THE COVARIANCE MATRIX OF A CONTINUOUS AUTOREGRESSIVE VECTOR TIME-SERIES

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Notation. We shall denote by x^* or A^* the complex conjugate of the transpose of a column-vector x or of a square matrix A. Thus, $y^*x = \sum x_\nu \bar{y}_\nu$ is the scalar product of vectors x and y; but xy^* is a square matrix with components $x_\mu \bar{y}_\nu (\mu, \nu = 1, \cdots, n)$.

Theorem 1. *Let A be an $n \times n$ matrix of real or complex constants a_{rs}. Suppose that all of the eigenvalues of A lie in the left half-plane. Let $f(t)$ be an n-component column-vector satisfying*

$$Ef(t)f^*(t-\tau) = (Ef_r(t)\bar{f}_s(t-\tau)) = \delta(\tau)C$$

where $\delta(\tau)$ is the delta-function and where C is a positive semi-definite Hermitian matrix. Let $x(t)$ be the stationary stochastic process defined by

$$\frac{dx(t)}{dt} = Ax(t) + f(t) \qquad (-\infty < t < \infty). \tag{1}$$

Then

$$Ex(t)x^*(t-\tau) = e^{A\tau}M \tag{2}$$

where the $n \times n$ covariance matrix M is uniquely determined by the system of n^2 linear equations in n^2 unknowns

$$-C = AM + MA^*. \tag{3}$$

Proof. The steady-state solution of the differential equation (1) is

$$x(t) = \int_{-\infty}^{t} e^{(t-\lambda)A} f(\lambda)\, d\lambda = \int_0^\infty e^{\lambda A} f(t-\lambda)\, d\lambda.$$

Therefore,

$$\begin{aligned} Ex(t)x^*(t-\tau) &= E\int_0^\infty \int_0^\infty e^{\lambda A} f(t-\lambda) f^*(t-\tau-\mu) e^{\mu A^*}\, d\lambda\, d\mu \\ &= \int_0^\infty \int_0^\infty e^{\lambda A} \delta(\tau+\mu-\lambda) C e^{\mu A^*}\, d\lambda\, d\mu \\ &= \int_0^\infty e^{(\mu+\tau)A} C e^{\mu A^*}\, d\mu = e^{\tau A} M \end{aligned}$$

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where

$$M = Ex(t)x^*(t) = \int_0^\infty e^{\mu A} C e^{\mu A^*} \, d\mu. \tag{4}$$

Let the integrand in (4) be called $Q(\mu)$. Differentiation gives

$$(d/d\mu)Q(\mu) = AQ(\mu) + Q(\mu)A^*. \tag{5}$$

Since A has all its eigenvalues in the left half-plane, the matrix $Q(\mu)$ tends exponentially to 0 as $\mu \to \infty$. Therefore, we may integrate (5) from 0 to ∞ to obtain

$$-C = AM + MA^*. \tag{6}$$

This is the classical Lyapunov equation of stability-theory. The uniqueness of the solution M for any fixed C is established by the argument that, for any C, Hermitian or non-Hermitian, the n^2 linear equations (6) in n^2 unknowns m_{rs} have a solution given by the convergent integral (4). Since *every* system (6) has *some* solution, the solution is unique, by the theory of linear algebraic equations. This completes the proof.

It was necessary to assume that A have its eigenvalues in the left half-plane to insure that the moments be finite and that the process $x(t)$ be stationary. The matrix C is necessarily positive semi-definite because for any vector v we have the quadratic form $v^*Cv = Ea^*a \geqq 0$ where a is the random number $a = (\int_0^1 f^*(t)\, dt)v$.

THEOREM 2. *Let $\phi(t)$ be the real stochastic process satisfying the differential equation*

$$\phi^{(n)}(t) + a_1\phi^{(n-1)}(t) + \cdots + a_n\phi(t) = w(t) \qquad (-\infty < t < \infty) \tag{7}$$

where $w(t)$ is real white noise, and where the a_ν are real numbers such that all the zeros of the characteristic polynomial $\zeta^n + a_1\zeta^{n-1} + \cdots + a_n$ lie in the left half-plane. Then for $r, s = 0, 1, \cdots, n-1$

$$\begin{aligned} E\phi^{(r)}(t)\phi^{(s)}(t) &= 0 && (r + s \text{ odd}) \\ &= (-)^{(s-r)/2} m_{(s+r)/2} && (r + s \text{ even}) \end{aligned} \tag{8}$$

where the n numbers $m_0, m_1, \cdots, m_{n-1}$ are uniquely determined by the n linear equations

$$\begin{aligned} (-)^k \sum_{k/2 \leqq q \leqq (n+k)/2} (-)^q a_{n-2q+k}\, m_q &= 0 && (k = 0, \cdots, n-2) \\ &= \tfrac{1}{2} && (k = n-1) \end{aligned} \tag{9}$$

where we define $a_0 = 1$.

EXAMPLE. For $n = 6$ the theorem states that

$$(E\phi^{(r)}\phi^{(s)})_{r,s=0,\cdots,5} = \begin{pmatrix} m_0 & 0 & -m_1 & 0 & m_2 & 0 \\ 0 & m_1 & 0 & -m_2 & 0 & m_3 \\ -m_1 & 0 & m_2 & 0 & -m_3 & 0 \\ 0 & -m_2 & 0 & m_3 & 0 & -m_4 \\ m_2 & 0 & -m_3 & 0 & m_4 & 0 \\ 0 & m_3 & 0 & -m_4 & 0 & m_5 \end{pmatrix}$$

where $m_0, \cdots, m_5$ are determined by the linear equations

$$\begin{pmatrix} a_6 & -a_4 & a_2 & -1 & 0 & 0 \\ 0 & a_5 & -a_3 & a_1 & 0 & 0 \\ 0 & -a_6 & a_4 & -a_2 & 1 & 0 \\ 0 & 0 & -a_5 & a_3 & -a_1 & 0 \\ 0 & 0 & a_6 & -a_4 & a_2 & -1 \\ 0 & 0 & 0 & a_5 & -a_3 & a_1 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

PROOF OF THE THEOREM. Define

$$(10) \qquad x = \begin{pmatrix} \phi \\ \phi' \\ \vdots \\ \phi^{(n-1)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot\cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ w(t) \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$M = (m_{rs}) = (E\phi^{(r)}\phi^{(s)})_{r,s=0,1,\cdots,n-1}.$$

Then

$$dx/dt = Ax + f, \qquad Ef(t)f^*(t-\tau) = C\delta(\tau), \qquad Exx^* = M.$$

By Theorem 1 the real, symmetric matrix M is uniquely determined by the equation $-C = AM + MA^*$, which states in this case:

$$(11) \qquad 0 = m_{i+1,j} + m_{i,j+1} \qquad (i, j = 0, \cdots, n-2)$$

$$(12) \qquad 0 = m_{i+1,n-1} - \sum_{\nu=0}^{n-1} a_{n-\nu} m_{i\nu} \qquad (i = 0, \cdots, n-2)$$

$$(12') \qquad 0 = m_{n-1,j+1} - \sum_{\nu=0}^{n-1} a_{n-\nu} m_{\nu j} \qquad (j = 0, \cdots, n-2)$$

$$(13) \qquad 1 = \sum_{\nu=0}^{n-1} a_{n-\nu} m_{n-1,\nu} + \sum_{\nu=0}^{n-1} a_{n-\nu} m_{\nu,n-1}.$$

Equation (11) states that on any south-west to north-east diagonal of the matrix M the components are of equal magnitudes with alternating signs. Therefore, since M is symmetric, $m_{i,i+1} = -m_{i+1,i} = 0$, so that $m_{rs} = 0$ when $r + s$ is odd. Define $m_r = m_{rr}$. Then

$$m_{rs} = \begin{cases} 0 & \text{if } r+s \text{ is odd} \\ (-)^{(s-r)/2} m_{(r+s)/2} & \text{if } r+s \text{ is even.} \end{cases} \tag{14}$$

By the symmetry of M, Equations (12) and (12′) are identical. By (14) we have for all $k = 0, \cdots, n-1$

$$\sum_{\nu=0}^{n-1} a_{n-\nu}\, m_{k\nu} = (-)^k \sum_{k/2 \leqq \lambda \leqq (n+k-1)/2} (-)^\lambda a_{n+k-2\lambda}\, m_\lambda . \tag{15}$$

Furthermore, by (14),

$$-m_{j+1,n-1} = \begin{cases} 0 & (j+n \text{ odd}) \\ (-)^{(n-j)/2} m_{(n+j)/2} & (j+n \text{ even}). \end{cases} \tag{16}$$

Therefore, if we define $a_0 = 1$, minus one times Equations (12) and minus one-half times Equation (13) may be written together in the form (9). This completes the proof.

A result equivalent to Theorem 2 was obtained, as the conclusion of a long argument involving the theory of residues, by R. S. Phillips [1] pp. 333–339. A similar derivation was obtained later by R. C. Booton, Jr., M. V. Matthews, and W. W. Seifert [2] pp. 366–371. The inverse of the covariance matrix M was discussed by Parzen [3]. For discrete-parametric autoregressive time-series the inverse of the covariance matrix was computed by M. M. Siddiqui [4].

THEOREM 3. *Let $\phi(t)$ be the stochastic process satisfying the differential equation*

$$\phi^{(n)}(t) + a_1\phi^{(n-1)}(t) + \cdots + a_n\phi(t) = w(t) \qquad (-\infty < t < \infty) \tag{17}$$

where $w(t)$ is white noise, and where the a_ν are complex numbers such that all the zeros of the characteristic polynomial $\zeta^n + a_1\zeta^{n-1} + \cdots + a_n$ lie in the left half-plane. Then for $r, s, = 0, 1, \cdots, n-1$

$$E\phi^{(r)}(t)\overline{\phi^{(s)}(t)} = i^{s-r}\mu_{r+s} \tag{18}$$

where $\mu_0, \mu_1, \cdots, \mu_{2n-2}$ are real numbers uniquely determined by the $2n - 1$ real linear equations

$$\sum_{q=j}^{n+j} \alpha_{jq}\mu_q = 0, \qquad \sum_{q=j}^{n+j} \beta_{jq}\mu_q = 0 \qquad (j = 0, \cdots, n-2)$$
$$\sum_{q=n-1}^{2n-2} \alpha_{n-1,q}\mu_q = \tfrac{1}{2}(-)^{n-1} \tag{19}$$

where α_{jq}, β_{jq} are the real numbers given by

$$\alpha_{jq} + i\beta_{jq} = (-i)^q a_{n-q+j} . \tag{20}$$

PROOF. For complex coefficients a_ν the relations (11)-(13) must be replaced by

$$0 = m_{r+1,s} + m_{r,s+1} \qquad (r, s = 0, \cdots, n-2) \tag{21}$$

$$0 = m_{r+1,n-1} - \sum_{\nu=0}^{n-1} \bar{a}_{n-\nu} m_{r\nu} \qquad (r = 0, \cdots, n-2) \tag{22}$$

$$0 = m_{n-1,j+1} - \sum_{\nu=0}^{n-1} a_{n-\nu} m_{\nu j} \qquad (j = 0, \cdots, n-2) \tag{22'}$$

$$1 = \sum_{\nu=0}^{n-1} \bar{a}_{n-\nu} m_{n-1,\nu} + \sum_{\nu=0}^{n-1} a_{n-\nu} m_{\nu,n-1}\,. \tag{23}$$

Since $m_{rs} = \bar{m}_{sr}$ in the complex case, relations (22) and (22′) are equivalent. Since $m_{ss} = E|\phi^{(s)}|^2$, we have $m_{ss} \geqq 0$. Therefore, by (21), terms m_{rs} for $r + s$ equal to an even constant are real, of equal magnitudes, and of alternating signs. For $r = s$ Equation (21) states that $m_{r+1,r} + m_{r,r+1} = 0$, or, since M is Hermitian, $2 \operatorname{Re} m_{r+1,r} = 0$. Therefore, terms m_{rs} for $r + s$ equal to an odd constant are pure imaginary and equal except for alternating signs. In summary, there exist real numbers $\mu_0, \mu_1, \cdots, \mu_{2n-2}$ such that

$$E\phi^{(r)}\overline{\phi^{(s)}} = m_{rs} = i^{s-r}\mu_{rs}\,. \tag{24}$$

These real numbers are determined by Equations (22′) and (23), which by the identity (24) may be written as

$$-i^{j-n}\mu_{n+j} - \sum_{\nu=0}^{n-1} a_{n-\nu} i^{j-\nu} \mu_{j+\nu} = 0 \qquad (j = 0, \cdots, n-2)$$

$$2 \operatorname{Re} \sum_{\nu=0}^{n-1} a_{n-\nu} i^{n-1-\nu} \mu_{\nu+n-1} = 1.$$

By rearranging the indices and taking real and imaginary parts, we find the required Equations (19).

THEOREM 4. *For $k = 0, \cdots, n-1$ let $\psi_k(t)$ be the solution of the initial-value problem*

$$\begin{aligned} \psi_k^{(n)}(t) + a_1\psi_k^{(n-1)}(t) + \cdots + a_n\psi_k(t) &= 0 \qquad (t > 0) \\ \psi_k^{(r)}(0) &= \delta_{rk} \qquad (r = 0, \cdots, n-1). \end{aligned} \tag{25}$$

Then under the hypotheses of Theorem 2

$$E\phi^{(r)}(t)\phi^{(s)}(t-\tau) = \sum_{\substack{k=0 \\ k+s \text{ even}}}^{n-1} \psi_k^{(r)}(\tau)(-)^{(s-k)/2} m_{(s+k)/2}\,.$$

More generally, under the hypotheses of Theorem 3,

$$E\phi^{(r)}(t)\overline{\phi^{(s)}(t-\tau)} = \sum_{k=0}^{n-1} \psi_k^{(r)}(\tau) i^{s-k} \mu_{ks}\,.$$

PROOF. Let A be defined by (10). The matrix $X = e^{At}$ is defined as the solution of the initial-value problem $(d/dt)X(t) = AX(t)\,(t > 0)$, $X(0) = I$. Therefore, by (25), $e^{At} = (\psi_s^{(r)}(t))_{r,s=0,\cdots,n-1}$. Theorem 4 now follows from formula (2) of Theorem 1.

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