CONDITIONED LIMIT THEOREMS[1]

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1. Introduction. Limit theorems for Markoff processes and suitable functionals defined on the processes occur in two principal contexts. The first category of applications treats a situation where the limit process is one of the classical stable processes. The usual approximating processes are sums of independent random variables. A second important class of examples is that of a limiting diffusion process of Bessel type (Section 2). Then the approximating processes may themselves either be of diffusion type (i.e., random walks, birth and death or bona fide diffusion) or processes almost of diffusion type [14], [23].

In both these cases under sufficient regularity conditions we have an invariance principle, i.e., the convergence of the processes entails the convergence in law of functionals continuous a.e. with respect to the limit process.

In this paper our objective is to develop several limit laws for random variables subject to conditioning on a recurrent event.

Such limit laws arise in a natural way in considering Kolmorogov-Smirnov statistics, and other related statistics as follows. Consider the Poisson process, $U(t)$, $t \geqq 0$, with stationary increments and $EU(1) = 1$. The event $U(n) = n$ for some n is a certain recurrent event $(n = 1, 2, \cdots)$. Let N_n denote the number of recurrences that have taken place up to time n. It is well known that

$$\lim_{n\to\infty} \Pr(N_n/n^{\frac{1}{2}} < t) = (2/\pi)^{\frac{1}{2}} \int_0^t e^{-x^2/2}\, dx, \qquad 0 \leqq t < \infty, \text{[20], [3].}$$

On the other hand, $P(N_n = k \mid U(n) = n)$ is just the probability that $F_n(x) = F(x)$ for k values of x where F_n is the empirical c.d.f. of n independent random variables each distributed according to the c.d.f., F. It follows, in particular, from the results of this paper that

$$\lim_{n\to\infty} \Pr(N_n/n^{\frac{1}{2}} < t \mid U(n) = n) = 1 - e^{-t^2/2}, \qquad 0 \leqq t < \infty, \text{[20].}$$

Similarly, let $M_n = \max_{0 \leqq t \leqq n} (U(t) - t)$, $A_n = \max_{0 \leqq t \leqq n} |U(t) - t|$. The limiting distribution of $M_n/n^{\frac{1}{2}}$, $A_n/n^{\frac{1}{2}}$ are well known [11]. The "conditioned" versions involve the limits,

$$\lim_{n\to\infty} \Pr(M_n/n^{\frac{1}{2}} < t \mid U(n) = n), \ \lim_{n\to\infty} \Pr(A_n/n^{\frac{1}{2}} < t \mid U(n) = n).$$

These, of course, are the well known limiting distributions of the one and two-sided Kolmogorov-Smirnov statistics, $n^{\frac{1}{2}}D_n^+$, $n^{\frac{1}{2}}D_n$. Similarly, instead of $U(t)$ one considers the process $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, where the X_i are independent and identically distributed random variables equaling 1 and -1

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with probabilities $\frac{1}{2}$, $\frac{1}{2}$. If N_n = the number of indices i for which $S_i = 0$, $1 \leqq i \leqq n$, $M_n = \max_{1 \leqq i \leqq n} S_i$, $A_n = \max_{1 \leqq i \leqq n} |S_i|$ then $P(N_{2n} = k \mid S_{2n} = 0)$, $P(M_{2n} = k \mid S_{2n} = 0)$, $P(A_{2n} = k \mid S_{2n} = 0)$ are exactly the probabilities $P\{F_n(x) = G_n(x)$ for k values of $x\}$, $P\{\max (F_n(x) - G_n(x)) = k/n\}$, $P\{\max |F_n(x) - G_n(x)| = k/n\}$ where F_n, G_n are independent empirical c.d.f.'s based on observations from the same population. Some of the results of this paper specialized to this case establish limit laws for $N_{2n}/n^{\frac{1}{2}}$, $M_{2n}/(2n)^{\frac{1}{2}}$, $A_{2n}/n^{\frac{1}{2}}$ under the condition that $S_{2n} = 0$, thus providing new proofs of the well-known limit laws for Kolmogorov-Smirnov statistics which arise from the comparison of two empirical c.d.f.'s.

Let $X(t)$, $t > 0$ be a Markoff stochastic process whose state space is the real line. We will consider two cases. The first is the situation where $X(t)$ is a process of a sum of independent and identically distributed random variables in discrete time. In this circumstance we adopt the traditional notation and write $X(n) = S_n$ where

$$S_0 = 0, \; S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \qquad n \geqq 1 \tag{1}$$

and ξ_i are independent identically distributed. In order to obtain nontrivial limit laws for the process $S_{[nt]}$ suitably normalized we will assume that $\{\xi_i\}$ belongs to the domain of attraction of a symmetric stable law of index α, $1 < \alpha \leqq 2$ and $E(\xi_i) = 0$. It is established in [19] under the conditions stated that the processes

$$Z^{(n)}(t) = S_{[nt]}/n^{1/\alpha} \tag{2}$$

converge to a symmetric stable process of index α (as $n \to \infty$) and the invariance principle holds.

Consider now the special case where ξ_i are integer valued non-lattice random variables. The conditions $1 < \alpha \leqq 2$ and $E(\xi_i) = 0$ imply (see [2]) that the process $S_{[nt]}$ is null recurrent and in particular the event $S_n = 0$ for some $n \geqq 1$ is certain recurrent. Henceforth whenever dealing with the case of (1) we assume that the convergence in (2) prevails and that ξ_i are integer valued non-lattice random variables.

We introduce the following random variables

$$M_n = \max_{0 \leqq k \leqq n} S_k, \qquad M_n^* = \{\max_{0 \leqq k \leqq n} S_k \mid S_n = 0\} \tag{3}$$

where the last notation signifies that we are considering the maximum variable restricted to the sample paths where $S_n = 0$. (The subsequent notation is to be interpreted analogously.)

$$\begin{aligned} M_n^+ &= \{\max_{0 \leqq k \leqq n} S_k \mid S_k > 0, k = 1, 2, \cdots, n\} \\ A_n &= \max_{0 \leqq k \leqq n} |S_k| \\ A_n^* &= \{\max_{0 \leqq k \leqq n} |S_k| \mid S_n = 0\} \\ Y_n &= n - \max\{k : S_k = 0, k = 0, 1, \cdots, n\} \\ N_n &= \text{Number of } S_k = 0 \; (k = 1, 2, \cdots, n) \\ N_n^* &= \{\text{Number of } S_k = 0, k = 1, \cdots, n \mid S_n = 0\}. \end{aligned} \tag{3}$$

It should be emphasized, to be precise, that the * random variables are defined on a different measure space than the unstarred random variables.

It is proved in [5] and independently in [12] under the hypothesis stated above that

$$\lim_{n\to\infty} \Pr\{Y_n/n \leqq u\} = F_{1/\alpha}(u) \tag{4}$$

where $F_c(u)$ represents a one parameter family of distributions whose explicit form is

$$F_c(u) = \frac{\sin \pi c}{\pi} \int_0^u \xi^{c-1}(1-\xi)^{-c}\, d\xi. \tag{5}$$

Moreover, the existence of

$$\lim_{n\to\infty} \Pr\{M_n/n^{1/\alpha} \leqq x\} \tag{6}$$

is demonstrated and the limit distribution is identified in terms of a double Laplace transform in [6]. Also limit laws for A_n and N_n are developed respectively in [19] and [3]. These assertions can be deduced partly with the aid of the general invariance principle [18]. Our present aim is to analyze the nature of the limit laws for M_n^*, A_n^* and N_n^*. Actually the emphasis of this paper is more on deriving relationships amongst the limit distributions of the variables (3) rather than on proving their existence. Nevertheless in most cases, we also obtain the existence of the limiting distributions.

In addition to elaborating the program of the preceding paragraph for the case of sums of independent random variables we will also develop the corresponding results for the case where the approximating processes are of diffusion type. For simplicity of exposition we assume that $X(t)$, $t > 0$, $X(0) = 0$ is a birth and death process on the integers; analogous arguments and constructions apply in the case of random walks on the integers or diffusion processes on the line. These processes are characterized as those Markoff processes on the line whose path functions are "continuous" (see [8] and [17]).

A birth and death process is a stationary Markoff process whose state space are the integers and whose transition probabilities $P_{ij}(t)$ satisfy the order relations

$$\begin{aligned} P_{ij}(t) &= \lambda_i t + o(t) & j &= i+1 \\ &= \mu_i t + o(t) & j &= i-1 \\ &= 1 - (\lambda_i + \mu_i)t + o(t) & j &= i \end{aligned} \tag{7}$$

where λ_i and $\mu_i > 0$ $(i = 0, \pm 1, \pm 2, \pm 3 \cdots)$. In the one-sided case, $\mu_i = 0$, $i \leqq 0$ and the relevant state space reduces to the nonnegative integers.

It is proved by Stone [23] that the only non-degenerate limit processes that arise by renormalization, i.e. $\lim_{c\to\infty} c^{-1}X(g(c)t) = Z(t)$, $t > 0$ are necessarily generalized Bessel processes (see Section 2). The existence of the limit entails that λ_i and μ_i possess growth properties of special algebraic structure and then $g(c) = c^\alpha L(c)$ where $L(\cdot)$ is a slowly varying function (see also [24] and [15]).

More specifically, if

$$(8)\qquad \begin{array}{ll} \pi_n \sim En^{\gamma-1}, & \pi_{-n} \sim E_1 n^{\gamma_1-1} \\ (\lambda_n \pi_n)^{-1} \sim Cn^{\beta-1} & (\lambda_{-n}\pi_{-n})^{-1} \sim C_1 n^{\beta_1-1} \end{array} \qquad n \to \infty$$

where

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \qquad \pi_{-n} = \frac{\mu_0 \mu_1 \cdots \mu_{-n+1}}{\lambda_{-1} \lambda_{-2} \cdots \lambda_{-n}}$$

and the C's, D's, β's and γ's are positive constants. In order to avoid technical complications and the presence of degenerate distributions we will henceforth assume $\beta = \beta_1$ and $\gamma = \gamma_1$. Then

$$(9)\qquad \lim_{c\to\infty} c^{-1} X(c^a t) = Z(t)$$

where $a = \beta + \gamma$. (The hypothesis (8) can be generalized allowing the introduction of slowly varying functions as multiplying factors and a corresponding version of (9) is obtained; see [23]. All the considerations of this paper carry over to this more general setting.) The process $Z(t)$ is a diffusion process on the line in the sense of Ito and McKean [7] whose infinitesimal operator is of the form

$$(10)\qquad Uf(x) = D_\eta D_\xi f(x)$$

where $\eta(x) = E|x|^\gamma/\gamma$ and $\xi(x) = C|x|^\beta/\beta$ (see [23] and also [24]). Such processes are labeled Bessel diffusion processes for reasons indicated later.

Henceforth, whenever we deal with the case (7) we assume the relation (8) and hence (9) holds. Under these conditions Stone also establishes the validity of the invariance principle.

We introduce analogous to (3), the variables

$$(11)\qquad M(t),\ M^*(t),\ M^+(t),\ A(t),\ A^*(t),\ A^+(t),\ N(t),\ N^*(t),\ Y(t)$$

defined in terms of the process $X(t)$ in the obvious way, for example (interpreting the notation as indicated earlier)

$$M^*(t) = \{\sup_{0\leqq\tau\leqq t} x(\tau) \mid X(t) = 0\},\ M^+(t) = \{\sup_{0\leqq\tau\leqq t} x(\tau) \mid X(\tau) \geqq 0,\ 0 \leqq \tau \leqq t\},$$

$$Y(t) = t - \max(\tau \mid X(\tau) = 0), \text{ etc.}$$

Our analysis concerning (11) is done analogously to that of (3). In particular one of the principal results of this paper is to point out distribution relations amongst the unconditioned variable, the associated conditioned variable and an appropriate arc sin law.

The results are elaborated for the two cases set forth above (sums of independent random variables, and diffusion processes), although it will be apparent to the reader that many of the corresponding results should be forthcoming for any sequence of approximating stochastic processes attracted to either a stable process or a Bessel type diffusion process.

We summarize briefly the contents of the paper. In Section 2 we develop various relationships amongst the distribution functions of the limit laws for the variables (3) and (11). Some extensions are indicated in Section 3. The actual problem of the existence of such limit laws is discussed in Sections 3 and 4. The nature of the limit law for the conditioned occupation time of a half line is described. Moreover, the nature of the limit laws for random variables of the type

$$\begin{aligned} M^*_{\rho n} &= \{\max_{0 \leqq k \leqq \rho n} S_k \mid S_n = 0\} \\ N^*_{\rho n} &= \{N_{\rho n} \mid S_n = 0\} \qquad (12) \\ Y^*_{\rho n} &= \{Y_{\rho n} \mid S_n = 0\} \end{aligned}$$

where $\rho < 1$ is given. Finally, a brief discussion of the limit laws for joint random variables amongst the variables (3) is also given.

2. Relationship of conditioned limit laws and standard limit laws. This section is devoted to a discussion of relations amongst the limit distributions of certain unconditioned random variables and the same random variables conditioned by a recurrent event. The examples of (3) and (11) are typical

A. *Conditioned occupation time random variable.* We first restrict attention to the process (1) and assume for simplicity of exposition that the characteristic function $C(t) = E(e^{it\xi_k})$ satisfies the property, as $t \to 0$,

$$C(t) \sim 1 - D|t|^{\alpha}, \qquad 1 < \alpha \leqq 2, D > 0. \tag{13}$$

We may deduce that the convergence of (2) is valid and also relation (4) holds [19]. In other words the process $S_{[nt]}$ belongs to the domain of attraction of the symmetric stable process of index α. Moreover, it is known that $\lim_{n\to\infty} \Pr\{N_n/Dn^{\delta} \leqq x\} = G_\delta(x)$, $\delta = 1 - 1/\alpha$, [3], where $G_\delta(x)$ is the Mittag-Leffler distribution whose Laplace transform is

$$\int_0^{\infty} e^{-su}\, dG_\delta(u) = \sum_{k=0}^{\infty} \frac{(-s)^k}{\Gamma(1 + \delta k)}\,.$$

Our immediate aim is to evaluate the limit $\lim_{n\to\infty} \Pr\{N_n/Dn^{\delta} \leqq x \mid S_n = 0\}$. The key to this analysis is the obvious relationship,

$$N_n = N_{n-Y_n}\,. \tag{14}$$

Using (14) and the fact that the conditional distribution of N_n given that $Y_n/n = u$ coincides with the conditional distribution of N_n given that $S_{[n(1-u)]} = 0$, we have

$$\Pr\{N_n \leqq xn^{\delta}D\} = \int_0^1 \Pr\{N_{[n(1-u)]} \leqq xn^{\delta}D \mid S_{[n(1-u)]} = 0\}\, d_u \Pr\{Y_n \leqq nu\}. \tag{15}$$

Here $[n(1-u)]$ denotes the smallest integer $\geqq n(1-u)$. A formal argument on behalf of (15) proceeds as follows. By the law of total probabilities, we have

$$\Pr\{N_n \leqq xn^\delta D\} = \int_0^1 \Pr\{N_{[n(1-u)]} \leqq xn^\delta D \mid Y_n = nu\}\, d_u \Pr\{Y_n \leqq nu\}.$$

But

$$\Pr\{N_{[n(1-u)]} \leqq xn^\delta D \mid Y_n = nu\}$$
$$= \frac{\Pr\{N_{[n(1-u)]} \leqq xn^\delta D,\, S_{[n(1-u)]} = 0,\, S_i \neq 0([n(1-u)] < i \leqq n)\}}{\Pr\{S_{[n(1-u)]} = 0,\, S_i \neq 0([n(1-u)] < i \leqq n)\}}$$

By the Markov nature of the S_n process this expression becomes

$$\frac{\Pr\{N_{[n(1-u)]} \leqq xn^\delta D \mid S_{[n(1-u)]} = 0\}\Pr\{S_i \neq 0; ([n(1-u)] < i \leqq n) \mid S_{[n(1-u)]} = 0\}}{\Pr\{S_i \neq 0; [n(1-u)] < i \leqq n \mid S_{[n(1-u)]} = 0\}}$$
$$= \Pr\{N_{[n(1-u)]} \leqq xn^\delta D \mid S_{[n(1-u)]} = 0\}.$$

The identity (15) clearly obtains by virtue of the last relation.

Assume now the existence of the limit

$$\lim_{n\to\infty} \Pr\{N_n/Dn^\delta \leqq x \mid S_n = 0\} = G_\delta^*(x). \tag{16}$$

(This will be proved in Section 3 under appropriate conditions.)

Passing to the Laplace transform in (15) yields

$$\varphi_n(s) = \int_0^1 \varphi^*_{[n(1-u)]}(s(1-u)^\delta)\, d_u \Pr\{Y_n \leqq nu\} \tag{17}$$

where $\varphi_n(x) = E(e^{-s(N_n/n^\delta)})$, $\varphi_n^*(s) = E(e^{-s(N_n/n^\delta)} \mid S_n = 0)$.

We recall the fact (see (4)) that under the conditions stated

$$\lim_{n\to\infty} \Pr\{Y_n/n \leqq u\} = F_{1-\delta}(u), \qquad 1-\delta = 1/\alpha.$$

Proceeding to a limit in (17) leads to the relation

$$\varphi(s) = \frac{\sin \pi\delta}{\pi}\int_0^1 \varphi^*(s(1-u)^\delta)u^{-\delta}(1-u)^{\delta-1}\, du \qquad s > 0 \tag{18}$$

where $\varphi(s)$, $\varphi^*(s)$ are the Laplace transforms of $G_\delta(u)$ and $G_\delta^*(u)$ respectively. We can write (18) equivalently in the form

$$G_\delta(x) = \frac{\sin \pi\delta}{\pi}\int_0^1 G_\delta^*\left(\frac{x}{(1-u)^\delta}\right)u^{-\delta}(1-u)^{\delta-1}\, du. \tag{19}$$

Let N and N^* denote random variables with Laplace transforms $\varphi(s)$ and $\varphi^*(s)$ respectively. We may compute the moments from (18). This yields

$$\frac{r!}{\Gamma(1+\delta r)} = E((N)^r) = \frac{\sin \pi\delta}{\pi}E((N^*)^r)\int_0^1 u^{-\delta}(1-u)^{\delta(r+1)-1}\, du$$

which simplifies to $E((N^*)^r) = r!\Gamma(\delta)/\Gamma(\delta(r+1))$.

The distribution function of the variable N^* can now be identified and indeed a direct computation of the moments will verify

$$(20) \qquad G_\delta^*(x) = \Gamma(1+\delta) \int_0^x \xi \, dG_\delta(\xi).$$

We may invert the relation (18) and express φ^* in terms of φ. This amounts, apart from a change of variable, to the Abel inversion formula. Thus, an obvious change of variable in (18) and some simple manipulations produce $\varphi(s^\delta) = (\sin \pi\delta/\pi) \int_0^s \varphi^*(v^\delta)(s-v)^{-\delta} v^{\delta-1} \, dv$.

The Abel inversion formula ([26], p. 40) yields

$$(21) \qquad \varphi^*(v^\delta)v^{\delta-1} = \frac{\pi}{\sin \pi\delta} \frac{d}{dv} \int_0^v \varphi(s^\delta)(v-s)^{\delta-1} \, ds.$$

By the same devices, the formula (19) can be inverted, thus expressing G_δ^* explicitly in terms of G_δ.

We summarize the preceding results as follows.

THEOREM. *Suppose that* $\{\xi_n\}$ *is a sequence of independent and identically distributed random variables,* $S_n = \xi_1 + \cdots + \xi_n$, *and* $E(e^{it\xi_n}) \sim 1 - D|t|^\alpha$, as $t \to 0$, $(1 < \alpha \leqq 2, D > 0)$. *Let* N_n *denote the number of* S_k *equal to* 0, $k = 1, \cdots, n$. *Then*

$$\lim_{n\to\infty} \Pr\{N_n/Dn^\delta \leqq x \mid S_n = 0\} = \Gamma(1+\delta) \int_0^x \xi \, dG_\delta(\xi),$$

assuming that this limit exists.[2] $G_\delta(x)$ *is the Mittag-Leffler distribution whose Laplace transform is* $\int_0^\infty e^{-su} \, dG_\delta(u) = \sum_{k=0}^\infty [(-s)^k/\Gamma(1+\delta k)]$, *and* $\delta = 1 - 1/\alpha$.

We may establish the relationship (18) also in the case where the underlying process is of type (7) and satisfies the conditions (8). It is proved in [9] subject to these stipulations that the recurrence distribution $F_*(t)$ of the zero state satisfies

$$(22) \qquad 1 - F_*(t) \sim A/t^\delta \qquad t \to \infty$$

where $\delta = \beta/(\beta+\gamma)$ (see the notation of (8)).

It is also known [3] (cf. [9]) that

$$(23) \qquad \lim_{t\to\infty} \Pr(N(t)/At^\delta \leqq x) = G_\delta(x)$$

and

$$(24) \qquad \lim_{t\to\infty} \Pr\{Y(t)/t \leqq u\} = F_{1-\delta}(u).$$

The same reasoning as above shows that

$$(25) \qquad \lim_{t\to\infty} \Pr\{N(t)/At^\delta \leqq x \mid X(t) = 0\} = G_\delta^*(x)$$

and of course (23), (24) and (25) are connected by the Formula (19).

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[2] In Section 3 sufficient conditions for this limit to exist are given which cover many cases of interest. See also the discussion in Section 4.

The limit law established for the conditioned variable of $N(t)$ is considerably general to the following extent. Consider any irreducible Markoff chain $U(t)$ (discrete or continuous time) whose state space are the integers. Let $F_*(t)$ be the recurrence time distribution of the zero state and suppose (22) holds for $1 - F_*(t)$. Let $N_I(t)$ be the occupation time of I, during the time interval $(0, t)$. It is known [3] that (23) holds provided that A is replaced by an appropriate constant (essentially a multiple of the stationary measure of the set I). A slight extension of the previous argument again proves (25), *assuming this limit exists.* We need not enter into details, but the argument of Section 3 based on a consideration of moments establishes (25) in an important special case.

B. *Conditioned maximum random variable.* We begin with the case of the birth and death process (7) under the hypotheses of (8). As remarked in the introduction, this case is typical of the situation where the underlying process is of diffusion type.

Applying the invariance theorem, we conclude that the joint distribution of $\{c^{-1}M(c^a t), c^{-1}X(c^a t)\}$, $(a = \beta + \gamma)$ converges as $c \to \infty$ to the joint distribution of $\{M_Z(t), Z(t)\}$ where $M_Z(t) = \max_{0 \leqq \tau \leqq t} Z(\tau)$. In particular, we conclude that the conditioned distribution law

$$\Pr\{M(t)/t^{1/a} \leqq x \mid X(t) \leqq 0\} = H_t(x) \tag{26}$$

converges to the distribution law $\Pr\{M_Z(1) \leqq x \mid Z(1) \leqq 0\} = H(x)$.

Now consider the obvious relation $U(t)M(t) = M(t - Y(t))U(t)$ where $U(t) = 1$ if $X(t) \leqq 0$ and zero otherwise, valid because of continuity of paths.

This relation expressed in terms of distribution functions becomes

$$\Pr\{M(t) \leqq xt^{1/a} \mid X(t) \leqq 0\}$$
$$= \int_0^1 \Pr\{M(t) \leqq xt^{1/a} \mid Y(t) = tu, X(t) \leqq 0\}\, d_u \Pr\{Y(t) \leqq tu\}.$$

But

$$\Pr\{M(t) \leqq xt^{1/a} \mid Y(t) = tu, X(t) \leqq 0\}$$
$$= \frac{\Pr\{M(t(1-u)) \leqq xt^{1/a}, X(t(1-u)) = 0, X(\tau) < 0; t(1-u) < \tau < t\}}{\Pr\{X(t(1-u)) = 0, X(\tau) > 0; t(1-u) < \tau < t\}}$$

and by the Markov property this reduces to

$$\Pr\{M(t(1-u)) \leqq xt^{1/a} \mid X(t(1-u)) = 0\}.$$

Combining, we have

$$\begin{aligned} &\Pr\{M(t) \leqq xt^{1/a} \mid X(t) \leqq 0\} \\ &= \int_0^1 \Pr\{M(t(1-u)) \leqq xt^{1/a} \mid X(t(1-u)) = 0\}\, d_u \Pr\{Y(t) \leqq tu\}. \end{aligned} \tag{27}$$

We postulate the convergence as $t \to \infty$ of

$$H_t^*(x) = \Pr\{M(t)/t^{1/a} \leqq x \mid X(t) = 0\} \quad \text{to} \tag{27a}$$
$$\Pr\{M_Z(1) \leqq x \mid Z(1) = 0\} = H^*(x).$$

(Conditions under which the existence of the limit holds will be given in Section 4.)

Now following the method of Part A of this section based on the relation (27), we obtain

$$H(x) = \frac{\sin \pi\delta}{\pi} \int_0^1 H^*\left(\frac{x}{(1-u)^{1/a}}\right) u^{-\delta}(1-u)^{\delta-1}\, du \tag{28}$$

where $H(\cdot)$ is the distribution function referred to above and $\delta = \beta/(\beta+\gamma)$, and $a = \beta + \gamma$. Formula (28) can be inverted along the lines of (21). This expresses H^* explicitly in terms of H.

We next turn to the question of evaluating and establishing the existence of the limit $(t \to \infty)$ of the quantities

$$\Pr\{M(t) \leqq xt^{1/a} \mid X(u) \geqq 0, 0 \leqq u \leqq t\}. \tag{29}$$

The analysis of (29) is quite simple. We convert the state 0 into a reflecting barrier and consider the associate birth and death process on the non-negative integers. Let $X^+(t)$ represent this new process and $M^+(t)$ the corresponding maximum variable. A little reflection shows that

$$\Pr\{M(t) \leqq xt^{1/a} \mid X(u) \geqq 0, 0 \leqq u \leqq t\} \tag{30}$$
$$= \Pr\{M^+(t) \leqq xt^{1/a}\} = H_t^+(x).$$

The asymptotic behavior of the coefficients in the $X^+(t)$ process obviously satisfy $\pi_n^+ \sim \pi_n \sim En^{\gamma-1}$, $(\lambda_n^+\pi_n^+)^{-1} \sim Cn^{\beta-1}$, $(n \to \infty)$.

According to [24], we know that the process $X_c^+(t) = c^{-1}X^+(c^a t)$ converges to a diffusion process $Z^+(t)$ on $[0, \infty)$ with a reflecting regular boundary at the origin and infinitesimal operator given by $Af(x) = D_\eta D_\xi f(x)$, $x > 0$ where $\xi(x) = Cx^\beta/\beta$; $\eta(x) = Ex^\gamma/\gamma$. The invariance theorem prevails (see [23]) and the limit in (30) is the distribution function of the maximum

$$M_{Z^+}(1) = \max_{0 \leqq t \leqq 1} Z^+(t). \tag{31}$$

An explicit expression of the distribution law of (31) can be determined in the following way. Let T_x be the first passage time starting from the origin of reaching $x > 0$ for the diffusion process $Z^+(t)$. The Laplace transform of T_x is identified in [9] (see also Ito and McKean [7]) as the reciprocal of the Bessel function

$$I_\delta(s) = \sum_{r=0}^{\infty} [\Gamma(1-\delta)/r!\Gamma(r+1-\delta)](CDs/a^2)^r$$

where $\delta = \beta/(\beta+\gamma)$, $a = \beta + \gamma$. The familiar relation $\Pr\{M_{Z^+}(t) \geqq x\} = \Pr\{T_x \leqq t\}$ serves now to compute the probability law of $M_{Z^+}(t)$.

Consider the suggestive relation

$$M(t) = \max\{M(t - Y(t)),\, U(t)M^+(Y(t))\}. \tag{32}$$

This expression is, of course, not to be interpreted literally since some of the random variables are defined on different probability spaces. The identity (32) is intended only to motivate the meaningful relationships involving the corresponding distribution functions of these variables.

A rigorous derivation of the corresponding distribution function identity for (32) proceeds as follows. With the aid of the law of total probabilities and conditioning on the values of $Y(t)$, we obtain

$$\text{(32a)} \quad \Pr\{M(t) \leqq t^{1/a}x\} = \int_0^1 \Pr\{M(t) \leqq t^{1/a}x \mid Y(t) = tu\}\, d\Pr_u\{Y(t) \leqq tu\}.$$

But, the Markov property yields (compare with (15) and (27))

$$\text{(32b)} \quad \begin{aligned} &\Pr\{M(t) \leqq t^{1/a}x \mid Y(t) = tu\} \\ &\quad = \Pr\{X(t) \leqq 0\} \Pr\{M(t(1-u)) \leqq t^{1/a}x \mid X(t(1-u)) = 0\} \\ &\quad\quad + \Pr\{X(t) > 0\} \Pr\{M(t(1-u)) \leqq t^{1/a}x \mid X(t(1-u)) = 0\} \\ &\quad\quad \times \Pr\{M(tu) \leqq t^{1/a}x \mid X(0) = 0, X(\tau) > 0 (0 < \tau \leqq tu)\}. \end{aligned}$$

We insert this last formula into (32a) which furnishes the desired expression.

Referring to (27a), the convergence stated by (31) and letting $t \to \infty$, we obtain

$$\text{(33)} \quad \begin{aligned} &\Pr\{M_Z(1) \leqq x\} \\ &\quad = \frac{\sin \delta\pi}{\pi} \int_0^1 H^*\left(\frac{x}{(1-u)^{1/a}}\right)\left[\lambda + (1-\lambda)H^+\left(\frac{x}{u^{1/a}}\right)\right] u^{-\delta}(1-u)^{\delta-1}\, du \end{aligned}$$

where $H^*(x) = \Pr\{M_Z(1) \leqq x \mid Z(1) = 0\}$, $H^+(x) = \Pr\{M_{Z^+}(1) \leqq x\}$ and $\lambda = \lim_{t\to\infty} \Pr\{X(t) \leqq 0\} = \Pr\{Z(1) \leqq 0\}$.

The proof of the existence of $\lim_{t\to\infty} \Pr\{X(t) \leqq 0\} = \lambda$ is simple and its value can be computed as follows. Let η be the sojourn time of the non-negative axis starting from state 1, i.e., η is the random variable equal to the first passage time from state 1 to -1, whose distribution function is denoted by $F_{1,-1}(t)$. Similarly let ζ denote the first passage time from state -1 to state $+1$. It is proved in [9] that

$$\begin{aligned} 1 - F_{1,-1}(t) &\sim C^+/t^\delta \qquad t \to \infty \\ 1 - F_{-1,1}(t) &\sim C^-/t^\delta \end{aligned}$$

for an appropriate pair of constants $C^+, C^- > 0$. A standard renewal argument shows that $\lim_{t\to\infty} \Pr\{X(t) \leqq 0\} = \lambda = C^-/(C^+ + C^-)$.

The preceding analysis exhibited relations of certain limit laws pertaining to the maximum variable when the underlying process is of diffusion type. Specifically, we had focused attention on the case of birth and death processes for ease of exposition. We now turn to examine the corresponding versions of these theorems where $X(n)$ is the process (1) obeying the restrictions (13).

If the summands ξ_i obey the property that $\Pr\{\xi_i \leqq -2\} = 0$ (recall that ξ_i are integer valued), then the paths $S_n(\omega)$ are "continuous" in movement to the left. In this case the arguments leading to (28) apply mutatis mutandis.

This situation is only possible if $\alpha = 2$ and then $S_{[nt]}$ is in the domain of attraction of the Brownian motion process. The expression (28) becomes

$$\begin{aligned} &\Pr\{M_Z(1) \leqq x \mid Z(1) \leqq 0\} \\ &\qquad = \pi^{-1} \int_0^1 \Pr\{M_Z(1) \leqq x/u^{\frac{1}{2}} \mid Z(1) = 0\} u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}\, du \end{aligned} \tag{34}$$

and $Z(t)$ is standard Brownian motion. Since the left hand side is $(2/\pi)^{\frac{1}{2}} \cdot \int_0^x e^{-\xi^2/2}\, d\xi$ an immediate verification shows that $\Pr\{M_Z(1) \leqq x \mid Z(1) = 0\} = 1 - e^{-x^2/2}$.

The methods employed above in deducing (34) depend on the validity of the convergence.

$$\lim_{n\to\infty} \Pr\{M_n/Dn^{\frac{1}{2}} \leqq x \mid S_n = 0\} = \Pr\{M_Z(1) \leqq x \mid Z(1) = 0\} \tag{35}$$

which may be established subject to some suitable mild restrictions with the aid of the invariance principle. The formula (34) can also be derived readily by direct considerations of Brownian motion.

In the general case where ξ_i has finite variance σ^2 and therefore $S_{[nt]}/\sigma n^{\frac{1}{2}}$ converges to Brownian motion we would expect (35). Undoubtedly the invariance principle also applies to the tied down approximating process $(S_{[nt]/\sigma n^{\frac{1}{2}}} \mid S_n = 0)$ which converges to Brownian motion $Z(\tau)$, $0 \leqq \tau \leqq 1$ conditioned so that $Z(1) = 0$ and then (35) obtains by considering the maximum functional. An example of this sort was studied in [4].

Returning to the case of (13) and $1 < \alpha < 2$, we postulate that

$$\begin{aligned} &\lim_{n\to\infty} \Pr\{M_n/Dn^{1/\alpha} \leqq x \mid S_n = 0\} = R_\alpha^*(x) \\ &\lim_{n\to\infty} \Pr\{M_n/Dn^{1/\alpha} \leqq x \mid S_i \neq 0, i = 1, \cdots, n\} = \tilde{R}_\alpha(x) \end{aligned} \tag{36}$$

both exist. It is known [19] that

$$\lim_{n\to\infty} \Pr\{M_n/Dn^{1/\alpha} \leqq x\} = R_\alpha(x). \tag{37}$$

Moreover, under the assumption (13) it follows easily that

$$\lim_{n\to\infty} \Pr\{S_n \leqq 0\} = \tfrac{1}{2}. \tag{38}$$

Following the method of (33) we obtain the relation

$$R_\alpha(x) = \frac{\sin \pi\alpha}{2\pi} \int_0^1 R_\alpha^*\left(\frac{x}{(1-u)^{1/\alpha}}\right)\left[1 + \tilde{R}_\alpha\left(\frac{x}{u^{1/\alpha}}\right)\right] u^{-\delta}(1-u)^{\delta-1}\, du, \quad \left(\delta = 1 - \frac{1}{\alpha}\right). \tag{39}$$

Of course, (39) can be interpreted as a relation amongst certain functionals defined on the paths of the symmetric stable process $Z^\alpha(t)$ of order α. Explicitly

(40) $R_\alpha(x) = \Pr\{M_{Z^\alpha}(1) \leqq x\}, \quad R_\alpha^*(x) = \Pr\{M_{Z^\alpha}(1) \leqq x \mid Z^\alpha(1) = 0\},$

and $\tilde{R}_\alpha(x) = \Pr\{M_{Z^\alpha}(1) \leqq x \mid Z^\alpha(\tau) \neq 0, 0 < \tau \leqq t\}.$

Under some mild conditions which guarantee the existence of certain local limit laws the proof of the convergence of (36) can be established as a consequence of the invariance principle (see Section 4).

We summarize the preceding analysis in the statement of the following theorem.

THEOREM. *Let* $X(t)$ *represent a process of type* (7) *satisfying the conditions of* (8). *Then*

$\lim_{t\to\infty} \Pr\{M(t) \leqq xt^{1/\alpha} \mid X(t) = 0\} = H^*(x),$

$\lim_{t\to\infty} \Pr\{M(t) \leqq xt^{1/\alpha} \mid X(u) \geqq 0, 0 \leqq u \leqq t, X(0) = 0\} = H^+(x),$

$\lim_{t\to\infty} \Pr\{X(t) \leqq 0\} = \lambda$

and (33) *hold.*

Moreover, if S_n *is a process of sums of independent random variables obeying* (13) *then* (36) *and* (38) *are valid.*

C. *Conditioned limit laws for* S_n. In this part we are interested in the limit probability distribution of

(41) $$\Pr\{M_n/Dn^{1/\alpha} \leqq x \mid M_n > M_{n-1}\}$$

where S_n is the process (1). We will assume that $E(\xi_i) = 0$,

(42) $$\lim_{n\to\infty} \Pr\{M_n/Dn^{1/\alpha} \leqq x\} = R_\alpha(x)$$

and

(43) $$\lim_{n\to\infty} \Pr\{S_n > 0\} = \lambda \qquad (0 < \lambda < 1).$$

The result to be proved is the following.

THEOREM. *Assuming* (42), (43) *and that the limit referred to below exists, then*

$$\lim_{n\to\infty} \Pr\{M_n/Dn^{1/\alpha} \leqq x \mid M_n > M_{n-1}\} = S_\alpha(x).$$

$S_\alpha(x)$ *is a distribution function described in terms of* $R_\alpha(x)$ *by relation* (50) *below.*

The proof proceeds as follows. We introduce the recurrent event $e: M_n > M_{n-1}$, i.e., we say e occurs at time n if $M_n > M_{n-1}$. Let $\{p_n\}_{n=1}^\infty$ be the probability distribution of the time T of the next occurrence of the event e. The generating function of T is given in [1]:

(44) $$E(x^T) = 1 - \exp\left(-\sum_{k=1}^\infty \frac{x^k}{k} \Pr\{S_k > 0\}\right).$$

Applying a standard Abelian type argument to (44) using (43) (actually Cesaró order convergence in (43) would suffice) we deduce

(45) $$1 - E(x^T) = (1 - x)^\lambda L(1/(1 - x)) \qquad x \uparrow 1$$

where $L(\cdot)$ is a slowly varying function. The slowing varying function $L(\cdot)$

arises by the following considerations. We assert that the function

$$L\left(\frac{1}{1-x}\right) = \exp\left(-\sum_{k=1}^{\infty} \frac{x^k}{k} a_k\right) \qquad 0 < x < 1$$

is slowly varying whenever $\lim_{k\to\infty} a_k = 0$. To prove this assertion, consider

$$\frac{L(cu)}{L(u)} = \exp\left(-\sum_{k=1}^{\infty}\left[\left(\frac{cu-1}{cu}\right)^k - \left(\frac{u-1}{u}\right)^k\right]\frac{1}{k} a_k\right) \qquad (0 < c \text{ fixed}).$$

Choose k large enough so that $|a_k| \leqq \epsilon$ for $k > K$. Then determine u sufficiently large so that $\max_{1 \leqq k \leqq K} |((cu-1)/cu)^k - ((u-1)/u)^k| \leqq \epsilon$. Since $(cu-1)/cu$ is an increasing function of its argument and $\log cu - \log u$ is bounded, we obtain for u large enough, the estimate $e^{-3\epsilon} \leqq L(cu)/L(u) \leqq e^{3\epsilon}$ verifying that $L(\cdot)$ is a slowly varying function as claimed.

Now we introduce the variable

(46) $\quad U_n$ = time elapsed since the last time e occurred counted from time n.

Appealing to the theorem of Dynkin, Lamperti [5], [12], we have $\lim_{n\to\infty} \Pr\{U_n/n \leqq u\} = F_{1-\lambda}(u)$.

We start with the identity:

$$M_n = M_{n-U_n}. \tag{47}$$

The method from here on proceeds identically to that of Part A.

If we postulate the existence of the limit

$$\lim_{n\to\infty} \Pr\{M_n/An^{1/\alpha} \leqq x \mid M_n > M_{n-1}\} = S_\alpha(x). \tag{48}$$

(We will discuss the truth of (48) in Section 4), then analogous to (19), we obtain

$$R_\alpha(x) = \frac{\sin \pi\lambda}{\pi} \int_0^1 S_\alpha\left(\frac{x}{(1-u)^{1/\alpha}}\right) u^{-\lambda}(1-u)^{\lambda-1}\, du. \tag{49}$$

Along the lines of (21) we can invert (49) and express $S_\alpha(x)$ in terms of $R_\alpha(x)$. Executing these manipulations, we obtain

$$v^{\lambda-1} S_\alpha(v^{-1/\alpha}) = \left(\frac{\pi}{\sin \pi\lambda}\right)\frac{d}{dv}\int_0^v R_\alpha(x^{-1/\alpha})(v-x)^{\lambda-1}\, dx. \tag{50}$$

It is a familiar trick to rewrite (41) in the form

$$\begin{aligned} &\Pr\{M_n/Dn^{1/\alpha} \leqq x \mid M_n > M_{n-1}\} \\ &\quad = \frac{\Pr\{S_n \leqq xDn^{1/\alpha}, S_n > S_{n-1}, S_n > S_{n-2}, \cdots, S_n > 0\}}{\Pr\{X_n > 0, X_n + X_{n-1} > 0, \cdots, X_n + \cdots + X_1 > 0\}} \\ &\quad = \Pr\{S_n \leqq xDn^{1/\alpha} \mid S_1 > 0, \cdots, S_n > 0\}. \end{aligned} \tag{51}$$

This proves under the conditions (42) and (43)

$$\lim_{n\to\infty} \Pr\{S_n \leqq xDn^{1/\alpha} \mid S_1 > 0, \cdots, S_n > 0\} = S_\alpha(x). \tag{52}$$

Consider now the quantity

$$\Pr\{M_n \leqq xDn^{1/\alpha} \mid S_n = M_n\}. \tag{53}$$

Again we postulate (42) and (43). Clearly $\Pr\{S_n = 0\} \to 0$ $(n \to \infty)$ and therefore $\lim_{n\to\infty} \Pr\{S_n \geqq 0\} = \lambda$. We say that the event $\tilde{e}$ occurs at time n if $S_n = M_n$. Let $\tilde{T}$ be the time of the next occurrence of the event $\tilde{e}$. Baxter [1] has determined the generating function of $\tilde{T}$:

$$E(x^{\tilde{T}}) = 1 - \exp\left(-\sum_{k=1}^{\infty} \frac{x^k}{k} \Pr\{S_k \geqq 0\}\right).$$

Again (45) follows and subsequently (49) and (50) (the factor $L(\cdot)$ may be different but it is still slowly varying). To sum up: If (42) and (43) hold then

$$\begin{aligned}\Pr\{M_n \leqq xDn^{1/\alpha} \mid S_n = M_n\}& \\ = \Pr\{S_n \leqq xDn^{1/\alpha} \mid S_1 \geqq 0, S_2 \geqq 0, \cdots, S_n \geqq 0\}&\end{aligned} \tag{54}$$

and the limit in (54) is the function $S_\alpha(x)$ that appears in (52).

We close this section citing the following example of the preceding theory. Suppose $\alpha = 2$, $\lambda = \frac{1}{2}$ and $R_2(x) = (2/\pi)^{\frac{1}{2}}\int_0^x e^{-\xi^2/2}\,d\xi$. This will be the case in particular when ξ satisfies $E(\xi) = 0, 0 < E(\xi^2) = \sigma^2 < \infty$. Comparing (49) with (34) we infer $S_2(x) = 1 - e^{-x^2/2}$, $(0 \leqq x < \infty)$ (cf. Spitzer [22] footnote, p. 162).

D. *Conditional occupation time of a half line.* Let $X(t)$ be a Markoff process with the following structure. The states of the process are divided into two classes I_+ and I_-, except for one special state σ. The assumptions are that occupation of state σ is a certain recurrent event and if $X(t_1) \,\varepsilon\, I_+$ and $X(t_2) \,\varepsilon\, I_-$, then for some intervening time (between t_1 and t_2), $X(t) = \sigma$.

An illustration of the above set up arises if $X(t)$ is a birth and death process on the integers (or random walk if time is discrete) and then put $I_+ = (0, \infty)$, $I_- = (-\infty, 0)$ and $\sigma = \{0\}$.

Let $N_+(t)$ denote the occupation time up to time t of the set I_+; $N_-(t)$ is defined similarly. Let $F(t)$ be the recurrence time distribution of the state σ. We will assume throughout this section

$$1 - F(t) = (1/t^\delta)L(t) \qquad (0 \leqq \delta < 1) \tag{55}$$

where $L(t)$ is slowly varying.

Lamperti [13] (see also Takács [25] who treats a more general situation) has determined all possible limit laws for

$$N_+(t)/t \qquad t \to \infty. \tag{56}$$

The class of limit distributions of (56) comprise a two parameter family which are described explicitly in [13].

Our objective in this section is to relate the limit laws of

$$N_+^*(t)/t \tag{57}$$

with those of (56) where $N_+^*(t)$ is the occupation time up to time t of I_+ conditioned that $X(t) = \sigma$.

The point of view is the same as before. We postulate the existence of the limit law of (57) and determine its form. It is an open problem as to the precise conditions under which $N_+^*(t)/t$ admits a limit. Presumably, the same hypotheses which yield results in the case (56) will work here as well.

To analyze (56) we start with the relations:

$$N_+(t) = N_+(t - Y(t)) + V(t)Y(t) \tag{58}$$

where

$$V(t) = 1 \quad \text{if} \quad X(t) \,\varepsilon\, I_+$$
$$= 0 \quad \text{if} \quad X(t) \,\varepsilon\, I_-$$

and $Y(t)$ denotes the time measured from t of the last visit to $\{\sigma\}$. The occupation $N_\sigma(t)$ of state σ in the time interval [0, t] is of the order of magnitude $t^\delta L(t)$ $(0 \leqq \alpha < 1)$ and this tends to zero when normalizing by t (see paragraph A of this section). Therefore it is irrelevant whether or not we include the value $N_\sigma(t)$ in $N_+(t)$. If we discard occupations of the state σ then the realizations of the process have the following form:

The process visits I_+ and I_- alternately. Let $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ be the successive sojourn times spent in I_+ and I_- starting from state σ. Since the states I_+ and I_- communicate through state σ, it is clear that $\{\xi_i\}_{i=1}^\infty$ are independent and identically distributed; similarly for $\{\eta_i\}_{i=1}^\infty$. Let $A(\xi)$ and $B(\eta)$ denote the distribution functions of ξ and η, respectively.

We postulate that

$$\begin{aligned} 1 - A(t) &= A/t^\delta L(t) && t > 0 \\ 1 - B(t) &= B/t^\delta L(t) && (0 \leqq \delta < 1, A > 0, B > 0). \end{aligned} \tag{59}$$

This is consistent with (55) and in fact it is an easy matter to express $F(t)$ in terms of $A(t)$, $B(t)$ and the quantities p = the probability that on leaving state σ the process moves to I_+ (see [13]). If the asymptotic growth properties in (59) were not of the same order of magnitude then the limit law of (56) would be degenerate.

Under the conditions (59) it is an easy matter to show (a renewal argument) that

$$\lim_{t\to\infty} \Pr\{V(t) = 1\} = A/(A + B) = \lambda. \tag{60}$$

Expressing (58) in terms of Laplace transforms of the corresponding distribution functions, the relation becomes meaningful and since $N_+(t - Y(t))$ and $Y(t)V(t)$ are conditionally independent given $Y(t)$ (cf. (32b)), we obtain $\psi_t(s) = \int_0^1 \psi_{t(1-u)}^*(s(1 - u))[E(e^{-sV(t)u})]\, d_u \Pr\{Y(t) \leqq tu\}$, $s > 0$, where

$$\psi_t(s) = E(e^{-s(N_+(t)/t)}), \qquad \psi_t^*(s) = E(e^{-s(N_+(t)/t)} \mid X(t) = \sigma).$$

Proceeding to a limit $(t \to \infty)$ we obtain

$$\psi(s) = \frac{\sin \pi\delta}{\pi} \int_0^1 \psi^*(s(1-u))[\lambda e^{-su} + (1-\lambda)]u^{-\delta}(1-u)^{\delta-1}\, du \tag{61}$$
$$\psi(s) = \lim_{t\to\infty} \psi_t(s) \qquad \psi^*(s) = \lim_{t\to\infty} \psi_t^*(s).$$

The change of variable $s(1-u) = v$ converts (61) into a convolution form. We then compute a second Laplace transform yielding

$$\hat{\psi}_\delta(z) = \frac{\sin \pi\delta}{\pi} \hat{\psi}_\delta^*(z) \left[\lambda \frac{\Gamma(1-\delta)}{(1+z)^{1-\delta}} + (1-\lambda) \frac{\Gamma(1-\delta)}{z^{1-\delta}}\right] \tag{62}$$

where

$$\hat{\psi}_\delta(z) = \int_0^\infty e^{-zs}\psi(s)\, ds = \int_0^1 \frac{dE_\delta(x)}{z+x} \tag{63}$$

and $$E_\delta(x) = \lim_{t\to\infty} \Pr\{N_+(t)/t \leqq x\}$$

$$\psi_\delta^*(z) = \int_0^\infty e^{-zs}\psi^*(s)s^{\delta-1}\, ds = \Gamma(\delta) \int_0^1 \frac{dE_\delta^*(x)}{(z+x)^\delta} \tag{64}$$

and $$E_\delta^*(x) = \lim_{t\to\infty} \Pr\{N_+(t)/t \leqq x \mid X(t) = \sigma\}.$$

Lamperti has explicitly determined (63) in [13]. His formula is

$$\int_0^1 \frac{dE_\delta(x)}{z+x} = \frac{(z+1)^{\delta-1} + \dfrac{1-\lambda}{\lambda} z^{\delta-1}}{(z+1)^\delta + \dfrac{1-\lambda}{\lambda} z^\delta}. \tag{65}$$

In the symmetric case $(A = B; \lambda = \frac{1}{2})$. On comparing (65) and (62) we deduce

$$\frac{\psi_\delta^*(z)}{\Gamma(\delta)} = \int_0^1 \frac{dE_\delta^*(x)}{(z+x)^\delta} = \frac{2}{[(z+1)^\delta + z^\delta]}. \tag{66}$$

Now, if we specialize even further, set $\delta = \frac{1}{2}$ then

$$\int_0^1 \frac{dE_{\frac{1}{2}}^*(x)}{(z+x)^{\frac{1}{2}}} = \frac{2}{(z+1)^{\frac{1}{2}} + z^{\frac{1}{2}}} = 2[(z+1)^{\frac{1}{2}} - z^{\frac{1}{2}}]$$

whose inversion is clearly $E_{\frac{1}{2}}^*(x) = x$. G. Latta (private communication) has succeeded in inverting (66). He gives the formula

$$e^*\left(\frac{1}{1+x}\right) = (1+x)^{2-\delta} \frac{d}{dx} \frac{1}{\pi i} \int_0^x (x-t)^{\delta-1} \left[\frac{e^{\pi i\delta}}{t^\delta + e^{\pi i\delta}} - \frac{e^{-i\pi\delta}}{t^\delta + e^{-\pi i\delta}}\right] dt$$

where $e^*(\xi)$ is the density function of $E^*(\xi)$.

3. Moment methods. One of the questions left open above is: What are the exact circumstances under which the conditioned distribution of N_n approaches a limit. It seems plausible that this should take place if and only if the distribution of N_n itself approaches a limit. A full discussion of this matter will have to be postponed to a later work.

It is possible to give a sufficient condition for the conditioned distribution of N_n to approach a limit which takes care of most examples of interest, as follows.

Consider a certain, null-recurrent event, on the positive integers. Let

$$U_n = 1 \quad \text{if a recurrence takes place at time } n$$
$$= 0 \quad \text{otherwise.}$$

We will assume that $u_n = EU_n$ behaves like a power of n for large n. In order to arrange the constants conveniently we make the following specific assumption.

ASSUMPTION I. $u_n \sim An^{\delta-1}/\Gamma(\delta)$, as $n \to \infty$ $(0 < \delta < 1)$.

One can also carry through the analysis with the addition of a slow varying factor in the statement of Assumption I. We leave the details to the reader.

It is elementary to verify that under Assumption I, the conditioned distribution of N_n converges to the distribution described in (19). To do this, we compute moments as follows

$$E([(U_1 + U_2 + \cdots + U_n)/An^{\delta}]^{\kappa} \mid U_n = 1)$$
$$= \kappa!/A^{\kappa}n^{\delta\kappa} \sum_{1 \leqq i_1 < i_2 < \cdots < i_\kappa \leqq n} E(U_{i_1}U_{i_2} \cdots U_{i_\kappa}U_n)/E(U_n) + o(1)$$

where the $o(1)$ contribution corresponds to the terms of the multinomial expression for which at least one equality occurs amongst the set $(i_1, i_2, \cdots, i_\kappa)$. An induction proof on κ verifies that the contribution of these terms is of smaller order of magnitude than the sum exhibited. Next, invoking the renewal property associated with this event, the multiple sum can be written in the form

$$\frac{\kappa!}{A^{\kappa}n^{\delta\kappa}} \sum_{1 \leqq i_1 < i_2 < \cdots < i_\kappa \leqq n} E(U_{i_1})E(U_{i_2-i_1}) \cdots E(U_{i_\kappa - i_{\kappa-1}})E(U_{n-i_\kappa})/E(U_n)$$
$$\to \frac{\kappa!}{\Gamma^{\kappa}(\delta)} \int \frac{dx_1 \cdots dx_\kappa}{x_1^{1-\delta}(x_2 - x_1)^{1-\delta} \cdots (1 - x_\kappa)^{1-\delta}} \quad \text{(by Assumption I)}$$
$$0 \leqq x_1 < x_2 < \cdots < x_\kappa \leqq 1$$
$$= \frac{\kappa!}{\Gamma(\kappa\delta)} \int_0^1 \frac{dx}{(1-x)^{1-\kappa\delta}x^{1-\delta}} = \frac{\Gamma(\delta)\kappa!}{\Gamma(\delta(\kappa+1))} \qquad \kappa = 1, 2, \cdots.$$

These moments agree with formula (19).

The computation goes over in the same way in a slightly more general setting. Suppose that $0, i_1, \cdots, i_r$ are states belonging to the same null class in a denumerable Markov chain. At time 0 the system is in state 0. $U_n^{(i)}$ is the indicator random variable which is 1 at time n if the system is in state i and is 0 otherwise. State 0 may be one of $i_1, \cdots, i_r$ but the latter are distinct. The analogue of Assumption I is that the transition probabilities satisfy $P_{ij}^{(n)} \sim \pi_j/n^{1-\delta}$, $(i, j = 0, i_1, \cdots, i_r)$, $0 < \delta < 1$. Then for $U_1^{(0)} + \cdots + U_n^{(0)}$, the occupation time of 0 we have

$$\lim_{n\to\infty} E\left(\left(\frac{U_1^{(0)} + \cdots + U_n^{(0)}}{\pi_0 \Gamma(\delta) n^\delta}\right)^\kappa \middle| U_n^{(i_1)} + \cdots + U_n^{(i_r)} = 1\right) = \frac{\Gamma(\delta)\kappa!}{\Gamma(\delta(\kappa+1))}$$

exactly as before.

Another extension is possible as follows. Let $0 \leqq \mu < \lambda$ and let $N_{\mu n, \lambda n}$ count the number of times that state 0 is occupied in the time interval $(\mu n, \lambda n)$. Then under the assumptions stated above

$$\lim_{n\to\infty} E\left(\left(\frac{N_{\mu n,\lambda n}}{\pi_0 \Gamma(\delta) n^\delta}\right)^\kappa \middle| U_{\lambda n}^{(i_1)} + \cdots + U_{\lambda n}^{(i_r)} = 1\right) = \frac{\kappa!}{\Gamma(\kappa\delta)} \int_\mu^\lambda \frac{dx}{(\lambda - x)^{1-\kappa\delta} x^{1-\delta}}.$$

With some computations which we omit here it can be shown that this last integral equals

$$(67) \qquad \frac{\kappa!\,\Gamma(\delta)}{\Gamma(\delta(\kappa+1))} \int_0^{(\lambda-\mu)/\mu} \frac{[\lambda - \mu(1+z)]^\kappa \, dz}{\Gamma(\delta)\Gamma(1-\delta)(1 - (\mu/\lambda)(1+z))^{1-\delta} z^\delta (1+z)}$$

which displays the limiting distribution as that of a product of two independent random variables, one having the modified Mittag-Leffler distribution (20), the other having density function

$$[\Gamma(\delta)\Gamma(1-\delta)(1 - (\mu/\lambda)(1+z))^{1-\delta} z^\delta (1-z)]^{-1} \qquad 0 \leqq z \leqq (\lambda - \mu)/\mu.$$

(This representation as the distribution of a product of random variables, of course, is not unique.)

A direct explanation of this fact which displays some interesting "conditioned arc sin laws" can be made as follows. For simplicity, we place ourselves within the context of the certain, null-recurrent event discussed at the beginning of the section and we suppose that Assumption I holds. Consider the random variables

Y_n = time elapsed since last recurrence, measuring from time n,

Z_n = time to elapse until next occurrence measuring from time n.

Let $0 \leqq \mu < \lambda$. A straightforward computation shows that under Assumption I

$$\lim_{n\to\infty} P\left(\frac{Y_{\mu n}}{\mu n} < t \middle| U_{\lambda n} = 1\right) = \lim_{n\to\infty} P\left(\frac{Z_{(\lambda-\mu)n}}{\mu n} < t \middle| U_{\lambda n} = 1\right)$$

$$= \frac{(1 - \mu/\lambda)^\delta}{\Gamma(\delta)\Gamma(1-\delta)} \int_0^t \frac{dy}{y^\delta (1-y)^{1-\delta}[1 - (\mu/\lambda)(1-y)]}.$$

The Brownian motion version of this fact had been observed by Levy [16, Chapter 6] for $\delta = \frac{1}{2}$.

Now, to find the limiting distribution of $N_{\mu n,\lambda n}/n^\delta$ conditioned on $U_{\lambda n} = 1$, by a different method from the one leading to (67), we proceed as follows.

$$E\left(\left(\frac{N_{\mu n,\lambda n}}{An^{\delta}}\right)^{\kappa}\middle| U_{\lambda n}=1\right)$$

$$= \int_0^{(\lambda-\mu)/\mu} E\left(\left(\frac{N_{\mu n,\lambda n}}{An^{\delta}}\right)^{\kappa}\middle| \frac{Z_{\mu n}}{\mu n}=z,\ U_{\lambda n}=1\right) dP\left(\frac{Z_{\mu n}}{\mu n}\leqq z\middle| U_{\lambda n}=1\right)$$

$$= \int_0^{(\lambda-\mu)/\mu} E\left(\left(\frac{N_{\mu n,\lambda n}}{An^{\delta}}\right)^{\kappa}\middle| \frac{Z_{\mu n}}{\mu n}=z\right) dP\left(\frac{Z_{\mu n}}{\mu n}\leqq z\middle| U_{\lambda n}=1\right)$$

$$= \int_0^{(\lambda-\mu)/\mu} E\left(\left(\frac{N_{n(\lambda-\mu(1+z))}}{An^{\delta}}\right)^{\kappa}\middle| U_{((\lambda-\mu)/\lambda)n}=1\right) dP\left(\frac{Z_{\mu n}}{\mu n}\leqq z\middle| U_{\lambda n}=1\right)$$

$$\to \frac{\kappa!\Gamma(\delta)}{\Gamma(\delta(\kappa+1))}\int_0^{(\lambda-\mu)/\mu} \frac{[\lambda-\mu(1+z)]^{\kappa}\,dz}{\Gamma(\delta)\Gamma(1-\delta)(1-(\mu/\lambda)(1+z))^{1-\delta}z^{\delta}(1+z)} \quad \text{as} \quad n\to\infty.$$

4. Convergence problem and related open questions. We close this paper with some general remarks concerning the general convergence question left unsettled for some of the variables studied in Section 2.

1. If the underlying process is of diffusion type, i.e., random walk, birth and death or bona fide diffusion, then a direct proof can be given for the convergence of $\Pr\{M(t) \leqq xt^{1/\alpha} \mid X(t) = 0\}$ under the conditions (8).

We sketch the analysis for the case of a birth and death symmetric process, i.e., we assume $P_{ij}(t) = P_{-j,-i}(t)$.

The local theorem proved in [24] asserts

$$\Pr\{X(t) = 0\} \sim ct^{\alpha-1} \qquad t\to\infty. \tag{68}$$

Now consider the joint probability

$$\Pr\{M(t) \geqq t^{1/\alpha}x, X(t) = 0\} = \int_0^t P(t-\tau;\ [t^{1/\alpha}x], 0)\ d_\tau P(T_{[t^{1/\alpha}x]} = \tau), \tag{69}$$

([ ] symbolizes, as customary, the integral part of) the last resulting because of the continuity of paths. Here, $P(t, x, y)$ denotes the transition function of the underlying process and T_y is the random variable denoting the first passage time from 0 to y. It is proved in [24] (see also [9]) that T_n/n^{α} converges in law (this fact also follows by the invariance principle) and also

$$\lim_{t\to\infty} t^{1-\alpha}P(tu, t^{1/\alpha}x, 0) = p(u, x, 0) \tag{70}$$

is valid uniformly in x for each u where p is the density function of the symmetric Bessel process (10). Using the limit relations, (68) and (70) in (69), we infer the convergence of $\lim_{t\to\infty} \Pr\{M(t) \geqq t^{1/\alpha}x \mid X(t) = 0\}$.

If the underlying process is a sum of independent identically distributed integer valued random variables like (1) obeying the conditions (13) and symmetrically distributed then the convergence of

$$\Pr\{M(n) \leqq n^{1/\alpha}x \mid S_n = 0\} \tag{71}$$

can be deduced as a consequence of the invariance principle for conditioned processes. In fact, it is known under the hypothesis (13) that S_n possesses a local limit law of the form $\Pr\{S_n = 0\} \sim cn^{1/\alpha - 1}$ where c is an appropriate constant. This enables one to prove the convergence of the finite dimensional distributions of the random variables

$$S_{[nt_1]}/n^{1/\alpha}, S_{[nt_2]}/n^{1/\alpha}, \cdots, S_{[nt_r]}/n^{1/\alpha} \quad t_1 < t_2 < \cdots < t_r$$

conditioned that $S_n = 0$. It is not too hard to show that Skorokhod's regularity property [19] is satisfied and so the invariance principle applies. The details of this argument will be elaborated elsewhere.

2. The convergence problem concerning the variables of Part C of Section 2 in the case where $E(\xi_1) = 0$ and $E(\xi_i^2) = \sigma^2 < \infty$ can be dealt with by standard Tauberian arguments applied to moments using the familiar generating function relations of Baxter [1]. The general problem is open. The convergence problem is also unsettled in the case of the conditional occupation time variable of a half line. However, if the underlying process is of diffusion type (8) then the convergence can be established, with the aid of an invariance principle.

It would be worthwhile to develop the invariance principle for the general case of stochastic processes converging to a stable or Bessel diffusion process $Z(t)$, tied down so that $Z(1) = 0$. In the case of the Bessel process (under the conditions (8)) this probably can be done by the methods indicated in Part 1 of this section. This permits the assertion of the convergence of various functionals, in particular, the existence of limit laws for the variables of Section 2, Parts A, B and D. The analysis presented there identifies the actual limit law. The development of the invariance principle in the case of conditioned sums of independent random variables (not necessarily symmetric) attracted to an appropriate conditioned stable process is open.

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