CUBIC DESIGNS

By D. Raghavarao[1] and K. Chandrasekhararao

University of Bombay and Elphinstone College, Bombay

1. Summary and introduction. Investigations in the Partially Balanced Incomplete Block (PBIB) designs having three or more associate classes have been limited to the works of Vartak [14], Raghavarao [7], Roy [9], Singh and Shukla [12] and Tharthare [13]. In this article we study the combinatorial properties, construction and non-existence of the cubic designs exhibiting a three associate class association scheme. The method of construction, discussed in this article, gives a new way of arranging a p^3 factorial experiment in blocks of sizes different from p and p^2.

2. Definition and preliminaries. Adverting to Bose and Mesner [2] we can define a three associate class association scheme for v treatments. The arrangement of these v treatments in b blocks of size k is said to be a PBIB design (cf. Bose and Nair [1]) if

(i) every treatment occurs at most once in a block,

(ii) every treatment occurs in exactly r blocks,

(iii) every pair of treatments which are ith associates occur together in λ_i $(i = 1, 2, 3)$ blocks.

Let there be $v = s^3$ treatments denoted by (α, β, γ) $(\alpha, \beta, \gamma = 1, 2, \cdots, s)$. We define the distance δ between two treatments (α, β, γ) and $(\alpha', \beta', \gamma')$ to be the number of non-null elements in $(\alpha - \alpha', \beta - \beta', \gamma - \gamma')$. Let us call two treatments to be 1st, 2nd or 3rd associates according as $\delta = 1, 2$ or 3 respectively.

Geometrically interpreting, the two treatments lying on the same axis are 1st associates, those lying on the same plane are 2nd associates and the rest are 3rd associates when the s^3 treatments are arranged in a cube of side s. Because of this geometric configuration we call the above association scheme, a cubic association scheme. PBIB designs whose treatments exhibit a cubic association scheme may be defined as cubic designs.

For the cubic association scheme, we easily get

$$n_1 = 3(s-1), \qquad n_2 = 3(s-1)^2, \qquad n_3 = (s-1)^3 \tag{2.1}$$

and

$$P_1 = (p_{jk}^1) = \begin{bmatrix} s-2 & 2(s-1) & 0 \\ & 2(s-1)(s-2) & (s-1)^2 \\ & & (s-1)^2(s-2) \end{bmatrix} \tag{2.2}$$

Received 27 May 1963.

[1] The work of this author was financially supported by a Government of India Research Fellowship.

$$P_2 = (p^2_{jk}) = \begin{bmatrix} 2 & 2(s-2) & (s-1) \\ & 2(s-1)+(s-2)^2 & 2(s-1)(s-2) \\ & & (s-1)(s-2)^2 \end{bmatrix}$$

(2.2)

$$P_3 = (p^3_{jk}) = \begin{bmatrix} 0 & 3 & 3(s-2) \\ & 6(s-2) & 3(s-2)^2 \\ & & (s-2)^3 \end{bmatrix}.$$

Let $N = (n_{ij})$ be the $v \times b$ incidence matrix of the cubic design where $n_{ij} = 1$ or 0 according as the ith treatment occurs in the jth block or not. The treatments can be so numbered as to render NN' take the form

$$NN' = I_s \times (P - Q) + E_{ss} \times Q \tag{2.3}$$

where I_s is an identity matrix of order s, E_{mn} is a $m \times n$ matrix with positive units elements everywhere, " $\times$ " is the symbol for the Kronecker product of matrices, and

$$P = I_s \times (A - B) + E_{ss} \times B, \qquad Q = I_s \times (B - C) + E_{ss} \times C,$$

$$A = (r - \lambda_1)I_s + \lambda_1 E_{ss}, \qquad B = (\lambda_1 - \lambda_2)I_s + \lambda_2 E_{ss} \tag{2.4}$$

$$C = (\lambda_2 - \lambda_3)I_s + \lambda_3 E_{ss}.$$

The determinant of NN' can be obtained as $\rho_0 \rho_1^{\alpha_1} \rho_2^{\alpha_2} \rho_3^{\alpha_3}$ where

$$\rho_0 = r + 3(s-1)\lambda_1 + 3(s-1)^2\lambda_2 + (s-1)^3\lambda_3 = rk,$$

$$\rho_1 = r + (2s-3)\lambda_1 + (s-1)(s-3)\lambda_2 - (s-1)^2\lambda_3,$$

$$\rho_2 = r + (s-3)\lambda_1 - (2s-3)\lambda_2 + (s-1)\lambda_3,$$

$$\rho_3 = r - 3\lambda_1 + 3\lambda_2 - \lambda_3,$$

$$\alpha_1 = 3(s-1), \qquad \alpha_2 = 3(s-1)^2, \qquad \alpha_3 = (s-1)^3.$$

It can be observed that ρ_0, ρ_1, ρ_2 and ρ_3 are the characteristic roots of NN' with respective multiplicities $\alpha_0 = 1$, $\alpha_1 = 3(s-1)$, $\alpha_2 = 3(s-1)^2$ and $\alpha_3 = (s-1)^3$. Since NN' is positive and at least semi-definite, ρ_i's must not be negative ($i = 1, 2, 3$). Hence we have the

THEOREM 2.1. *A necessary condition for the existence of a cubic design is that* $\rho_i \geqq 0$ ($i = 1, 2, 3$).

Cubic designs with the following parameters violate the above necessary condition and hence are non-existing. The characteristic root with the negative value is shown in brackets against the parameters

(i) $s = 3, b = 12, r = 4, k = 9, \lambda_1 = 4, \lambda_2 = 0, \lambda_3 = 1$. ($\rho_3$)

(ii) $s = 4, b = 64, r = 18 = k, \lambda_1 = 1, \lambda_2 = 6, \lambda_3 = 5$. ($\rho_1$).

3. Analysis. The analysis of the PBIB designs can, in certain cases, be obtained, very elegantly, with the help of the characteristic roots and vectors of NN' (cf. Raghavarao [8], Tharthare [13]). In this case we shall obtain the analysis of the cubic designs by a similar method.

With the usual intrablock model, the normal equations giving the column vector of the intrablock estimates of the treatment effects $\hat{\mathbf{t}}$ are

$$\mathbf{Q} = C\hat{\mathbf{t}} \tag{3.1}$$

where

$$\mathbf{Q} = \mathbf{T} - (1/k)N\mathbf{B} \quad \text{and} \quad C = rI_v - (1/k)NN', \tag{3.2}$$

T and **B** being the column vectors of the treatment totals and block totals respectively.

The characteristic roots of C can be seen to be $0, \phi_1 = r - \rho_1/k, \phi_2 = r - \rho_2/k, \phi_3 = r - \rho_3/k$ with respective multiplicities $\alpha_0 = 1, \alpha_1, \alpha_2$ and α_3. The spectral decomposition (cf. Perlis [6]) of C is

$$C = \phi_1 A_1 + \phi_2 A_2 + \phi_3 A_3, \tag{3.3}$$

where

$$\begin{aligned} s^3 A_1 &= [sE_{s^2s^2} \times I_s + sE_{ss} \times I_s \times E_{ss} + sI_s \times E_{s^2s^2} - 3E_{s^3s^3}] \\ s^3 A_2 &= [s^2\{I_{s^2} \times E_{ss} + I_s \times E_{ss} \times I_s + E_{ss} \times I_{s^2}\} \\ &\quad - 2s\{I_s \times E_{s^2s^2} + E_{ss} \times I_s \times E_{ss} + E_{s^2s^2} \times I_s\} + 3E_{s^3s^3}] \\ s^3 A_3 &= [s^3 I_{s^3} - s^2\{I_{s^2} \times E_{ss} + I_s \times E_{ss} \times I_s + E_{ss} \times I_{s^2}\} \\ &\quad + s\{I_s \times E_{s^2s^2} + E_{ss} \times I_s \times E_{ss} + E_{s^2s^2} \times I_s\} - E_{s^3s^3}]. \end{aligned} \tag{3.4}$$

Using Shah's result [10] that $\hat{\mathbf{t}} = (C + aE_{vv})^{-1}\mathbf{Q}$, where a is any real number, is a solution of (3.1) and after simplification we get

$$\begin{aligned} t_i &= (1/\phi_3)Q_i + (1/s^2)(1/\phi_1 - 2/\phi_2 - 1/\phi_3)(3Q_i + 2\textstyle\sum Q_{i1} + \sum Q_{i2}) \\ &\quad + (1/s)(1/\phi_2 - 1/\phi_3)(3Q_i + \textstyle\sum Q_{i1}), \qquad (i = 1, 2, \cdots, v) \end{aligned} \tag{3.5}$$

where $\sum Q_{ij}$ is the sum of the Q's of the treatments which are jth associates of the ith treatment. The intrablock analysis can now be completed in the usual manner as given in Kempthorne [4]. From (3.5) we can readily find that

$$\begin{aligned} \text{Var}(\hat{t}_i - \hat{t}_j) &= (2\sigma^2/s^2)[1/\phi_1 + 2(s-1)/\phi_2 + (s-1)^2/\phi_3], \quad \text{or} \\ &= (2\sigma^2/s^2)[2/\phi_1 + (3s-4)/\phi_2 + (s-1)(s-2)/\phi_3], \quad \text{or} \\ &= (2\sigma^2/s^2)[3/\phi_1 + 3(s-2)/\phi_2 + (s^2 - 3s + 3)/\phi_3] \end{aligned} \tag{3.6}$$

according as the ith and jth treatments are first, second or third associates respectively, where σ^2 is the intrablock error variance. The average variance of the design is

$$[2\sigma^2/(s^2 + s + 1)][3/\phi_1 + 3(s-1)/\phi_2 + (s-1)^2/\phi_3] \tag{3.7}$$

and its efficiency is

$$(3.8) \qquad (s^2 + s + 1)[3/\phi_1 + 3(s - 1)/\phi_2 + (s - 1)^2/\phi_3]^{-1}r^{-1}.$$

4. Combinatorial properties of certain cubic designs. We prove

THEOREM 4.1. *In a cubic design with $\rho_1 = 0$, k is divisible by s and further every block of the design contains k/s treatments of the form $(\alpha, \beta, \gamma)(\beta, \gamma = 1, 2, \cdots, s)$ for every $\alpha = 1, 2. \cdots, s$.*

PROOF. Let $e_{\alpha i}$ be the number of treatments of the form $(\alpha, \beta, \gamma)(\beta, \gamma = 1, 2, \cdots, s)$ occurring in the ith block $(i = 1, 2, \cdots, s)$ of the design. Then

$$(4.1) \qquad \sum_i e_{\alpha i} = s^2 r$$
$$\sum_i e_{\alpha i}(e_{\alpha i} - 1) = 2s^2(s - 1)\lambda_1 + s^2(s - 1)^2\lambda_2 .$$

Define $e_{\alpha \cdot} = \sum_i e_{\alpha i}/b = s^2 r/b = k/s$. Then

$$(4.2) \qquad \sum_i (e_{\alpha i} - e_{\alpha \cdot})^2 = s^2[r + 2(s - 1)\lambda_1 + (s - 1)^2\lambda_2] - bk^2/s^2$$
$$= 0, \quad \text{since} \quad \rho_1 = 0.$$

Hence $e_{\alpha 1} = e_{\alpha 2} = \cdots = e_{\alpha b} = e_{\alpha \cdot}$. Since $e_{\alpha i}$ must be an integer k is divisible by s. Therefore the result of the theorem is established.

In a similar way to the above theorem we can prove

THEOREM 4.2. *In a cubic design with $\rho_1 = 0$ and $\rho_2 = 0$, k is divisible by s^2 and further every block of the design contains k/s^2 treatments of the form (α, β, γ) $(\gamma = 1, 2, \cdots, s)$ for every α and β $(\alpha, \beta = 1. 2, \cdots, s)$.*

From the above theorem we deduce

COROLLARY 4.1.

(i) *A necessary condition for the existence of a cubic design with $\rho_1 = 0$ is that k is divisible by s.*

(ii) *A necessary condition for the existence of a cubic design with $\rho_1 = 0$, $\rho_2 = 0$ is that k is divisible by s^2.*

5. Construction of cubic designs. The three dimensional lattice designs (cf. Kempthorne [4]) in blocks of size s can be seen to be cubic designs with parameters $v = s^3$, $b = 3s^2$, $r = 3$, $k = s$, $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 0$, if the basic pattern is taken only once.

In this section we give a method of constructing cubic designs from the balanced incomplete block (BIB) designs. This method provides us, many times, a tool of constructing cubic designs in blocks of plot sizes different from s and s^2.

THEOREM 5.1. *If M is the incidence matrix of a* BIB *design with parameters $v^* = s, b^*, r^*, k^*, \lambda^*$ then*

$$(5.1) \qquad N = M \times M \times M$$

is the incidence matrix of a cubic design having its parameters

$$v = s^3, \quad b = b^{*3}, \quad r = r^{*3}, \quad k = k^{*3} \tag{5.2}$$
$$\lambda_1 = r^{*2}\lambda^*, \quad \lambda_2 = r^*\lambda^{*2}, \quad \lambda_3 = \lambda^{*3}.$$

Proof. The parameters v, b, r and k require no explanation. Observing that

$$MM' = (r^* - \lambda^*)I_s + \lambda^* E_{ss} \tag{5.3}$$

we have,

$$\begin{aligned} NN' = (r^* - \lambda^*)^3 I_v &+ \lambda^*(r^* - \lambda^*)^2[I_{s^2} \times E_{ss} + I_s \times E_{ss} \times I_s \\ &+ E_{ss} \times I_{s^2}] + \lambda^{*2}(r^* - \lambda^*)[I_s \times E_{s^2s^2} \\ &+ E_{ss} \times I_s \times E_{ss} + E_{s^2s^2} \times I_s] + \lambda^{*3}E_{vv}. \end{aligned} \tag{5.4}$$

It is now easy to verify that λ_1, λ_2, and λ_3 are as given in (5.2).

Illustration 5.1. Starting with the BIB design having parameters

$$v^* = 3 = b^*, \quad r^* = 2 = k, \quad \lambda^* = 1, \tag{5.5}$$

by the method of Theorem 5.1, we obtain the design

(111, 112, 121, 122, 211, 212, 221, 222)
(111, 113, 121, 123, 211, 213, 221, 223)
(112, 113, 122, 123, 212, 213, 222, 223)
(111, 112, 131, 132, 211, 212, 231, 232)
(111, 113, 131, 133, 211, 213, 231, 233)
(112, 113, 132, 133, 212, 213, 232, 233)
(121, 122, 131, 132, 221, 222, 231, 232)
(121, 123, 131, 133, 221, 223, 231, 233)
(122, 123, 132, 133, 222, 223, 232, 233)
(111, 112, 121, 122, 311, 312, 321, 322)
(111, 113, 121, 123, 311, 313, 321, 323)
(112, 113, 122, 123, 312, 313, 322, 323)
(111, 112, 131, 132, 311, 312, 331, 332)
(111, 113, 131, 133, 311, 313, 331, 333)
(112, 113, 132, 133, 312, 313, 332, 333)
(121, 122, 131, 132, 321, 322, 331, 332)
(121, 123, 131, 133, 321, 323, 331, 333)
(122, 123, 132, 133, 322, 323, 332, 333)
(211, 212, 221, 222, 311, 312, 321, 322)
(211, 213, 221, 223, 311, 313, 321, 323)
(212, 213, 222, 223, 312, 313, 322, 323)
(211, 212, 231, 232, 311, 312, 331, 332)
(211, 213, 231, 233, 311, 313, 331, 333)
(212, 213, 232, 233, 312, 313, 332, 333)
(221, 222, 231, 232, 321, 322, 331, 332)
(221, 223, 231, 233, 321, 323, 331, 333)
(222, 223, 232, 233, 322, 323, 332, 333)

which can be easily verified to be a cubic design with parameters

(5.6) $v = 27 = b, \quad r = 8 = k, \quad \lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = 1.$

The efficiency of the above design can be seen, using (3.8), to be 0.899. The efficiency of the usual 3-dimensional lattice for testing a 3^3 treatments in blocks of 3 plots can be seen to be 0.591. Thus, the design proposed in the above illustration is more efficient than the usual 3-dimensional lattice design for testing 3^3 treatments.

6. Non-existence of certain symmetrical cubic designs. We shall call, cubic designs with $\rho_i \neq 0$ $(i = 1, 2, 3)$, to be regular cubic designs. From Shrikhande's [11] and Connor and Clatworthy's [3] results, it follows that

THEOREM 6.1. *A necessary condition for the existence of symmetrical regular cubic designs is that* $\rho_1^{\alpha_1}\rho_2^{\alpha_2}\rho_3^{\alpha_3}$ *should be a perfect square.*

It is obvious that the above theorem can be used only when s is even. In that case, we get

COROLLARY 6.1. *A necessary condition for the existence of symmetrical regular cubic designs when s is even is that* $\rho_1\rho_2\rho_3$ should be a perfect square.

The following designs violate the condition of the above corollary and hence are non-existing:

(i) $s = 4,\ b = 64,\ r = 9 = k,\ \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 1.$
(ii) $s = 4,\ b = 64,\ r = 18 = k,\ \lambda_1 = 7,\ \lambda_2 = 5,\ \lambda_3 = 4.$
(iii) $s = 4,\ b = 64,\ r = 18 = k,\ \lambda_1 = 4,\ \lambda_2 = 5,\ \lambda_3 = 5.$
(iv) $s = 4,\ b = 64,\ r = 27 = k,\ \lambda_1 = 12,\ \lambda_2 = 11,\ \lambda_3 = 11.$
(v) $s = 6,\ b = 216,\ r = 30 = k,\ \lambda_1 = 8,\ \lambda_2 = 5,\ \lambda_3 = 3.$
(vi) $s = 6,\ b = 216,\ r = 20 = k,\ \lambda_1 = 7,\ \lambda_2 = 2,\ \lambda_3 = 1.$

Further necessary conditions for the existence of symmetrical, regular cubic designs can be obtained with the help of the Hasse-Minkowski invariant. For a brief resumé of the properties of the Legendre symbol, Hilbert norm residue and the Hasse-Minkowski invariant we refer to Ogawa [5].

Following Ogawa, we can show that

$$C_p(NN') = C_p\{\text{diag}\,(\rho_0 v, \rho_1 Q_1, \rho_2 Q_2, \rho_3 Q_3)\} \tag{6.1}$$

where diag $(a_1, a_2, \cdots, a_n)$ stands for a diagonal matrix with its diagonal positions being filled by the elements or matrices $a_1, a_2, \cdots, a_n$, and Q_i is the gramian of the rational, independent vectors corresponding to the root ρ_i $(i = 1, 2, 3)$. We can show further that

$$|Q_1|\cdot|Q_2|\cdot|Q_3| \sim v, \tag{6.2}$$

and

$$(|Q_1|, |Q_2|)_p(|Q_1|, |Q_3|)_p(|Q_2|, |Q_3|)_p\cdot C_p(Q_1)C_p(Q_2)C_p(Q_3) = (-1, -1)_p \tag{6.3}$$

for all primes p, where $a \sim b$ means that the square free parts of a and b are the same. Using (6.2), (6.3) and the properties of the Hilbert norm residue symbol and the Hasse-Minkowski invariant, (6.1) becomes

$$C_p(NN') = (-1, -1)_p \left\{ \prod_{i=1}^{3} (-1, \rho_i)_p^{\alpha_i(\alpha_i+1)/2} \right\} (\rho_1, \rho_2)_p^{\alpha_1\alpha_2} \cdot (\rho_1, \rho_3)_p^{\alpha_1\alpha_3} (\rho_2, \rho_3)_p^{\alpha_2\alpha_3} (|Q_1|, \rho_1\rho_3)_p (|Q_2|, \rho_2\rho_3)_p (\rho_3, v)_p. \tag{6.4}$$

We now find $|Q_1|$ and $|Q_2|$. Let us number the treatments (α, β, γ) by $(\alpha - 1)s^2 + (\beta - 1)s + \gamma$. Define the vth order vectors $\xi_{\alpha 1}, \xi_{\beta 2}, \xi_{\gamma 3}$ ($\alpha, \beta, \gamma = 1, 2, \cdots, s$) as follows: The vectors $\xi_{\alpha 1}$ have s^2 unit entries in the positions $(\alpha - 1)s^2 + 1, (\alpha - 1)s^2 + 2, \cdots, \alpha s^2$ and zeros elsewhere ($\alpha = 1, 2, \cdots, s$). The vectors $\xi_{\beta 2}$ have s^2 unit entries in the positions $(\beta - 1)s + 1, (\beta - 1)s + 2, \cdots, \beta s$; $s^2 + (\beta - 1)s + 1, s^2 + (\beta - 1)s + 2, \cdots, s^2 + \beta s; \cdots; s^2(s - 1) + (\beta - 1)s + 1, s^2(s - 1) + (\beta - 1)s + 2, \cdots, s^2(s - 1) + \beta s$ and zeros elsewhere ($\beta = 1, 2, \cdots, s$). The vectors $\xi_{\gamma 3}$ have s^2 unit entries in the positions $\gamma, s + \gamma, 2s + \gamma, \cdots, (s - 1)s + \gamma; s^2 + \gamma, s^2 + s + \gamma, \cdots, s^2 + (s - 1)s + \gamma; \cdots; s^2(s - 1) + \gamma, s^2(s - 1) + s + \gamma, \cdots, s^2(s - 1) + s(s - 1) + \gamma$ and zeroes elsewhere ($\gamma = 1, 2, \cdots, s$).

We can easily see that among $\xi_{\alpha 1}, \xi_{\beta 2}, \xi_{\gamma 3}$ only $3(s - 1) + 1$ are linearly independent vectors and E_{v1} lies in the vector space generated by $\xi_{\alpha 1}, \xi_{\beta 2}, \xi_{\gamma 3}$. The vector space generated by $\xi_{\alpha 1}, \xi_{\beta 2}, \xi_{\gamma 3}$ and orthogonal to E_{v1} can be seen to be the proper space corresponding to the root ρ_1 of NN'. Hence

$$\begin{bmatrix} v & \\ & Q_1 \end{bmatrix} \sim \begin{bmatrix} s^3 & s^2E_{1p} & s^2E_{1p} & s^2E_{1p} \\ s^2E_{p1} & s^2I_p & sE_{pp} & sE_{pp} \\ s^2E_{p1} & sE_{pp} & s^2I_p & sE_{pp} \\ s^2E_{p1} & sE_{pp} & sE_{pp} & s^2I_p \end{bmatrix} \tag{6.5}$$

where $p = s - 1$.

Evaluating the determinant on the right hand side of (6.5) we get

$$|Q_1| \sim s. \tag{6.6}$$

Let us now define $3s^2$ vectors $\mathfrak{n}_{\alpha\beta 1}, \mathfrak{n}_{\alpha\gamma 2}, \mathfrak{n}_{\beta\gamma 3}$ of order v ($\alpha, \beta, \gamma = 1, 2, \cdots, s$) as follows: $\mathfrak{n}_{\alpha\beta 1}$ has s unit entries in the positions corresponding to $(\alpha - 1)s^2 + (\beta - 1)s + 1, (\alpha - 1)s^2 + (\beta - 1)s + 2, \cdots, (\alpha - 1)s^2 + \beta s$ and zeros elsewhere ($\alpha, \beta = 1, 2, \cdots, s$). $\mathfrak{n}_{\alpha\gamma 2}$ has s unit entries in the positions corresponding to $(\alpha - 1)s^2 + \gamma, (\alpha - 1)s^2 + s + \gamma, \cdots, (\alpha - 1)s^2 + (s - 1)s + \gamma$ and zeros elsewhere ($\alpha, \gamma = 1, 2, \cdots, s$). $\mathfrak{n}_{\beta\gamma 3}$ has s unit entries in the positions corresponding to $(\beta - 1)s + \gamma, s^2 + (\beta - 1)s + \gamma, \cdots, (s - 1)s^2 + (\beta - 1)s + \gamma$ and zeros elsewhere ($\beta, \gamma = 1, 2, \cdots, s$).

Among the $\mathfrak{n}$ vectors only $3(s - 1)^2 + 3(s - 1) + 1$ are linearly independent. We can easily show that the vector space generated by $\mathfrak{n}$'s and orthogonal to the vector space generated by ξ's is the proper space of NN' corresponding to the root ρ_2. Hence

$$(6.7)\qquad \begin{bmatrix} v & & \\ & Q_1 & \\ & & Q_2 \end{bmatrix} \sim$$

$$\begin{bmatrix}
s^3 & s^2E_{1p} & s^2E_{1p} & s^2E_{1p} & sE_{1p^2} & sE_{1p^2} & sE_{1p^2} \\
s^2E_{p1} & s^2I_p & sE_{pp} & sE_{pp} & sI_p \times E_{1p} & sI_p \times E_{1p} & E_{pp^2} \\
s^2E_{p1} & sE_{pp} & s^2I_p & sE_{pp} & sE_{1p} \times I_p & E_{pp^2} & sI_p \times E_{1p} \\
s^2E_{p1} & sE_{pp} & sE_{pp} & s^2I_p & E_{pp^2} & sE_{1p} \times I_p & sE_{1p} \times I_p \\
sE_{p^21} & sI_p \times E_{p1} & sE_{p1} \times I_p & E_{p^2p} & sI_{p^2} & I_p \times E_{pp} & E_{p1} \times I_p \times E_{1p} \\
sE_{p^21} & sI_p \times E_{p1} & E_{p^2p} & sE_{p1} \times I_p & I_p \times E_{pp} & sI_{p^2} & E_{pp} \times I_p \\
sE_{p^21} & E_{p^2p} & sI_p \times E_{p1} & sE_{p1} \times I_p & E_{1p} \times I_p \times E_{p1} & E_{pp} \times I_p & sI_{p^2}
\end{bmatrix}.$$

Evaluating the determinant, and using (6.6) we get

$$(6.8)\qquad |Q_2| \sim s^{(s-1)}.$$

Substituting the square free parts of $|Q_1|$ and $|Q_2|$ in (6.4) and simplifying we get

$$(6.9)\qquad \begin{aligned} C_p(NN') = (-1, -1)_p \left\{\prod_{i=1}^{3} (-1, \rho_i)_p^{\alpha_i(\alpha_i+1)/2}\right\} (\rho_1, \rho_2)_p^{\alpha_1\alpha_2} \\ \cdot (\rho_1, \rho_3)_p^{\alpha_1\alpha_3} (\rho_2, \rho_3)_p^{\alpha_2\alpha_3} (s, \rho_1)_p (s, \rho_2\rho_3)_p^{(s-1)}. \end{aligned}$$

Since $NN' \sim I_v$, we should have $C_p(NN') = (-1, -1)_p$ for all primes. Thus

THEOREM 6.2. A necessary condition *for the existence of a regular symmetrical cubic design is that*

$$\left\{\prod_{i=1}^{3} (-1, \rho_i)_p^{\alpha_i(\alpha_1+1)/2}\right\} (\rho_1, \rho_2)_p^{\alpha_1\alpha_2} (\rho_1, \rho_3)_p^{\alpha_1\alpha_3} \cdot (\rho_2, \rho_3)_p^{\alpha_2\alpha_3} (s, \rho_1)_p (s, \rho_2\rho_3)_p^{(s-1)} = +1,$$

for all primes p.

The following corollary can be deduced easily.

COROLLARY 6.2.

(i) *Necessary conditions for the existence of regular symmetrical cubic designs when s is odd are that* $(s, \rho_1)_p = +1$, *when* $s \equiv 1 \pmod 4$; *and* $(-1, \rho_1)_p(s, \rho_1)_p = +1$ *when* $s \equiv 3 \pmod 4$.

(ii) *Necessary conditions for the existence of regular, symmetrical cubic designs when s is even are that* $\rho_1\rho_2\rho_3$ *must be a perfect square and further* $(\rho_2, -\rho_3)_p = +1$ *when* $s \equiv 0 \pmod 4$; *and* $(\rho_2, -\rho_3)_p = +1$ *when* $s \equiv 2 \pmod 4$.

The following designs are non-existing in view of the above corollary

(i) $s = 3$, $v = 27 = b$, $r = 8 = k$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$

(ii) $s = 5$, $v = 125 = b$, $r = 16 = k$, $\lambda_1 = 8$, $\lambda_2 = 3$, $\lambda_3 = 0$.

REFERENCES

[1] Bose, R. C. and Nair, K. R. (1938–40). Partially balanced incomplete block designs. *Sankhyā* **4** 337–372.

[2] Bose, R. C. and Mesner, Dale M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs. *Ann. Math. Statist.* **30** 21–38.

[3] Connor, W. S. and Clatworthy, W. H. (1954). Some theorems for partially balanced designs. *Ann. Math. Statist.* **25** 100–112.

[4] Kempthorne, Oscar (1952). *The Design and Analysis of Experiments.* Wiley, New York.

[5] Ogawa, Junjiro (1960). A necessary condition for existence of regular and symmetrical experimental designs of triangular type with partially balanced incomplete blocks. *Ann. Math. Statist.* **30** 1063–1071.

[6] Perlis, S. (1952). *Theory of Matrices.* Addison-Wesley.

[7] Raghavarao, Damaraju (1960). A generalization of group divisible designs. *Ann. Math. Statist.* **31** 756–771.

[8] Raghavarao, Damaraju (1963). On the use of latent vectors in the analysis of group divisible and L_2 designs. *J. Indian Soc. Agric. Statist.* **14** 138–144.

[9] Roy, Purnendu Mohon (1953–54). Hierarchical group divisible incomplete block designs with m-associate classes. *Sci. Culture.* **19** 210–211.

[10] Shah, B. V. (1959). A generalisation of partially balanced incomplete block designs. *Ann. Math. Statist.* **30** 1041–1050.

[11] Shrikhande, S. S. (1950). The impossibility of certain symmetrical balanced incomplete block designs. *Ann. Math. Statist.* **21** 106–111.

[12] Singh, N. K. and Shukla, G. C. (1963). Non-existence of some PBIBD. *J. Indian Statist. Assoc.* **1** 71–78.

[13] Tharthare, Suresh K. (1963). Right angular designs. *Ann. Math. Statist.* **34** 1057–1067.

[14] Vartak, Manohar Narhar (1959). The non-existence of certain PBIB designs. *Ann. Math. Statist.* **30** 1051–1062.