GENERATING FUNCTIONS FOR MARKOV RENEWAL PROCESSES

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A general matrix representation is given for the multivariate transition probability generating functions of a Markov renewal process with a finite number of states. It is indicated how numerous derived probability distributions can be obtained by simple substitutions. Finally an application is made to the distribution of the maximum length of a busy $M/M/1$ queue.

Let $\mathbf{N}(t) = (N_1(t), \cdots, N_m(t))$ denote a Markov renewal process with a finite number m of states and with a matrix of transition probability distributions $\mathbf{Q} = \{Q_{ij}\}$ [3], [4]. Set $H_i = \sum_{j=1}^{m} Q_{ij}$. The $Q_{ij}(t)$ are nondecreasing right-continuous functions satisfying $Q_{ij}(0-) = 0$ and $H_i(\infty) = 1$ for all $i, j = 1, \cdots, m$. The random variable $N_i(t)$ is equal to the number of visits to state i during the time interval $[0, t)$. The stochastic process Z_t is referred to as the semi-Markov process (S-M P) associated with the Markov renewal process. $Z_t = i$ when state i is being visited at time t. We assume that $P\{Z_0 = i\} = p_i$ with $\sum_{i=1}^{m} p_i = 1$. Let $\mathbf{k} = (k_1, \cdots, k_m)$ denote an m-tuple of non-negative integers and define $T(\mathbf{k}) = \inf\{t: N_1(t) = k_1, \cdots, N_m(t) = k_m\}$, where it is interpreted to be $+\infty$ if the set is empty. Thus, $T(\mathbf{k})$ is the random time at which the Markov renewal process enters the state $\mathbf{k} = (k_1, \cdots, k_m)$. Let $Z(\mathbf{k}) = Z_{T(\mathbf{k})}$ and $T'(\mathbf{k})$ denote the time instant at which the S-M P leaves the state $Z(\mathbf{k})$. We define the following transition probabilities for the Markov renewal process.

$$C_j(\mathbf{k}, t) = P\{T(\mathbf{k}) \leqq t \text{ and } Z(\mathbf{k}) = j\} \tag{1}$$

and

$$D_j(\mathbf{k}, t) = P\{T(\mathbf{k}) \leqq t < T'(\mathbf{k}) \text{ and } Z_t = j\}. \tag{2}$$

It is clear that these distribution functions all vanish for $t < 0$. In the sequel we shall only consider them for nonnegative values of t. Let $\mathbf{e}_i$ denote the ith unit vector. The probabilities defined in (1) and (2) satisfy the following relations.

$$\begin{aligned} C_j(\mathbf{e}_i, t) &= \delta_{ij} p_j \\ C_j(\mathbf{k}, t) &= \sum_{\nu=1}^{m} C_\nu(\mathbf{k} - \mathbf{e}_j, t) * Q_{\nu j}(t) \text{ for } \mathbf{k} \neq \mathbf{e}_i \\ C_\nu(\mathbf{k} - \mathbf{e}_j, t) &\equiv 0 \text{ if } k_j = 0 \\ D_j(\mathbf{k}, t) &= [1 - H_j(t)] * C_j(\mathbf{k}, t). \end{aligned} \tag{3}$$

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Let a superscript asterisk affixed to a function denote its Laplace-Stieltjes transform. For example, $H_j^*(s) = \int_{0-}^{\infty} e^{-st}\, dH_j(t)$. We introduce the following multivariate probability generating functions in which $\mathbf{z} = (z_1, z_2, \cdots, z_m)$:

$$G_j^*(\mathbf{z}, s) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} C_j^*(\mathbf{k}, s) z_1^{k_1} \cdots z_m^{k_m}$$

and

$$K_j^*(\mathbf{z}, s) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} D_j^*(\mathbf{k}, s) z_1^{k_1} \cdots z_m^{k_m}$$

and the column-vectors

$$\mathbf{G}^* = [G_1^*(\mathbf{z}, s), \cdots, G_m^*(\mathbf{z}, s)], \mathbf{K}^* = [K_1^*(\mathbf{z}, s), \cdots, K_m^*(\mathbf{z}, s)].$$

The column-vectors $\mathbf{G}^*$ and $\mathbf{K}^*$ now have the following matrix representation.

Theorem. *For* $|z_i| < 1, i = 1, \cdots, m,$

$$\begin{aligned} \mathbf{K}^*(\mathbf{z}, s) &= [I - \Delta(H^*)]\mathbf{G}^*(\mathbf{z}, s) \\ &= [I - \Delta(H^*)][I - \Delta(\mathbf{z})\mathbf{Q}^{*\prime}]^{-1}\Delta(\mathbf{z})\mathbf{p} \end{aligned} \tag{4}$$

in which

$$\Delta(H^*) = \text{diag}\,(H_1^*(s), \cdots, H_m^*(s)), \Delta(\mathbf{z}) = \text{diag}\,(z_1, \cdots, z_m),$$

$\mathbf{Q}^* = \{Q_{ij}^*\}$ *and* $\mathbf{p}$ *is the column-vector* $[p_1, \cdots, p_m]$.

Proof. The first equality is simply the transform of the fourth equality of (3). Moreover it follows from the other equalities in (3) that

$$G_j^*(\mathbf{z}, s) = p_j z_j + z_j \sum_{\nu=1}^{m} Q_{\nu j}^*(s) G_\nu^*(\mathbf{z}, s)$$

which implies the second equality in (4) if the inverse of $I - \Delta(\mathbf{z})\mathbf{Q}^{*\prime}$ exists. This is easily seen to be so since $|z_j Q_{\nu j}^*(s)| \leqq Q_{\nu j}(\infty)$. The numbers $Q_{\nu j}(\infty)$ form an $m \times m$ stochastic matrix which therefore has spectral radius equal to one. A theorem of Wielandt [5] then implies that the spectral radius of $\Delta(\mathbf{z})\mathbf{Q}^{*\prime}$ is not less than one, which completes the proof.

A particular case. Discrete time finite Markov chains. If $C_j(\mathbf{k})$ denotes the probability that in $k_1 + \cdots + k_m - 1$ transitions a discrete Markov chain reaches state j and has visited state ν exactly k_ν times ($\nu = 1, \cdots m$) then

$$C_j^*(\mathbf{k}, s) = C_j(\mathbf{k}) \exp[-s(k_1 + \cdots + k_m - 1)]$$

$\mathbf{Q}^* = e^{-s}\mathbf{P}$, $G_j^*(\mathbf{z}, s) = e^s G_j(\mathbf{z}e^{-s})$ in which $G_j(\xi)$ is the generating function of the $C_j(\mathbf{k})$. After setting $z_i e^{-s} = \xi_i$ we obtain

$$\mathbf{G}(\xi) = [I - \Delta(\xi)\mathbf{P}']^{-1}\Delta(\xi)\mathbf{p}$$

which was proved earlier by Neuts [2].

Generating functions derivable from formula (4). Generating functions for many related probabilities can be derived from $\mathbf{K}(\mathbf{z}, s)$ by an appropriate choice of the variables z_i. If we set some of the z_i equal to the same variable u we find the transition probabilities of the S-M P which specify only the number of visits to certain but not all states. If we set certain variables z_i in (4) equal to zero, we find generating functions for taboo-probabilities, i.e. transition probabilities of events in which one specifies that certain states should not be visited.

Finally if we perform the substitutions $z_\nu = \rho e^{\alpha_\nu i v}$, $(0 < \rho < 1)$, for all or some of the variables z in which α_ν is equal to zero, plus or minus one we obtain generating functions for events defined with respect to algebraic sums of the random variables $N_i(t)$. Some detailed examples of these substitutions have been worked out in the case of finite Markov chains in Neuts [2].

An application. We consider a single server Poisson queue with input rate λ and service rate μ. We wish to evaluate the probabilities $\pi^n_{ij}(t)$ that in the time-interval $[0, t)$ there have been n transitions in the queue, the queue length at time t is j and neither of the queue lengths zero and b have been attained, given that the initial queue length was i, $(0 < i, j < b)$.

Let us consider the $b + 1$ state S-M P in which

$$
\begin{aligned}
Q_{ij}(t) &= 1 - e^{-\lambda t} && i = 0, j = 1\\
&= [\lambda/(\lambda + \mu)][1 - e^{-(\lambda+\mu)t}] && j = i + 1, i = 1, \cdots, b - 1;\\
&= [\mu/(\lambda + \mu)][1 - e^{-(\lambda+\mu)t}] && j = i - 1, i = 1, \cdots, b - 1;
\end{aligned}
$$

Q_{ij} may be defined arbitrarily for $i = b$, for purposes of the problem studied here. If we substitute this $\mathbf{Q}$ into formula (4) and set $\mathbf{p} = \mathbf{e}_i$ and $z_0 = z_b = 0$, $z_1 = \cdots = z_{b-1} = u$ we obtain

$$K_j^*(0, u, \cdots u, 0, s) = u \sum_{n=0}^{\infty} u^n \int_0^{\infty} e^{-st}\, d\pi^n_{ij}(t)$$

for $j = 1, \cdots, b - 1$. Set

$$\sum_{n=0}^{\infty} u^n \int_0^{\infty} e^{-st}\, d\pi^n_{ij}(t) = P^*_{ij}(u, s);$$

then

$$P^*_{ij}(u, s) = (1/u)K^*(0, u, \cdots, u, 0, s) = [1 - H_j^*(s)][(I - \Delta\mathbf{Q}^{*\prime})^{-1}]_{ji}$$

where $\Delta = \operatorname{diag}(0, u, \cdots, u, 0)$. After inversion of the jacobi-matrix $I - \Delta\mathbf{Q}^{*\prime}$ we find

$$
\begin{aligned}
P^*_{ij}(u, s) &= \frac{s}{s + \lambda + \mu}\left(\frac{\lambda u}{s + \lambda + \mu}\right)^{j-i} \frac{(\xi_1^i - \xi_2^i)(\xi_1^{b-j} - \xi_2^{b-j})}{(\xi_1 - \xi_2)(\xi_1^b - \xi_2^b)}, && \text{for } j \geqq i,\\
&= \frac{s}{s + \lambda + \mu}\left(\frac{\mu u}{s + \lambda + \mu}\right)^{i-j} \frac{(\xi_1^j - \xi_2^j)(\xi_1^{b-i} - \xi_2^{b-i})}{(\xi_1 - \xi_2)(\xi_1^b - \xi_2^b)}, && \text{for } j \leqq i,
\end{aligned}
$$

where

$$\xi_1 = \tfrac{1}{2}\{1 + [1 - 4\lambda\mu u^2(s + \lambda + \mu)^{-2}]^{\frac{1}{2}}\}, \quad \xi_2 = \tfrac{1}{2}\{1 - [1 - 4\lambda\mu u^2(s + \lambda + \mu)^{-2}]^{\frac{1}{2}}\}.$$

If we set $1/\cos\alpha = 2(\lambda\mu)^{\frac{1}{2}}u/(s + \lambda + \mu)$ we find for $i \leqq j$, ($j \leqq i$ is analogous),

$$P^*_{ij}(u, s) = s\lambda^{\frac{1}{2}(j-i-1)}\mu^{\frac{1}{2}(i-j-1)}u^{-1}\frac{\sin i\alpha \sin (b - j)\alpha}{\sin \alpha \sin b\alpha}.$$

Now $u^{-1} \sin i\alpha \sin (b - j)\alpha\, (\sin \alpha \sin b\, \alpha)^{-1}$ is a rational function of u with $b - 1$ distinct poles at $u_\rho = \cos \rho\pi/b$ $\rho = 1, \cdots, b - 1$. Partial fraction expansion and comparison of coefficients yield

$$\pi^n_{ij}(t) = \{2^n\lambda^{\frac{1}{2}(n+j-i)}\mu^{\frac{1}{2}(n+i-j)}n^{-1}t^n e^{-(\lambda+\mu)t}\} \times \sum_{\rho-1}^{b-1} (2/b) \sin (i\rho\pi/b) \sin [(b - j)\rho\pi/b][\cos (\rho\pi/b)]^n.$$

We note that $\pi^n_{ij}(t)$ is precisely $[t(\lambda + \mu)]^n e^{-(\lambda+\mu)t}/n!$ times the analogous absorption probability for symmetric discrete time random walks. (See, for example, Equation (8.12) of [1]). This relationship between the continuous and discrete random walk could be used to derive $\pi^n_{ij}(t)$ from the discrete result.

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