MINIMAX THEOREMS ON CONDITIONALLY COMPACT SETS

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1. Introduction. Conditionally compact sets in minimax theorems were first considered by A. Wald [2]: Let $K(x, y)$ be a real-valued bounded function defined on the product of two arbitrary sets X and Y. The distances

$$d(x_1, x_2) = \sup_Y |K(x_1, y) - K(x_2, y)| \quad \text{for } x_1, x_2 \varepsilon X$$

$$d(y_1, y_2) = \sup_X |K(x, y_1) - K(x, y_2)| \quad \text{for } y_1, y_2 \varepsilon Y$$

define metric topologies for X and Y respectively which will be referred tos a the *intrinsic* topologies or, briefly, the (I)-*topologies* for X and Y with respect to the function K. In general these topologies are pseudo-metric only, but we assume that a reduction to equivalent classes has made them properly metric.

Now let P be the set of all probability measures p on $\mathcal{J}_X$, i.e. the σ-algebra, generated by the (I)-open sets in X. Similarly, Q is the set of all probability measures q on $\mathcal{J}_Y$, the σ-algebra, generated by the (I)-open sets in Y. Then, if $K(p, q) = \int K(x, y)\, dp(x) \times dq(y)$, we have [2]:

Theorem 1.1. *If one of the spaces X and Y is (I)-conditionally compact, then both spaces are (I)-conditionally compact and* $\sup_P \inf_Q K(p, q) = \inf_Q \sup_P K(p, q)$.

A metric space is said to be conditionally compact if and only if, given any $\epsilon > 0$, there exists a finite subset $\{x_1, \cdots, x_n\}$ of X such that the class of spheres $S(x_i, \epsilon) = \{x : d(x, x_i) \leqq \epsilon\}$ $(i = 1, \cdots, n)$ is a covering for X.

A concept which is equivalent to (I)-conditionally compactness is that of almost periodic functions defined as follows [1]: A real-valued bounded function $K(p, q)$ defined on the product of two sets P and Q is *left almost periodic* if and only if, given $\epsilon > 0$, there exists a finite subset $\{p_1, \cdots, p_n\}$ of P such that for any $p \varepsilon P$ there is some p_i, $1 \leqq i \leqq n$, for which $|K(p, q) - K(p_i, q)| \leqq \epsilon$, for all $q \varepsilon Q$.

An analogous definition holds for *right almost periodicity*. Obviously, right almost periodicity follows from left almost periodicity, and vice versa.

The following definitions are due to Ky Fan [1]: A real-valued function $K(p, q)$ is said to be *concave-like* in p if and only if, given any $t \varepsilon [0, 1]$ and any $p_1, p_2 \varepsilon P$, there exists $p_0 \varepsilon P$ such that the inequality $tK(p_1, q) + (1 - t)K(p_2, q) \leqq K(p_0, q)$ holds for all $q \varepsilon Q$.

$K(p, q)$ is said to be *convex-like* in q if and only if, given any $t \varepsilon [0, 1]$ and any $q_1, q_2 \varepsilon Q$ there exists $q_0 \varepsilon Q$ such that the inequality $tK(p, q_1) + (1 - t)K(p, q_2) \geqq K(p, q_0)$ holds for all $p \varepsilon P$.

$K(p, q)$ is *concave-convex-like* if it is concave-like in p and convex-like in q.

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Ky Fan has proved the following theorem [1]:

THEOREM 1.2. *If the function $K(p, q)$ is left almost periodic and concave-convex-like then* $\sup_P \inf_Q K(p, q) = \inf_Q \sup_P K(p, q)$.

2. A generalization of the (*I*)-topology. We are now going to introduce topologies for X and Y which will lead to more general theorems than Theorem 1.1 and Theorem 1.2.

LEMMA 2.1. *Let $K(x, y)$ be a (not necessarily bounded) real-valued function defined on $X \times Y$. Then the class $\mathcal{U}$ of the subsets:*

$$U(x_0, a) = \{x : \sup_Y [K(x, y) - K(x_0, y)] < a\} \qquad a > 0,\ x_0 \,\varepsilon\, X,$$

is a base for a topology for X. Similarly, the class $\mathcal{V}$ of the subsets:

$$V(y_0, b) = \{y : \sup_X [K(x, y_0) - K(x, y)] < b\} \qquad b > 0,\ y_0 \,\varepsilon\, Y$$

is a base for a topology for Y.

The proof is trivial.

We shall refer to these topologies as the *semi-intrinsic topologies* or the (*S*)-*topologies* for X and Y with respect to the function K. We define:

The space $(X, \mathcal{U})$ is said to be (*S*)-*conditionally compact* if and only if to any $\epsilon > 0$, there exists a finite subset $\{x_1, \cdots, x_n\}$ of X such that the elements $U(x_i, \epsilon)$ of the class $\mathcal{U}$ form a covering for X.

We observe that every (S)-open subset of X is (I)-open. Also, every (I)-conditionally compact set is (S)-conditionally compact. Finally, the given function $K(x, y)$ is (S)-upper semi-continuous in x for each fixed $y \,\varepsilon\, Y$ and (S)-lower semi-continuous in y for each fixed $x \,\varepsilon\, X$.

LEMMA 2.2. *If X is conditionally compact in the (S)-topology determined by the function $K(x, y)$, then given $\epsilon > 0$ there exist a finite subset $\{x_1, \cdots, x_n\}$ and a finite class of subsets $W_1, \cdots, W_n$ of X such that*

(i) $x_k \,\varepsilon\, W_k$ *for* $k = 1, \cdots, n$

(ii) $\bigcup_{k=1}^{n} W_k = X$

(iii) $W_k \cap W_m = \phi$ *for* $k \neq m$

(iv) *W_k is $(\mathcal{S}_X)$-measurable, i.e. it belongs to $\mathcal{S}_X$, the σ-algebra generated by the (S)-open sets of X.*

(v) $K(x, y) - K(x_k, y) \leqq \epsilon$ *for all* $x \,\varepsilon\, W_k$ *and all* $y \,\varepsilon\, Y$.

A similar statement holds for Y.

PROOF. Let the class of subsets $U_j = U(\xi_j, \epsilon/2)$ be a covering for X, $(j = 1, \cdots, p)$. If P is the set $\{\xi_1, \cdots, \xi_p\}$, then define

$$W_1 = \bigcup \{U_j : \xi_j \,\varepsilon\, U_1, (U_j - U_1) \cap P = \phi\}$$

$p_2 = \min \{j : \xi_j \,\varepsilon\, X - W_1\}$;

$$W_2 = \bigcup \{U_j : \xi_j \,\varepsilon\, U_{p_2}, (U_j - U_{p_2}) \cap P = \phi\} - W_1$$

$p_3 = \min \{j : \xi_j \,\varepsilon\, X - (W_1 \cup W_2)\}$;

$$W_3 = \bigcup \{U_j : \xi_j \,\varepsilon\, U_{p_3}, (U_j - U_{p_3}) \cap P = \phi\} - (W_1 \cup W_2)$$

and so on. Then there will be an $n \geqq 1$ such that the set $X - \bigcup_{k=1}^{n} W_k$ does not contain any ξ_j. This set is empty. Defining $x_1 = \xi_1$, $x_k = \xi_{p_k}$ $(k = 2, \cdots, n)$ we see that x_k and W_k $(k = 1, \cdots, n)$ satisfy the hypotheses of the lemma.

LEMMA 2.3. *If X is (I)-separable, then $\mathcal{S}_X = \mathcal{I}_X$.*

PROOF. Clearly $\mathcal{S}_X \subset \mathcal{I}_X$. To prove the reverse relation, consider the (I)-open sphere

$$S(x_0, a) = \{x : \sup_Y [K(x, y) - K(x_0, y)] < a\}$$
$$\cup \{x : \inf_Y [K(x, y) - K(x_0, y)] > -a\}.$$

Now $\inf_Y [K(x, y) - K(x_0, y)]$ is (S)-upper semi-continuous. So the sets $E_n = \{x : \inf_Y [K(x, y) - K(x_0, y)] < -a + 1/n\}, n = 1, 2, \cdots$ are (S)-open. Consequently, the set $\bigcap_{n=1}^{\infty} E_n = \{x : \inf_Y [K(x, y) - K(x_0, y)] \leqq -a\}$ and its complement $\{x : \inf_Y [K(x, y) - K(x_0, y)] > -a\}$ belong to $\mathcal{S}_X$. This proves that $S(x_0, a) \varepsilon \mathcal{S}_X$. The lemma follows from the (I)-separability of X.

We define $\mathcal{I}$ to be the smallest σ-algebra in $X \times Y$, containing the rectangles $A \times B$, $A \varepsilon \mathcal{I}_X$ and $B \varepsilon \mathcal{I}_Y$. $\mathcal{S}$ is defined similarly. Wald ([3], p. 34) has proved that the function $K(x, y)$ is $(\mathcal{I})$-measurable, if X is (I)-separable. The next lemma is a generalization of this statement.

LEMMA 2.4. *If X is (I)-separable, then $K(x, y)$ is $(\mathcal{S})$-measurable.*

PROOF. We define:

$$Z = \{(x, y) : K(x, y) > a\} \quad a \text{ real}$$
$$C(x_1, r) = \{x : d(x_1, x) < r\} \quad r \text{ positive rational.}$$
$$X_0 = \{x : K(x, y) > a \text{ for at least one } y \varepsilon Y\}$$

Since $K(x, y)$ is (I)-continuous for each fixed y, the set X_0 is (I)-open. Let X_1 be a countable dense subset in X_0 and (x_0, y_1) be a point of Z. Then there exist a point $x_1 \varepsilon X_1$ and a positive rational r such that

$$d(x_0, x_1) < r \quad \text{and} \quad r < K(x_1, y_1) - a.$$

Let ϵ be a positive number such that $0 < r + \epsilon < K(x_1, y_1) - a$. Then, if (x, y) is a point of the rectangle $C(x_1, r) \times V(y_1, \epsilon)$, we have

$$K(x_1, y_1) - K(x, y) \leqq |K(x_1, y_1) - K(x, y_1)| + [K(x, y_1) - K(x, y)]$$
$$< r + \epsilon < K(x_1, y_1) - a.$$

It follows that $K(x, y) > a$ or $C(x_1, r) \times V(y_1, \epsilon) \subset Z$. If $Y(x_1, r)$ is defined to be the set

$$Y(x_1, r) = \bigcup \{V(y, \epsilon) : y \varepsilon Y, \epsilon > 0 \quad \text{and} \quad 0 < r + \epsilon < K(x_1, y) - a\}$$

then it has been proved that for any point $(x, y) \varepsilon Z$ there exist a point $x_1 \varepsilon X_1$ and a rational $r > 0$ such that $(x, y) \varepsilon C(x_1, r) \times Y(x_1, r) \subset Z$. Since $C(x_1, r) \varepsilon \mathcal{S}_X$ and the class of rectangles $C(x_1, r) \times Y(x_1, r)$ is countable, the set Z is $(\mathcal{S})$-measurable.

3. Minimax theorems.

THEOREM 3.1. *Let*

(i) $K(x, y)$ *be a real-valued function defined on* $X \times Y$ *such that* X *and* Y *are conditionally compact in the* (S)*-topologies determined by* K.

(ii) P *(*Q *respectively) be the set of all probability measures on* $\mathcal{S}_X(\mathcal{S}_Y)$.

(iii) $K(p, q) = \int_Y \int_X K(x, y)\, dp(x)\, dq(y) = \int_X \int_Y K(x, y)\, dq(y)\, dp(x)$ *for all* $p \varepsilon P$ *and* $q \varepsilon Q$.

Then $\sup_P \inf_Q K(p, q) = \inf_Q \sup_P K(p, q)$.

PROOF. Let $\epsilon > 0$. Then, according to Lemma 2.2 there exist a finite subset $\{x_1, \cdots, x_n\}$ of X and a finite class of subsets $W_1, \cdots, W_n$ of X, satisfying the hypotheses of the lemma. It follows that for each i, $1 \leqq i \leqq n$, the inequalities $K(x, y) - K(x_i, y) \leqq \epsilon$ hold for all $y \varepsilon Y$ and all $x \varepsilon W_i$. If $K(x_i, q)$ stands for the integral $\int_Y K(x_i, y)\, dq(y)$, then it follows from hypotheses (ii), (iii) and (iv) of Lemma 2.2:

$$K(p, q) \leqq \sum_1^n K(x_i, q) p(W_i) + \epsilon \quad \text{for all } p \varepsilon P \text{ and all } q \varepsilon Q.$$

Let P' be the set of all probability measures p' on the set $\{x_1, \cdots, x_n\}$, such that $p'(\{x_i\}) = p(W_i)$ for $i = 1, \cdots, n$, and $p \varepsilon P$. Then $K(p, q) \leqq K(p', q) + \epsilon$ for all $p \varepsilon P$, and all $q \varepsilon Q$. It follows that

$$\inf_Q \sup_P K(p, q) \leqq \inf_Q \max_{P'} K(p', q) + \epsilon. \tag{3.1.1}$$

Similarly, there exist points $y_1, \cdots, y_m$ and subsets $Z_1, \cdots Z_m$ in Y which satisfy Lemma 2.2. Again, define Q' to be the set of all probability measures q' on the set $\{y_1, \cdots, y_m\}$, such that $q'(\{y_j\}) = q(Z_j)$ for $j = 1, \cdots, m$ and $q \varepsilon Q$. Then $K(p, q') \leqq K(p, q) + \epsilon$ for all $p \varepsilon P$ and all $q \varepsilon Q$. It follows that

$$\max_{P'} \min_{Q'} K(p', q') \leqq \sup_P \inf_Q K(p, q) + \epsilon. \tag{3.1.2}$$

Since $\max_{P'} \min_{Q'} K(p', q') = \min_{Q'} \max_{P'} K(p', q')$ it follows from (3.1.1) and (3.1.2), $\inf_Q \sup_P K(p, q) \leqq \sup_P \inf_Q K(p, q)$. This proves the theorem since the inverse inequality always holds.

REMARKS.

1. Hypothesis (iii) of Theorem 3.1 is fulfilled if $K(x, y)$ is bounded and X is (I)-separable, which is a necessary condition for X being (I)-conditionally compact.

2. The following example shows that (S)-conditional compactness of one of the spaces X and Y is not sufficient for the other space having the same topological property: Let X be the set of the positive integers and Y the class of all subsets of X. $K(x, y) = -1$ for $x \varepsilon y$ and $= +1$ for $x \not\varepsilon y$. It is readily seen that the Y-space is (S)-conditionally compact but the X-space is not.

However, the next lemma is easily proved:

LEMMA 3.1. *If* X *is finite and* $K(x, y)$ *is bounded below in* y *for each fixed* x, *then* Y *is* (S)*-conditionally compact.*

Theorem 3.2 is an extension of Theorem 1.2. It is an extension of Theorem 3.1

as well. In linear cases however, the latter is still useful because of the relatively simpler structures of the X- and Y-spaces in comparison with the spaces of the probability measures P and Q.

THEOREM 3.2. *If*

(i) $K(p, q)$ *is a real-valued function on the product of the arbitrary spaces* P *and* Q,

(ii) $K(p, q)$ *is bounded below in* q *for each fixed* p,

(iii) $K(p, q)$ *is concave-convex-like,*

(iv) P *is* (S)*-conditionally compact,*

then $\sup_P \inf_Q K(p, q) = \inf_Q \sup_P K(p, q)$.

PROOF. We need the following lemma [1]: *If* $K(p, q)$ *is concave-convex-like, then to every finite subset* $\{p_1, \cdots, p_n\}$ *of* P *and every finite subset* $\{q_1, \cdots, q_m\}$ *of* Q *there exist* $p_0 \varepsilon P$ *and* $q_0 \varepsilon Q$ *such that* $K(p_i, q_0) \leqq K(p_0, q_j)$, *for* $i = 1, \cdots, n$ *and* $j = 1, \cdots, m$.

Because of the conditional compactness of P there exists to every $\epsilon > 0$ a finite subset $P_0 = \{p_1, \cdots, p_n\}$ of P, such that for any $p \varepsilon P$ there exists an index i, $1 \leqq i \leqq n$ for which $K(p, q) < K(p_i, q) + \epsilon$ for all $q \varepsilon Q$. It follows

$$\inf_Q \sup_P K(p, q) \leqq \inf_Q \max_{P_0} K(p_i, q) + \epsilon. \tag{3.2.1}$$

According to Lemma 3.1, Q is (S)-conditionally compact with respect to the restriction of the function K to $P_0 \times Q$. Thus there exists a finite subset $Q_0 = \{q_1, \cdots, q_m\}$ of Q with the property that given any $q \varepsilon Q$, there is $q_j \varepsilon Q_0$ for which $K(p_i, q_j) < K(p_i, q) + \epsilon$, $i = 1, \cdots, n$. It follows:

$$\max_{P_0} \min_{Q_0} K(p_i, q_j) \leqq \sup_P \inf_Q K(p, q) + \epsilon. \tag{3.2.2}$$

According to the lemma, mentioned at the beginning of the proof, there exist $p' \varepsilon P$ and $q' \varepsilon Q$ such that $K(p_i, q') \leqq K(p', q_j)$ for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Since P is conditionally compact, it follows $K(p_i, q') \leqq K(p', q_j) < K(p_k, q_j) + \epsilon$ for some $p_k \varepsilon P_0$ and consequently

$$\inf_Q \max_{P_0} K(p_i, q) \leqq \max_{P_0} \min_{Q_0} K(p_i, q_j) + \epsilon. \tag{3.2.3}$$

From (3.2.1), (3.2.2) and (3.2.3) it follows

$$\inf_Q \sup_P K(p, q) \leqq \sup_P \inf_Q K(p, q) + 3\epsilon$$

and thus

$$\inf_Q \sup_P K(p, q) = \sup_P \inf_Q K(p, q).$$

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