D. R. Cox, *Renewal Theory*. Methuen and Company, Ltd., London, 1962. 142 pp.

Review by C. DERMAN

Columbia University

A *renewal process* is a sequence $\{X_i, i = 1, 2, \cdots\}$ of independent random variables with $\{X_i, i \geqq 2\}$ identically distributed. In applications the X_i's are non-negative and denote lengths of time between certain recurring events (renewals); $\tau_n = \sum_{i=1}^{n} X_i$ denotes the time at which the nth $(n = 1, 2, \cdots)$ renewal takes place.

Renewal theory is concerned with the chance behavior of such quantities as the number of renewals that occur within a given interval of time, the length of time at any given instant since (until) the occurrence of the previous (next) renewal, and asymptotic properties of these quantities as the interval of time or given instant becomes infinite. This monograph treats the case where the X_i's are non-negative and continuous. The emphasis is on material of use in applications.

In chapter one are the usual definitions and preliminaries. The Laplace Transform is introduced as a prelude to its extensive use in subsequent chapters. Certain special distributions such as the exponential and Erlang are discussed—these distributions being amenable to Laplace transform methods and useful as approximations to more general distributions.

The renewal process is introduced in chapter two. Three types are distinguished according to the distribution function of X_1 : *ordinary*, if X_1 has the same distribution function as X_i, $i \geqq 2$; *equilibrium*, if X_1 has the density function $[1 - F(x)]/\mu$, where $F(x)$ is the common distribution function of X_i and $\mu = EX_i$, $i \geqq 2$; *modified*, if X_1 has any other distribution function. Some advantage accrues from working with the equilibrium process in that certain asymptotic properties of the ordinary or modified processes become exact with the proper choice of the initial distribution function. The Poisson process is also introduced in this chapter.

In chapter three the distribution of the number of renewals is obtained in terms of its Laplace transform. The transform can be inverted in special cases. The asymptotic distribution of the number of renewals is also obtained by way of the Central limit theorem using the relationship between the number of renewals and the sums of the X_i's. There is also some discussion of the number of renewals in a random interval of time.

Chapter four deals with the moments of the number of renewals. In particular, the well-known asymptotic result about the mean number of renewals, together with some sharper approximations, is obtained. Backward (forward) recurrence time at time t is defined as the length of time from (until) the last (next) renewal. These distributions along with the limiting distributions as $t \rightarrow \infty$ are treated in chapter five.

Chapter six is concerned with the superposition of renewal processes. This refers to situations where a number of independent renewal processes are considered simultaneously and renewals are recorded whenever a renewal takes place in any one of the processes. Under certain conditions with the number of independent processes large, the "pooled" process is approximately a Poisson process.

In chapter seven alternating renewal processes are treated. That is, renewal type processes of the form $\{X_1, Y_1, X_2, Y_2, \cdots\}$ where the X_i's have one common distribution and the Y_i's another. For example, one can think of X_1 as the length of time a machine operates before it breaks down, Y_1 as the length of time until it is repaired, X_2 as the length of time it operates until the second failure, etc. Results are also obtained which are relevant to calculating the precision of systematic sampling of such a process.

Cumulative processes are discussed in chapter eight. These arise out of recurrent event processes; i.e. renewals are recurrent events (in the sense of Feller) of some underlying stochastic process and there are random variables $\{W_i, i = 1, 2, \cdots\}$ whose respective values are determined by the behavior of the process between the $(i - 1)$th and ith renewal. The W_i's are assumed to be independent and identically distributed. The Cumulative process is defined as $Z_t = \sum_{i=1}^{N_t} W_i, t > 0$, where N_t is the number of renewals in the interval $(0, t)$.

Chapter nine alludes to stronger results than those contained within the book. In chapter ten various probabilistic models of failure are dealt with in which it is possible to deduce the time between failure (failures are renewals). Chapter eleven deals with the evaluation of certain policies for the replacement of components in systems. Formulas are obtained which are helpful for such evaluations. Bibliographical notes and exercises appear in the appendices.

Though the author has chosen not to present the more mathematically interesting aspects of renewal theory, he has made a valuable contribution to the selection and organization of material suitable for applications. In particular, those engaged in developing curricula for Industrial Engineers and Operations Researchers welcome such a contribution.