s e^{-ix\lambda_s}. \tag{2.2}$$

Then for each integer s,

$$\lim_{T\to\infty} (\text{q.m.})\, T^{-1} \int_0^T X^2(x) e^{ix\lambda_s} = \gamma_s. \tag{2.3}$$

III.3. An approach to direct variance estimation. The possibility of direct estimation of f_t will be examined heuristically in this section. Direct estimation of $f_t = \operatorname{var} X_t$, when X_t is a Gaussian cyclical sequence, does not pose much of a problem (at least asymptotically) when f_t is purely periodic, with period N, say, since one could estimate f_t with

$$m^{-1} \sum_{k=1}^{m} X_{t+kN}^2, \tag{3.1}$$

given mN observations $X_1, X_2, \cdots, X_{mN}$. However difficulties arise when f_t is not purely periodic. We consider the case where f_t is limit-periodic in the sense of Bohr.

DEFINITION III.3.1. A sequence f_t is *limit-periodic* if it is the limit of a uniformly convergent sequence $f_t^{(k)}$ ($k = 1, 2, \cdots$) of purely periodic sequences. It can be shown that limit-periodic sequences are u.a.p.

THEOREM III.3.1. *f_t is limit-periodic if, and only if, its Fourier series has form*

$$f_t = \sum_{-\infty}^{\infty} \gamma_s z^{-tqr_s}, \tag{3.2}$$

where q is a real number and the r_s are all rational numbers.

THEOREM III.3.2. *If f_t is a* u.a.p. *sequence, then*

$$\lim_{N\to\infty} N^{-1} \sum_{k=0}^{N-1} f_{t+kp} = f_t^{(p)} \tag{3.3}$$

exists uniformly in t and is a purely periodic sequence whose Fourier series consists of those terms of the Fourier series of f_t which have a period p. If f_t is limit-periodic, it can be shown that

$$\lim_{p\to\infty} f_t^{(p)} = f_t, \tag{3.4}$$

uniformly in t.

This theorem and its predecessor are specializations, to the case of sequences, of theorems in Besicovitch (op. cit.).

A heuristic approach to the problem of direct variance estimation for Gaussian cyclical random sequences, in the limit-periodic variance case, will be based upon the following theorem, together with Theorem III.3.2.

THEOREM III.3.3. *If X_t is a Gaussian cyclical random sequence with variance sequence f_t, then*

$$\lim_{N\to\infty} (\text{q.m.}) N^{-1}\sum_{k=0}^{N-1} X_{t+kp}^2 = f_t^{(p)}, \tag{3.5}$$

where $f_t^{(p)}$ is defined by (3.3).

PROOF. We have

$$\lim_{N\to\infty} \text{ave}\,\{N^{-1}\sum_{k=0}^{N-1} X_{t+kp}^2\} = \lim_{N\to\infty} N^{-1}\sum_{k=0}^{N-1} f_{t+kp} = f_t^{(p)}, \tag{3.6}$$

and

$$\lim_{N\to\infty} N \,\text{var}\,(N^{-1}\sum_{k=0}^{N-1} X_{t+kp}^2) = 2\lim_{N\to\infty} N^{-1}\sum_{k=0}^{N-1} f_{t+kp}^2 = F_t^{(p)}, \tag{3.7}$$

say, which is finite since f_t^2 is u.a.p. Hence

$$\lim_{N\to\infty} \text{var}\, N^{-1}\sum_{k=0}^{N-1} X_{t+kp}^2 = 0, \tag{3.8}$$

completing the proof.

The preceding theorems suggest the following procedure, not rigorously justified, for estimating f_t in the limit-periodic case. Form

$$f_{t,N}^{(p)} = N^{-1}\sum_{k=0}^{N-1} X_{t+kp}^2. \tag{3.9}$$

For sufficiently large N, p, one hopes that $f_{t,N}^{(p)}$ would closely approximate (in q.m.) to f_t.

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