LINEAR FORMS IN THE ORDER STATISTICS FROM AN EXPONENTIAL DISTRIBUTION

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1. Introduction and summary. In testing hypotheses about an exponential distribution with probability density function

$$p(x;\theta,A) = (1/\theta)e^{-(x-A)/\theta}, \qquad \text{for } x \geqq A,$$
$$= 0, \qquad \text{for } x < A,$$

where $\theta > 0$, the following questions arise:

(1) Are certain linear forms in the order statistics of a random sample of size n from this distribution distributed as chi-square random variables?

(2) Are certain linear forms in the order statistics stochastically independent? The first two theorems in Section 2 answer these questions. As a consequence of these two theorems, several results follow which are similar to those pertaining to quadratic forms in normally distributed variables.

The characterization theorem in Section 3 was suggested by a result for normally distributed variables. Lukacs [4] proved that if a random sample is taken from a continuous type distribution with finite variance, then the independence of the sample mean and the sample variance characterizes the normal distribution. That is, the independence of the estimates of the two parameters of the normal distribution characterizes that distribution. Now if $X_1 < X_2 < \cdots < X_n$ are the order statistics of a random sample from the exponential distribution $p(x;\theta,A)$, then X_1 and $(1/n)\sum_{i=1}^{n}(X_i - X_1)$ are estimates of the parameters A and θ, respectively. In Section 3, we prove that the independence of these two statistics characterizes this exponential distribution.

2. Chi-square and independence theorems. Let $X_1 < X_2 < \cdots < X_n$ denote the order statistics of a random sample of size n from $p(x;\theta,A)$. Let $U' = (u_1, u_2, \cdots, u_n)$ be a vector in n-dimensional Euclidean space, E^n, and let $X' = (X_1, X_2, \cdots, X_n)$ be the vector composed of the order statistics. Epstein and Sobel [2] proved that $(2/\theta)U'X$ has a chi-square distribution for particular vectors, U'. Theorem 1 gives a necessary and sufficient condition on U' so that $(2/\theta)U'X$ has a (translated) chi-square distribution. A *translated chi-square random variable* Y is equal to $Z + b$ where Z has a chi-square distribution and b is a real number. The number b is called the translation parameter. The characteristic function of Y is given by

$$E[\exp(itY)] = \exp(bit)/(1-2it)^r,$$

where $2r$ is equal to the number of degrees of freedom of Z.

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THEOREM 1. *Let $X' = (X_1, X_2, \cdots, X_n)$ denote the order statistics of a random sample of size n from $p(x; \theta, A)$. Let $U' = (u_1, u_2, \cdots, u_n)$ be a vector in E^n and let $\alpha_k = \sum_{i=k}^n u_i$, $k = 1, 2, \cdots, n$. Then $(2/\theta)U'X$ is distributed as a (translated) chi-square variable if and only if $\alpha_k = 0$ or $\alpha_k = n - k + 1$ for $k = 1, 2, \cdots, n$; the number of degrees of freedom is equal to twice the number of positive α_k's and the translation parameter is equal to $2\alpha_1 A/\theta$.*

PROOF. The characteristic function of $(2/\theta)U'X$ is given by

$$E[\exp(2itU'X/\theta)] = n!\exp[2\alpha_1 Ait/\theta]/\prod_{k=1}^{n} (n - k + 1 - 2it\alpha_k).$$

A proof of this can be based on a lemma proved by Rényi [5] or by evaluating the iterated integral

$$\int_A^\infty \cdots \int_{x_{n-1}}^\infty (n!/\theta^n) \exp\left[\left\{-\sum_{k=1}^n (1 - 2itu_k)x_k + nA\right\}\Big/\theta\right] dx_n \cdots dx_1 .$$

To prove the necessity, assume that $(2/\theta)U'X$ has a translated chi-square distribution with $2p$ depress of freedom and translation parameter b. Then $E[\exp(2itU'X/\theta)] = \exp[bit]/(1 - 2it)^p$. Thus

$$\exp[bit]/(1 - 2it)^p = \exp[2\alpha_1 Ait/\theta]/\prod_{k=1}^{n} [1 - 2it\alpha_k/(n - k + 1)].$$

So, $b = 2\alpha_1 A/\theta$, and by the unique factorization theorem, either $1 - 2it = 1 - 2it\alpha_k/(n - k + 1)$, which implies that $\alpha_k = n - k + 1$, or $1 = 1 - 2it\alpha_k/(n - k + 1)$, which implies that $\alpha_k = 0$.

To prove the sufficiency, we have, by hypothesis,

$$\begin{aligned} n - k + 1 - 2it\alpha_k &= n - k + 1, & \text{if } \alpha_k = 0, \\ &= (n - k + 1)(1 - 2it), & \text{if } \alpha_k = n - k + 1. \end{aligned}$$

Thus $E[\exp(2itU'X/\theta)] = \exp[2\alpha_1 Ait/\theta]/(1 - 2it)^p$, where p is equal to the number of positive α_k's. This completes the proof.

We note that if $\alpha_1 = 0$, the translation parameter is equal to zero, and hence $(2/\theta)U'X$ has a chi-square distribution. Also, sometimes in applications, $A = 0$, in which case the translation parameter is always equal to zero.

It follows easily from Theorem 1 that if $(2/\theta)U'X$ has a (translated) chi-square distribution, then (i) $u_k = 0$ or $u_k = -(n - k)$ if and only if $\alpha_k = 0$ and (ii) $u_k = 1$ or $u_k = n - k + 1$ if and only if $\alpha_k = n - k + 1$. Thus the number of degrees of freedom is equal to twice the number of positive u_k's.

Let U' and V' be vectors in E^n and let $X' = (X_1, X_2, \cdots, X_n)$ be the vector composed of the order statistics of a random sample of size n from $p(x; \theta, A)$. Epstein and Sobel [2] and Carlson [1] have proved that $U'X$ and $V'X$ are independent for particular vectors, U' and V'. The following theorem gives a necessary and sufficient condition for the independence of $U'X$ and $V'X$.

THEOREM 2. *Let $X' = (X_1, X_2, \cdots, X_n)$ denote the order statistics of a* *random sample of size n from* $p(x; \theta, A)$. *Let* $U' = (u_1, u_2, \cdots, u_n)$ *and* $V' = (v_1, v_2, \cdots, v_n)$ *be vectors in* E^n. *Let* $\alpha_k = \sum_{i=k}^n u_i$ *and let* $\beta_k = \sum_{i=k}^n v_i$ *for* $k = 1, 2, \cdots, n$. *Then* $(2/\theta)U'X$ *and* $(2/\theta)V'X$ *are independent if and only if* $\alpha_k\beta_k = 0, k = 1, 2, \cdots, n$.

PROOF. We prove first the necessity. The joint characteristic function of $(2/\theta)U'X$ and $(2/\theta)V'X$ is given by

$$\begin{aligned}\phi(t_1, t_2) &= E[\exp(2it_1U'X/\theta + 2it_2V'X/\theta)] \\ &= n!\exp[2Ai(t_1\alpha_1 + t_2\beta_1)/\theta]/\prod_{k=1}^n (n - k + 1 - 2it_1\alpha_k - 2it_2\beta_k).\end{aligned}$$

Since $(2/\theta)U'X$ and $(2/\theta)V'X$ are independent, $\phi(t_1, t_2) = \phi(t_1, 0)\phi(0, t_2)$. That is,

$$\frac{n!\exp[2Ai(t_1\alpha_1 + t_2\beta_1)/\theta]}{\prod_{k=1}^n (n - k + 1 - 2it_1\alpha_k - 2it_2\beta_k)} = \frac{n!\exp(2Ait_1\alpha_1/\theta)}{\prod_{k=1}^n (n - k + 1 - 2it_1\alpha_k)} \frac{n!\exp(2Ait_2\beta_1/\theta)}{\prod_{k=1}^n (n - k + 1 - 2it_2\beta_k)}.$$

Equivalently,

$$\prod_{k=1}^n [1 - 2i(t_1\alpha_k + t_2\beta_k)/(n - k + 1)] = \prod_{k=1}^n [1 - 2it_1\alpha_k/(n - k + 1)]\prod_{k=1}^n [1 - 2it_2\beta_k/(n - k + 1)]$$

for all real values of t_1 and t_2. Then if we take $t_1 = t_2 = t$, we have

$$\prod_{k=1}^n [1 - 2it(\alpha_k + \beta_k)/(n - k + 1)] = \prod_{k=1}^n [1 - 2it(\alpha_k + \beta_k)/(n - k + 1) - 4\alpha_k\beta_kt^2/(n - k + 1)^2].$$

Each factor on the left is equal to some factor on the right; this implies that $\alpha_k\beta_k = 0$ for $k = 1, 2, \cdots, n$.

In proving the sufficiency, since $\alpha_k\beta_k = 0$, $k = 1, 2, \cdots, n$, it follows that $\phi(t_1, t_2) = \phi(t_1, 0)\phi(0, t_2)$, and thus $(2/\theta)U'X$ and $(2/\theta)V'X$ are independent, completing the proof.

The following examples, which were considered by Epstein and Sobel [2], will illustrate the use of these theorems.

(i) In life testing, quite often only the first r items are observed. Then the statistic $(1/r)U'X$, where $U'X = \sum_{i=1}^r (X_i - X_1) + (n - r)(X_r - X_1)$, is an estimate of the spread parameter θ. For this statistic

$$\alpha_k = 0, \qquad \text{for } k = 1 \text{ and for } r < k \leqq n,$$
$$= n - k + 1, \qquad \text{for } 1 < k \leqq r.$$

Thus $(2/\theta)U'X$ has a chi-square distribution with $2r - 2$ degrees of freedom.

(ii) The statistic X_1 is an estimate of the location parameter A. It follows from Theorem 1 that $(2/\theta)nX_1$ has a translated chi-square distribution with 2 degrees of freedom and translation parameter $2nA/\theta$.

The independence of $U'X$ and X_1 follows immediately from Theorem 2.

Epstein and Sobel [2] also pointed out that the statistics $X_1, X_2 - X_1, \cdots, X_n - X_{n-1}$ are mutually independent. It is interesting to note that Theorem 2 implies that this is essentially the only set of n linear forms in the order statistics that are mutually independent.

It also follows from Theorem 2 that if m linear forms in the order statistics are pairwise independent, then they are mutually independent.

The following two theorems are a consequence of Theorems 1 and 2. They are analogous to theorems concerning quadratic forms in independent normally distributed variables.

THEOREM 3. *Let* $X' = (X_1, X_2, \cdots, X_n)$ *denote the order statistics of a random sample of size* n *from* $p(x; \theta, A)$. *Let* $U' = \sum_{j=1}^{m} V_j'$, *where* $V_j' = (v_{1j}, v_{2j}, \cdots, v_{nj})$ *is a vector in* E^n. *For each* $j = 1, 2, \cdots, m$, *let* $\beta_{kj} = \sum_{i=k}^{n} v_{ij}$, $k = 1, 2, \cdots, n$. *Let* $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, *be mutually independent. Then* $(2/\theta)U'X$ *has a (translated) chi-square distribution if and only if each* $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, *has a (translated) chi-square distribution.*

PROOF. The sufficiency of this condition is obvious. To prove the necessity, we note that $\alpha_k = \sum_{j=1}^{m} \beta_{kj}$, where $\alpha_k = \sum_{i=k}^{n} u_i$, $k = 1, 2, \cdots, n$. Since $(2/\theta)U'X$ has a (translated) chi-square distribution, either $\alpha_k = n - k + 1$ or $\alpha_k = 0$ by Theorem 1. Since the $(2/\theta)V_j'X$ are mutually independent, Theorem 2 implies that, for each k, there exists at most one j such that $\beta_{kj} \neq 0$. Thus if $\sum_{j=1}^{m} \beta_{kj} = n - k + 1$, there exists a $j = j(k)$ such that $\beta_{kj(k)} = n - k + 1$ and $\beta_{kj} = 0$ for $j \neq j(k)$. If $\sum_{j=1}^{m} \beta_{kj} = 0$, then $\beta_{kj} = 0$ for $j = 1, 2, \cdots, m$. So for each $j = 1, 2, \cdots, m$, either $\beta_{kj} = n - k + 1$ or $\beta_{kj} = 0$ for $k = 1, 2, \cdots, n$. Thus, by Theorem 1, each $(2/\theta)V_j'X$ has a (translated) chi-square distribution.

In the statement and proof of Theorem 4, we use the notation of Theorem 3.

THEOREM 4. *Let* $U' = \sum_{j=1}^{m} V_j'$. *Let each* $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, *have a (translated) chi-square distribution. Then* $(2/\theta)U'X$ *has a (translated) chi-square distribution if and only if* $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, *are mutually independent.*

PROOF. The sufficiency of this condition is obvious. To prove the necessity, we again note that $\alpha_k = \sum_{j=1}^{m} \beta_{kj}$ where $\alpha_k = \sum_{i=k}^{n} u_i$, $k = 1, 2, \cdots, n$. Since $(2/\theta)U'X$ has a (translated) chi-square distribution, Theorem 1 implies that $\alpha_k = n - k + 1$ or $\alpha_k = 0$, $k = 1, 2, \cdots, n$. Since $\beta_{kj} = n - k + 1$ or $\beta_{kj} = 0$ for $j = 1, 2, \cdots, m$, $\sum_{j=1}^{m} \beta_{kj} = n - k + 1$ implies that there exists one $j = j(k)$ such that $\beta_{kj(k)} = n - k + 1$ and $\beta_{kj} = 0$ for $j \neq j(k)$. $\sum_{j=1}^{m} \beta_{kj} = 0$ implies that $\beta_{kj} = 0$ for $j = 1, 2, \cdots, m$. The $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, are thus pairwise independent and hence are mutually independent.

The following theorem is somewhat analogous to Cochran's Theorem for quadratic forms in normally distributed variables. We again use the notation of Theorem 3.

THEOREM 5. *Let $U' = \sum_{j=1}^m V_j'$. Let r equal the number of positive α_k's and let r_j equal the number of positive β_{kj}'s, $j = 1, 2, \cdots, m$. Then $(2/\theta)U'X$ has a (translated) chi-square distribution and $r = \sum_{j=1}^m r_j$ if and only if each $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, has a (translated) chi-square distribution and $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, are mutually independent.*

PROOF. The sufficiency of the condition is clear. We now prove the necessity. Since $(2/\theta)U'X$ has a (translated) chi-square distribution, either $\alpha_k = n - k + 1$ or $\alpha_k = 0$. Since $r = \sum_{j=1}^m r_j$, for each k such that $\alpha_k = n - k + 1$, there is exactly one $j = j(k)$ such that $\beta_{kj(k)} \neq 0$. Thus $\sum_{j=1}^m \beta_{kj} = n - k + 1$ implies that $\beta_{kj(k)} = n - k + 1$ and $\beta_{kj} = 0, j \neq j(k)$. If $\sum_{j=1}^m \beta_{kj} = 0$, then $\beta_{kj} = 0$, $j = 1, 2, \cdots, m$. Thus each of $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m$, has a (translated) chi-square distribution and they are mutually independent by Theorem 4.

The following theorem is similar to a decomposition theorem of Hogg and Craig [3] which deals with quadratic forms in normally distributed variables. In the statement and proof of Theorem 6, we use the notation of Theorem 5. The following notation is also used. If Y has a translated chi-square distribution with r degrees of freedom and translation parameter b, we say that Y is $\chi^2(r)$ with translation parameter b.

THEOREM 6. *Let $U' = \sum_{j=1}^m V_j'$, where $(2/\theta)U'X$ is $\chi^2(2r)$, $0 < r \leqq n$, with translation parameter $2\alpha_1 A/\theta$. If $(2/\theta)V_j'X$ is $\chi^2(2r_j)$ with translation parameter $2\beta_{1j}A/\theta$, $j = 1, 2, \cdots, m - 1$, and if $\beta_{km} \geqq 0$, $k = 1, 2, \cdots, n$, then $(2/\theta)V_m'X$ is $\chi^2(2[r - \sum_{j=1}^{m-1} r_j])$ with translation parameter $2(\alpha_1 - \sum_{j=1}^{m-1} \beta_{1j})A/\theta$. Moreover, the m linear forms $V_j'X$, $m = 1, 2, \cdots, m$, are mutually independent.*

PROOF. Since $U' = \sum_{j=1}^m V_j'$, $\beta_{km} = \alpha_k - \sum_{j=1}^{m-1} \beta_{kj}$. Thus

$$\begin{aligned} \beta_{km} &= 0, \quad \text{if} \quad \alpha_k = 0 \quad \text{and} \quad \sum_{j=1}^{m-1} \beta_{kj} = 0, \\ &= 0, \quad \text{if} \quad \alpha_k = n - k + 1 \quad \text{and} \quad \sum_{j=1}^{m-1} \beta_{kj} = n - k + 1, \\ &= n - k + 1, \quad \text{if} \quad \alpha_k = n - k + 1 \quad \text{and} \quad \sum_{j=1}^{m-1} \beta_{kj} = 0. \end{aligned}$$

These are the only possible values because $\beta_{km} \geqq 0$ for $k = 1, 2, \cdots, m$, and $(2/\theta)U'X$ and $(2/\theta)V_j'X$, $j = 1, 2, \cdots, m - 1$, have (translated) chi-square distributions. Thus $\beta_{km} = n - k + 1$ if and only if $\alpha_k = n - k + 1$ and $\beta_{kj} = 0$, $j = 1, 2, \cdots, m - 1$. Hence the number of positive β_{km}'s is equal to $r - \sum_{j=1}^{m-1} r_j$ and $\beta_{1m} = \alpha_1 - \sum_{j=1}^{m-1} \beta_{1j}$; this implies, by Theorem 5, the conclusion of the theorem.

We note that if $U'X = \sum_{i=1}^n X_i$ and $m = 2$, then the hypothesis $\beta_{k2} \geqq 0$, $k = 1, 2, \cdots, n$, can be removed.

3. A characterization theorem. In the examples of the previous section, we saw that, when sampling from an exponential distribution, the statistics X_1 and $(1/n)\sum_{i=1}^n (X_i - X_1)$ are independent. In the next theorem we will prove that the independence of these statistics characterizes the exponential distribution.

THEOREM 7. *Let F be an absolutely continuous distribution function of the random variable X with $F(A) = 0$ and $F(x) > 0$ for $x > A$. Let $X_1 < X_2 < \cdots < X_n$ denote the order statistics of a random sample of size n from this distribution. Then the probability density function (hereafter,* p.d.f.) *of X is given by $p(x; \theta, A)$ if and only if X_1 and $\sum_{i=1}^n (X_i - X_1)$ are independent.*

PROOF. The necessity of this condition follows immediately from Theorem 2.

We prove the sufficiency of this condition first for the case $n = 2$. Let $Y = X - A$. If $Y_1 = X_1 - A$ and $Y_2 = X_2 - A$, then $Z_1 = Y_1$ and $Z_2 = Y_2 - Y_1$ are independent. The joint p.d.f. of Z_1 and Z_2 is given by $h(z_1, z_2) = 2f(z_1)f(z_1 + z_2)$, $z_1 \geqq 0$, $z_2 \geqq 0$, where f is the p.d.f. of Y. Because Z_1 and Z_2 are independent, $h(z_1, z_2) = h_1(z_1)h_2(z_2)$, where h_i is the p.d.f. of Z_i, $i = 1, 2$. Thus it follows that $f(z_1 + z_2) = f_1(z_1)f_2(z_2)$, where f_i is a function of z_i alone, $i = 1, 2$. This relation along with the inequalities $z_1 \geqq 0$ and $z_2 \geqq 0$ imply that f must satisfy (almost everywhere) the functional equation

$$f(0)f(x + y) = f(x)f(y), \qquad x \geqq 0, \quad y \geqq 0, \tag{1}$$

where $f(0) \neq 0$.

Since Equation (1) implies that $f(y) > 0$ for $y > 0$ (almost everywhere), we have, by taking logarithms, $\log f(0) + \log f(x + y) = \log f(x) + \log f(y)$. Letting $g(y) = \log f(y) - \log f(0)$, this bceomes

$$g(x + y) = g(x) + g(y), \qquad x \geqq 0, \quad y \geqq 0. \tag{2}$$

Sierpiński [6] proved that if a measurable function, g, satisfies the functional Equation (2), then $g(y) = cy$, $y \geqq 0$, where c is a constant. That is, $\log f(y) - \log f(0) = cy$, $y \geqq 0$. Thus, $f(y) = f(0)e^{cy}$, $y \geqq 0$. Since f is a probability density function, $c = -f(0)$. Letting $f(0) = 1/\theta$, we see that the p.d.f. of Y is given by $p(y; \theta, 0)$ and hence the p.d.f. of X is given by $p(x; \theta, A)$.

In the proof for $n > 2$, we first make the observation that $\sum_{i=1}^n (X_i - X_1) = X_n + \cdots + X_2 - (n - 1)X_1$ does not depend on the ordering of $X_2, \cdots, X_n$. Thus, if we take $X_1 < X_2, X_3, \cdots, X_n$, it is still true that X_1 and $\sum_{i=1}^n (X_i - X_1)$ are independent. Let f denote the p.d.f. of X. The conditional distribution of $X_2, X_3, \cdots, X_n$, given $X_1 = x_1$, is

$$g(x_2, \cdots, x_n \mid x_1) = f(x_2) \cdots f(x_n)/[1 - F(x_1)]^{n-1}, \qquad x_1 < x_2, \cdots, x_n.$$

The characteristic function of the conditional distribution of $\sum_{i=1}^n (X_i - X_1)$, given $X_1 = x_1$, is

$$E\left\{\exp\left[it \sum_{i=1}^n (X_i - X_1)\right] \mid x_1\right\} = \left\{\int_{x_1}^{\infty} \exp\,[it(x_2 - x_1)]f(x_2)\,dx_2/[1 - F(x_1)]\right\}^{n-1}.$$

This follows from the fact that each X_i, $i = 2, 3, \cdots, n$, given $X_1 = x_1$, has the same distribution and the X_i, $i = 2, 3, \cdots, n$, are conditionally mutually independent. Because of the independence of X_1 and $\sum_{i=1}^{n} (X_i - X_1)$, this characteristic function must be free of x_1. Thus

$$\int_{x_1}^{\infty} \exp [it(x_2 - x_1)]f(x_2)\, dx_2/[1 - F(x_1)]$$

must be free of x_1. But this is the characteristic function of the conditional distribution of $W_2 - W_1$, given $W_1 = x_1$, where $W_1 < W_2$ are the order statistics of a random sample of size 2 from $f(x)$. Thus W_1 and $W_2 - W_1$ are independent because this characteristic function is free of $W_1 = x_1$. Thus, by the case for $n = 2$, the p.d.f. of X is given by $p(x; \theta, A)$.

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