NOTES

A LOCAL LIMIT THEOREM[1]

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0. Introduction. In this note we shall obtain asymptotic estimates for $c_n = P\{a \leqq S_n \leqq b\}$, where a and b are fixed, $a < b$, and $S_n = X_1 + \cdots + X_n$ is a sum of independent random variables with a common distribution and finite variance. It is shown that for X_1, nonlattice, with mean zero and variance σ^2, $c_n \sim (b - a)(2\pi n\sigma^2)^{-\frac{1}{2}}$. This will appear as an application of the central limit theorem of Cramér and Esseen. On the other hand, in the special lattice case (integer-valued X_1), we have $c_n = \nu[a, b](2\pi n\sigma^2)^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$, where $\nu[a, b]$ is the number of integers in $[a, b]$.

The theorems appear unified in the following formulation. If F_n is the measure induced on the real line by S_n, then $(2\pi n\sigma^2)^{\frac{1}{2}}F_n$ converges in the sense of distributions to Lebesgue measure λ in the nonlattice case, and to the measure ν in the special lattice case. Each of these measures may be viewed as a Fourier transform of Dirac's distribution. The remaining general lattice case will be treated similarly.

These results are simple but do not appear to have been published previously. They are related to some known results as follows: Under certain additional assumptions on distributions in the domain of attraction of a nonnormal stable law ($\sigma = \infty$), Gnedenko ([2], p. 236; [3]) has obtained strong local theorems. Our results complement those of Gnedenko in the lattice case ([2], p. 233).

1. Statement of results. If with probability one, X_1 has only the values $\alpha + k\beta$, $k = 0, \pm 1, \cdots$, with α and β fixed, then X_1 is said to have a lattice distribution. We suppose $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$.

THEOREM 1. *If X_1 does not have a lattice distribution, then for all continuous functions, g, with compact support*

$$\int g(y)H_n\{dy\} \to \int g(y)\,dy, \tag{1.1}$$

where $H_n = (2\pi n\sigma^2)^{\frac{1}{2}}F_n$.

THEOREM 2. *If X_1 does not have a lattice distribution, then*

$$P\{a \leqq S_n \leqq b\} \sim (b - a)(2\pi n\sigma^2)^{-\frac{1}{2}}. \tag{1.2}$$

In the lattice case, assuming first that β is chosen maximally, we may set our

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units so that $\beta = 1$. We shall prove the following theorem, otherwise obtainable as a corollary of results of Gnedenko ([2], p. 233). Here $\{x\}$ denotes the fractional part of x and ν is the measure assigning unit mass to each integer.

THEOREM 3. *If X_1 has a lattice distribution, then*

$$(1.3) \qquad (2\pi n\sigma^2)^{\frac{1}{2}} P\{a \leqq S_n \leqq b\} = \nu[a - \{n\alpha\}, b - \{n\alpha\}] + o(1).$$

If α is rational, we may take $\alpha = 0$, and then for all continuous functions, g, with compact support,

$$(1.4) \qquad \int g(y)H_n\{dy\} \to \int g(y)\nu\{dy\}.$$

If α is irrational, we have convergence in the Césaro sense,

$$(1.5) \qquad \int g(y)H_n\{dy\} \to \int g(y)\,dy, \qquad (C, 1).$$

Here, (1.5) *also holds if g is the indicator of an interval.*

2. Nonlattice case with finite third moment. If X_1 does not have a lattice distribution and $E|X_1|^3 < \infty$, then an application of a deep theorem (stated below) of Cramér and Esseen ([2], p. 210) proves Theorem 1 for smooth functions, g. This theorem has been previously applied to similar situations by Kallianpur and Robbins [7] and by Bahadur and Rao [1].

THEOREM (Cramér-Esseen). *If X_1 does not have a lattice distribution and has finite third moment, then uniformly in x,*

$$(2.1) \qquad F_n(x\sigma n^{\frac{1}{2}}) = \Phi(x) + \alpha_3(1 - x^2)(72\pi n\sigma^2)^{-\frac{1}{2}} \exp\{-x^2/2\} + o(n^{-\frac{1}{2}}),$$

where $\alpha_3 = EX_1^3$.

If $b \in L$ is absolutely continuous and vanishes at infinity, we obtain by integration by parts, applying (2.1),

$$(2.2) \qquad \int b(y)H_n\{dy\} = -(2\pi n\sigma^2)^{\frac{1}{2}} \int F_n(y)b'(y)\,dy = \int b(y)\,dy + o(1).$$

The absolute continuity assumption is easily removed, but to prove (1.1) for general X_1, we must adopt a different attack.

We shall assume first that $g \in C^2$ (i.e., has two continuous derivatives). The transform, $\gamma(z) = \int g(y)e^{-iyz}\,dy = O(1 + z^2)^{-1}$ and so $\gamma \in L$. The characteristic function, $\varphi(z) = E(e^{iX_1 z})$, satisfies Parseval's identity,

$$(2.3) \qquad \int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{\frac{1}{2}} \int \gamma(z)\phi^n(z)\,dz.$$

For any $K > 0$, by *normal convergence* ([2], p. 181),

$$(2.4) \qquad (n\sigma^2/2\pi)^{\frac{1}{2}} \int_{|z| \leqq K} \gamma(z)\varphi^n(z)\,dz \to \gamma(0) = \int g(y)\,dy,$$

using the fact that sup $\{|\varphi(z)|^n : 0 < \delta \leqq |z| \leqq K\}$ is exponentially small ([2], p. 59), as $n \to \infty$.

The proof of Theorem 1 is completed by showing that

$$n^{\frac{1}{2}} \int_{|z|>K} \gamma(z)\varphi^n(z)\, dz = o(1). \tag{2.5}$$

We remark that (2.5) holds if X_1 satisfies condition C of Cramér, lim sup $|\varphi(z)| < 1$, as $z \to \infty$.

3. Nonlattice case with infinite third moment. We now assume that X_1 is nonlattice and $E|X_1|^3 = \infty$. We shall prove (1.1) by a truncation method. Assume first that g has four continuous derivatives. By integrations by parts, $|\gamma(z)| < c\beta(z)$, where $\beta(z) = (1 + z^4)^{-1}$, and $c > 0$. We put $\psi(z) = |\varphi(z)|^2$.

LEMMA. *If ψ is a nonnegative, nonlattice characteristic function, then for every $K > 0$, we have*

$$n^{\frac{1}{2}} \int_{|z|\geqq K} \beta(z)\psi^n(z)\, dz = o(1). \tag{3.1}$$

If X is a random variable with characteristic function ψ, let

$$\begin{aligned} Y &= X \quad \text{if} \quad |X| < T, \\ &= 0 \quad \text{else} \end{aligned} \tag{3.2}$$

where T is chosen so large that Y is nonlattice. Let ψ_1 denote the characteristic function of Y. For all z,

$$\psi(z) \leqq \psi_1(z). \tag{3.3}$$

If G_n denotes the distribution function corresponding to ψ_1^n, then by (2.2) with $b(x) = (2\pi)^{-1}\int e^{ixz}\beta(z)\, dz$, we obtain

$$2\pi n^{\frac{1}{2}} \int b(x) G_n\{dx\} \to (\sigma_1^2/2\pi)^{-\frac{1}{2}} \int b(x)\, dx, \tag{3.4}$$

where $\sigma_1 = EY^2$, since $E|Y|^3 < \infty$. By (2.4), we obtain

$$n^{\frac{1}{2}} \int_{|z|\leqq K} \beta(z)\psi_1^n(z)\, dz \to (\sigma_1^2/2\pi)^{-\frac{1}{2}} \int b(x)\, dx. \tag{3.5}$$

Subtracting (3.5) from (3.4) and using (2.3), we obtain

$$n^{\frac{1}{2}} \int_{|z|\geqq K} \beta(z)\psi_1^n(z)\, dz \to 0. \tag{3.6}$$

The lemma follows from (3.6) and (3.3). Since γ is bounded by β, (2.5) follows from the lemma, and so (1.1) holds whenever $EX_1^2 < \infty$, and $g \in C^4$. To remove the latter restriction, we employ the usual technique, proving somewhat more than necessary for Theorem 1. Let f_m, h_m, $m = 1, 2, \cdots$, be functions of C^4

with compact support, for which,

$$f_m \leqq g \leqq h_m \quad \text{and} \quad \int (h_m - f_m) \to 0. \tag{3.7}$$

If g satisfies the hypothesis of Theorem 1, or if g is the indicator of an interval, then such sequences f_m, h_m exist. Hence, for $m = 1, 2, \cdots$, $n = 1, 2, \cdots$, we have

$$\int f_m(y)H_n\{dy\} \leqq \int g(y)H_n\{dy\} \leqq \int h_m(y)H_n\{dy\}. \tag{3.8}$$

Letting $n \to \infty$, then $m \to \infty$, we learn that $\int g(y)H_n\{dy\} \to \int g(y)\,dy$. This proves Theorems 1 and 2.

4. Lattice case. We consider first the simple lattice case, when β may be chosen so that $\alpha = 0$. We assume that β is maximal, and then adjust units so that $\beta = 1$. With $\gamma(z) = \sum g(n)e^{-inz}$, where g has compact support, we obtain

$$\int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{\frac{1}{2}} \int_{-\pi}^{\pi} \gamma(z)\varphi^n(z)\,dz. \tag{4.1}$$

Arguing as in (2.4) gives immediately, since we are now dealing with a compact interval,

$$\int g(y)H_n\{dy\} \to \gamma(0) = \sum g(n) = \int g\,dv. \tag{4.2}$$

This proves (1.4). In the general lattice case, α is irrational, and (1.3) receives a direct proof based on the same argument.

If $g \in C^2$ has compact support, we have, with $\gamma(z) = \int e^{-ixz}g(x)\,dx$,

$$\int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{\frac{1}{2}} \int \gamma(z)e^{i\alpha zn}(e^{-i\alpha z}\varphi(z))^n\,dz. \tag{4.3}$$

Since $e^{-i\alpha z}\varphi(z)$ is periodic with period $2\pi(\beta = 1)$, we may write

$$\int g(y)H_n\{dy\} = (n\sigma^2/2\pi)^{\frac{1}{2}} \int_{-\pi}^{\pi} \gamma_n^*(z)\varphi^n(z)\,dz \tag{4.4}$$

where $\gamma_n^*(z) = \sum \gamma(z + 2\pi k)e^{i\alpha 2\pi kn}$. Applying the normal convergence and the equicontinuity of the sequence γ_n^*, $n = 1, 2, \cdots$,

$$\int g(y)H_n\{dy\} = \gamma_n^*(0) + o(1). \tag{4.5}$$

Using the Poisson summation formula ([8], Section 2.85), we have

$$\gamma_n^*(0) = \sum \gamma(2\pi k)e^{i\alpha 2\pi kn} = \sum g(k + n\alpha). \tag{4.6}$$

By Weyl's Lemma ([6], pp. 115–117), for any irrational α,

$$\frac{1}{n}\sum_{k=1}^{n} \gamma_k^*(0) \to \int g(y)\,dy. \tag{4.7}$$

Thus, for $g \varepsilon C^2$, with compact support,

$$\int g(y)H_n\{dy\} \to \int g(y)\,dy, \qquad (C, 1). \tag{4.8}$$

By imitating the steps in (3.7) and (3.8), we extend the class of functions, g, for which (4.8) holds. This proves (1.5) and Theorem 3.

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