A REVIEW OF THE LITERATURE ON A CLASS OF COVERAGE PROBLEMS

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0. Introduction. In recent years a large number of publications have appeared on probability problems arising from ballistic applications. Many of these papers and reports are concerned with topics which are often referred to as coverage problems. Some of the results are found only in obscure sources which are not readily available. Consequently it is a difficult and time consuming task to locate the numerous publications on this subject, a fact that has led to considerable duplication of effort and waste of time. It is hoped that this review will improve the situation and be of some use to those who have an interest in problems of this type.

The discussion of a simple example will serve the purpose of introducing some of the ideas and language needed for the definition of coverage problems which is presented later in the introduction. Suppose that a point target is located at the origin of a two dimensional coordinate system. A weapon with killing radius R is aimed at the origin with the intention of destroying the point target. When the weapon arrives at the target, the latter is located at (x_1', x_2'), a randomly selected position within or on a circle of radius D. That is, the probability density function of (x_1', x_2') is $g(x_1', x_2') = (\pi D^2)^{-1}$, $0 \leqq x_1'^2 + x_2'^2 \leqq D^2$. Assume that aiming errors are circularly normally distributed with unit variance so that the center of the lethal circle, (x_1, x_2), has p.d.f.,

$$f(x_1, x_2) = (2\pi)^{-1} \exp[-\tfrac{1}{2}(x_1^2 + x_2^2)].$$

Now a given point (x_1', x_2') will be destroyed if the impact point of the weapon is within R units of (x_1', x_2'). The probability that this happens is

$$h(x_1', x_2') = \iint\limits_{(x_1-x_1')^2+(x_2-x_2')^2 \leqq R^2} f(x_1, x_2)\, dx_1\, dx_2 .$$

The probability of destroying the target (that is, the probability that the impact point is within R units of the target given that the target is as likely to be at one point as any other in a circle of radius D) is

$$P(R, D) = \iint\limits_{x_1'^2+x_2'^2 \leqq D^2} h(x_1', x_2') g(x_1', x_2')\, dx_1'\, dx_2' .$$

The evaluation of this integral will be discussed in Paragraph 2.2.

Received 25 May 1962; revised 10 April 1963.

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Next we turn to the task of defining coverage problems. Let $X = (x_1, \cdots, x_n)$ be the impact point of the weapon, $X' = (x_1', \cdots, x_n')$ be the position of the target at the time of impact, $P_1(X, X') =$ probability of damaging the target for given values of X and X' (sometimes referred to as a damage function), and $F(x) =$ the distribution function of the impact point. Then $P_2(X') = \int_{-\infty}^{\infty} P_1(X, X')dF(X) =$ probability that a given X' is destroyed. Let $G(X') =$ the distribution function of X'. Then $P(\cdot) = \int_{-\infty}^{\infty} P_2(X')dG(X') =$ probability of destroying a point target whose position is governed by $G(X')$ when aiming errors have distribution function $F(X)$.

We will define a coverage problem as the computation of a probability of the type $P(\cdot)$, that is the evaluation of

$$P(\cdot) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_1(X, X')\, dF(X)\, dG(X'). \tag{0.1}$$

All three functions $P_1(X, X')$, $F(X)$, and $G(X')$ (and consequently $P(\cdot)$) will in general depend upon parameters. The integral $P(\cdot)$ may also be interpreted as the expected proportion of a target destroyed.

A number of special cases of (0.1) are discussed in this review. Section 1 is devoted to probability content problems. In this situation we have

$$\begin{aligned} P_1(X, X') &= 1, \qquad && X \in \text{ region } C_1 \\ &= 0, && \text{otherwise} \end{aligned}$$

(sometimes called a zero-one damage function) and the random variable X' assumes the position $X' = B = (b_1, \cdots, b_n)$, a fixed point, with probability 1. Under these conditions (0.1) takes the form

$$P(\cdot) = \int_{C_1} dF(X). \tag{0.2}$$

The same damage function is used in Section 2 but the distribution of X' does not concentrate all the probability at one point. Hence, for these cases (0.1) reduces to

$$P(\cdot) = \int_{-\infty}^{\infty} \int_{C_1} dF(X)\, dG(X'). \tag{0.3}$$

Finally, in Section 3 some examples are considered in which the damage function is not of the zero-one type. Although the latter two situations are more interesting than the one considered in Section 1, much more literature is available for the former.

In each of the special cases considered, $P(\cdot)$ is used to stand for the probability. After it becomes apparent which parameters determine $P(\cdot)$, we will often replace the dot by these parameters. Thus in the introductory example we used $P(R, D)$ to denote the fact that the probability depends only on R and D.

In most of the evaluations X will have the density function

$$(0.4) \qquad f(x_1, \cdots, x_n) = \left[(2\pi)^{\frac{1}{2}n}\prod_{i=1}^{n}\sigma_i\right]^{-1}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i^2/\sigma_i^2)\right].$$

Thus $(y_1, \cdots, y_n)$, where $y_i = x_i/\sigma_i$, $i = 1, 2, \cdots, n$, has an n-dimensional standard normal distribution.

A bibliography entitled "The Coverage Problem" [51] has been published by the Sandia Corporation. It contains a list of several hundred references that are either directly or indirectly related to this field. The majority of papers covered in this review are listed in that report.

1. Probability content problems.

1.0 *Introduction.* In order to limit this section to a reasonable length, we will consider only those probability content problems which have received the most attention in ballistic applications. The region C_1 will be spherical and the point $B = (b_1, \cdots, b_n)$ will be captured by a sphere of radius R whenever X falls within or on $\sum_{i=1}^{n}(x_i - b_i)^2 = R^2$. Thus if X has the p.d.f. given by (0.4), then (0.2) can be written

$$P(\cdot) = \int_{\sum_{i=1}^{n}(x_i-b_i)^2 \leqq R^2}\cdots\int f(x_1, \cdots, x_n)\,dx_1\cdots dx_n$$

$$(1.1) \qquad = \int_{\sum_{i=1}^{n}[(y_i-b_i/\sigma_i)^2/(R/\sigma_i)^2]\leqq 1}\cdots\int [(2\pi)^{\frac{1}{2}n}]^{-1}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}y_i^2\right)dy_1\cdots dy_n$$

$$(1.2) \qquad = \Pr\left[\sum_{i=1}^{n}\frac{(y_i - b_i/\sigma_i)^2}{(R/\sigma_i)^2} \leqq 1\right].$$

We note that it is not necessary to consider separately the case in which C_1 is elliptical for if C_1 contains the points within or on $\sum_{i=1}^{n}(x_i - b_i)^2/c_i^2 = 1$, then $P(\cdot)$ reduces to $\Pr[\sum_{i=1}^{n}(y_i - b_i/\sigma_i)^2/(c_i/\sigma_i)^2 \leqq 1]$, the same evaluation as (1.2).

If X has a non-singular multivariate normal distribution and the axes of an elliptical region are not parallel to the coordinate axes, $P(\cdot)$ can still be reduced to an evaluation of the type (1.2) even though some of the covariances are not zero (i.e. see [49], Section 1).

1.1 *Region centered at the origin, variances equal.* The probability given by (1.2) reduces to

$$(1.3) \qquad P(R/\sigma) = \Pr\left[\sum_{i=1}^{n}y_i^2 \leqq R^2/\sigma^2\right] = \Pr[w^2 \leqq R^2/\sigma^2]$$

where w^2 has a chi-square distribution with n degrees of freedom and σ^2 is the common variance.

There are a number of ways to evaluate (1.3). The most obvious is to use one of the numerous tables of the chi-square distribution, the limiting factor being the number of entries. Since $\Pr[w^2 \leqq R^2/\sigma^2] = \Pr[w^2/n \leqq R^2/n\sigma^2]$, the chi-square divided by degrees of freedom table of Dixon and Massey [8] may also be used for an approximate solution. It contains more probability levels than most chi-square tables. Another expression equivalent to (1.3) is

$$\Pr[u \leqq R^2/2\sigma^2] = \int_0^{R^2/2\sigma^2} [1/\Gamma(\tfrac{1}{2}n)]u^{\frac{1}{2}n-1}e^{-u}\,du$$

which can be found in Pearson's [42] table of the incomplete Γ-function. Finally, for $n = 2$ and $n = 3$ it is easy to verify that

$$P(R/\sigma) = 1 - \exp(-R^2/2\sigma^2), \qquad n = 2, \tag{1.4}$$

and

$$P(R/\sigma) = 2\Phi(R/\sigma) - 1 - 2(R/\sigma)\phi(R/\sigma), \qquad n = 3, \tag{1.5}$$

where

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)\,dt, \qquad \phi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2).$$

Formulas similar to (1.4) and (1.5) can be obtained for any n by repeated integration by parts. Tables are available which permit (1.4) and (1.5) to be evaluated to a high degree of accuracy. In [34] the function e^{-x} is given to eighteen decimal places for $x = 0(.0001)2.5$ and to twenty decimal places in [35] for $x = 2.5(.001)10$. In [36] the integral $\int_{-x}^{x}(2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}t^2)dt$ and $\phi(x)$ are given to fifteen decimal places for $x = 0(.0001)1(.001)7.800(\text{various})8.285$.

Reference [38] contains two single page tables that can be used for the evaluation of (1.4) and (1.5). For the case $n = 2$ Table 8.1 (p. 171) gives R/σ to four decimal places for $P(R/\sigma) = .01(.01).99(.001).999(.0001).9999$, .99995, .99999, .999995, .999999, .9999995, .9999999, .99999995, .99999999, .999999995, .999999999. For the case $n = 3$ Table 8.6 (p. 203) gives R/σ to four decimal places for $P(R/\sigma) = .01(.01).99(.001).999(.0001).9999$, .99999, .999999, .9999999.

1.2 *Region not centered at the origin, variances equal.* If we let $Z_i = (x_i - b_i)/\sigma$, $i = 1, 2, \cdots, n$ in (1.1) we get

$$P(\cdot) = \int \cdots \int_{\sum_{i=1}^{n} Z_i^2 \leqq R^2/\sigma^2} [(2\pi)^{\frac{1}{2}n}]^{-1} \exp\left[-\frac{1}{2}\sum_{i=1}^{n} (Z_i + b_i/\sigma)^2\right] dZ_1 \cdots dZ_n \tag{1.6}$$

which is $\Pr[\sum_{i=1}^{n} Z_i^2 \leqq R^2/\sigma^2] = \Pr[W^2 \leqq R^2/\sigma^2]$ where W^2 has a non-central chi-square distribution with n degrees of freedom and non-centrality parameter $\lambda^2 = \sum_{i=1}^{n} b_i^2/\sigma^2 = r^2/\sigma^2$.

A number of derivations are available for obtaining the p.d.f. of W^2. Perhaps the simplest is the non-constructive method used by Graybill [17], pages 74–76. He observes that $\sum_{i=1}^n Z_i^2$ and a random variable W^2 with p.d.f.

$$(1.7) \qquad h(W^2;n, \lambda^2) = \exp[-\tfrac{1}{2}(W^2 + \lambda^2)]\sum_{j=0}^{\infty} \lambda^{2j}W^{n+2j-2}/j!\, 2^{\frac{1}{2}n+2j}\Gamma(\tfrac{1}{2}n + j)$$

both have the same moment generating function and must have the same distribution. Mann's book [27] contains a constructive derivation. In his proof he makes an orthogonal transformation on the Z_i and shows that $W^2 = v^2 + w^2$, where v and w^2 are independent random variables, the first normal and the second chi-square with $n - 1$ degrees of freedom. Then to get the p.d.f. of W^2 he makes the transformation $w^2 = W^2 \cos^2 \theta$, $v = W \sin \theta$ in the joint density of v and w^2 and integrates out θ. A recent geometrical derivation by Ruben [48] leads to the form

$$(1.8) \qquad h(W^2; n, \lambda^2) = \tfrac{1}{2}(W/\lambda)^{\frac{1}{2}(n-2)} \exp[-\tfrac{1}{2}(W^2 + \lambda^2)]I_{\frac{1}{2}(n-2)}(\lambda W)$$

where $I_{\frac{1}{2}(n-2)}(x)$ is the modified Bessel function of order $\frac{1}{2}(n - 2)$. Of course (1.8) could be obtained from (1.7) by using the well known series expansion for the Bessel function ([57], p. 77).

We seek

$$(1.9) \qquad P(R/\sigma, r/\sigma) = H(R^2/\sigma^2; n, r^2/\sigma^2) = \int_0^{R^2/\sigma^2} h(W^2; n, \lambda^2)\, dW^2.$$

Before discussing numerical evaluations it is interesting to note that a recursion formula for (1.9) can be derived by integrating by parts with

$$dv = \tfrac{1}{2}\exp[-\tfrac{1}{2}(W^2 + \lambda^2)]\, dW^2, \qquad u = (W/\lambda)^{\frac{1}{2}(n-2)}I_{\frac{1}{2}(n-2)}(\lambda W).$$

This yields

$$(1.10) \qquad \begin{aligned} H(R^2/\sigma^2; n, \lambda^2) &= H(R^2/\sigma^2; n - 2, \lambda^2) \\ &\quad - (R/\lambda\sigma)^{\frac{1}{2}(n-2)} \exp\{-\tfrac{1}{2}[(R^2/\sigma^2) + \lambda^2]\}I_{\frac{1}{2}(n-2)}(R\lambda/\sigma), \end{aligned}$$

a result observed by Guenther [19] and Quenouille [43].

1.2.1 *Two dimensional case.* When $n = 2$, Formula (1.9) becomes

$$(1.11) \qquad P(R/\sigma, r/\sigma) = e^{-r^2/2\sigma^2} \int_0^{R/\sigma} te^{-\frac{1}{2}t^2}I_0((r/\sigma)t)\, dt.$$

An alternate derivation of this result is obtained from (1.1) (with $n = 2$ and $b_1 = b_2 = 0$) by rotating the axes through an angle $\alpha = \arctan (b_1/b_2)$ followed by translating the origin to the center of the circle. These operations yield

$$(1.12) \qquad \iint_{x_1^2+x_2^2 \leq R^2} (2\pi\sigma^2)^{-1} \exp\left[-(2\sigma^2)^{-1}(x_1^2 + 2rx_1 + r^2 + x_2^2)\right] dx_1\, dx_2\,.$$

Switching to polar coordinates (1.12) becomes

$$\int_0^{2\pi}\int_0^R \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2\sigma^2}(t^2+2rt\cos\theta+r^2)\right]t\,dt\,d\theta$$
$$= e^{-r^2/2\sigma^2}\int_0^R \frac{t}{\sigma^2}e^{-t^2/2\sigma^2}\left[\frac{1}{\pi}\int_0^\pi e^{-rt\cos\theta/\sigma^2}\,d\theta\right]dt$$
$$= e^{-r^2/2\sigma^2}\int_0^R \frac{t}{\sigma^2}e^{-t^2/2\sigma^2}I_0\left(\frac{rt}{\sigma^2}\right)dt$$

which is equivalent to (1.11). Programs for the numerical integration of (1.11) are discussed in [44] and [3].

Several extensive tables of (1.11) have been prepared. The Bell Aircraft Corporation has tabulated $P(R/\sigma, r/\sigma)$ in [3] to five decimal places for $r/\sigma = 0.00(0.01)3.00$, $R/\sigma = 0.01(0.01)4.59$. Marcum [28] has tabulated $q(R/\sigma, r/\sigma) = 1 - P(R/\sigma, r/\sigma)$ to six decimal places for $R/\sigma = 0.10(0.10)20.0$ over values of r/σ by intervals of .05 as are necessary to cover the range $q(R/\sigma, r/\sigma) = 0$ to $q(R/\sigma, r/\sigma) = 1$. Thus the Bell table covers smaller intervals of R/σ and r/σ but the ranges of these parameters are much greater in the Marcum table. Both are rather bulky covering 301 and 185 pages respectively. Abridgements of [28] appear in [14], [31], [38], [44], and [55].

Di Donato and Jarnagin [6] have published extensive inverse tables of R/σ (to seven significant figures) for given values of $P(R/\sigma, r/\sigma)$ and r/σ. Values of R/σ are given for $r/\sigma = 0.0(0.1)5.00(0.2)10.00(2.0)20.0(5.0)120.0$ and $P(R/\sigma, r/\sigma) = 0.01(0.01)0.99$. Also included is a table of high probabilities for $r/\sigma = 0$, .05, .10, .25, .50, .75, 1.00, 1.50, 2.00, 3.00, 4.00, 5.00, 6.00, 8.00, 10.00, 20.00, 30.00, 50.00, 80.00, 120 and $P(R/\sigma, r/\sigma) = .99(.0005).999(.0001).9999(.00001).99999(.000001).999999$. This form of tabulation would be particularly useful for finding the radius R required if it is desired that the probability of capturing a point r units from the origin be at least some specified figure, say $1 - \alpha$. An abbreviated table appears in [7]. Both references include a discussion of computational procedures.

A graph has been prepared by Solomon [53] from which it is possible to approximate (1.11). Curves are given for $P(R/\sigma, r/\sigma) = .05(.05).95$ graphed over ranges $0 \leqq r/\sigma \leqq 10$, $0 \leqq R/\sigma \leqq 8$. To use the graph (his Figure 1), relabel the vertical axis R/σ and the horizontal axis r/σ.

1.2.2 *Three dimensional case.* Using the formula

$$I_{\frac{1}{2}}(x) = (2\pi x)^{-\frac{1}{2}}(e^x - e^{-x})$$

(see [57], p. 80), it is easy to verify by straight forward integration that (1.9) can be written as

$$P(R/\sigma, r/\sigma) = \Phi((r+R)/\sigma) - \Phi((r-R)/\sigma) - (r/\sigma)^{-1}[\phi((r-R)/\sigma) - \phi((r+R)/\sigma)]. \tag{1.13}$$

Formula (1.13) can be derived directly from (1.1) by choosing $(r, 0, 0)$ as the center of the sphere (which can be done with no loss of generality) followed by the substitution $x_1 = r + Z_1$, $x_2 = Z_2$, $x_3 = Z_3$ and a switch to spherical coordinates.

An abbreviated table of (1.13) has been prepared by Guenther [18]. It gives $P(R/\sigma, r/\sigma)$ to two decimal places for $R/\sigma = 0(0.5)3.5$, $r/\sigma = 0(.5)3.5$. Other entries are easy to compute since exponentials and standard normal distributions are well tabulated (i.e., see [34], [35], and [36]). The table and the approximation discussed in Paragraph 1.2.3 could also be used for the evaluation of (1.13).

1.2.3 *n-dimensional case.* The most extensive tables of the non-central chi-square are apparently those prepared by Haynam and Leone [21]. Their Table I gives $H(R^2/\sigma^2; n, r^2/\sigma^2)$ to five decimal places for all combinations of values of

$$r^2/\sigma^2 = 0(0.1)1.0(0.2)3.0(0.5)5.0(1.0)34.0$$

$$n = 1(1)30(2)50(5)100$$

$$R^2/\sigma^2 = 0.01(0.01)0.1(0.1)1.0(0.2)3.0(.5)10.0(1.0)30.0(2)50.0(5.0)165.0,$$

over 5,000 entries for each degree of freedom and more than one quarter of a million entries all together. In addition they have a Table II and Table III specifically arranged to give power for chi-square tests, but these are not useful for coverage problems. The biggest objection to these tables is their bulk. It is hoped that the authors will find some convenient way to condense them so that they will be more readily publishable.

The Haynam and Leone tables are computed from a series expansion. They use (1.7) which allows them to replace (1.9) by

$$H(R^2/\sigma^2; n, \lambda^2) = \int_0^{R^2/\sigma^2} \frac{e^{-\frac{1}{2}x}e^{-\frac{1}{2}\lambda^2}}{2^{\frac{1}{2}n}} \sum_{j=0}^{\infty} \frac{x^{\frac{1}{2}n+j-1}(\lambda^2)^j}{\Gamma(\frac{1}{2}n + j)2^{2j}j!}\, dx.$$

After some fairly routine manipulations, they show that

$$(1.14)\quad H(R^2/\sigma^2; n, \lambda^2) = \sum_{i=0}^{\infty} P_{i+\frac{1}{2}n}(R^2/2\sigma^2) \sum_{j=0}^{i} P_j(\tfrac{1}{2}\lambda^2), \qquad n \text{ even}$$

$$= \sum_{i=0}^{\infty} Q_{i+\frac{1}{2}(n+1)}(R^2/2\sigma^2) \sum_{j=0}^{i} P_j(\tfrac{1}{2}\lambda^2), \qquad n \text{ odd}$$

where $P_i(y) = e^{-y}y^i/i!$, the Poisson density function and

$$Q_i(y) = e^{-y}y^{i-\frac{1}{2}}/\Gamma(i + \tfrac{1}{2}), \qquad i \geqq 1$$

$$= (2/\pi^{\frac{1}{2}}) \int_y^{\infty} e^{-u^2}\, du. \qquad i = 0.$$

Formula (1.14) is used as the basis of their calculations.

If tables are not available, then one of the many approximations for the non-central chi-square distribution could be used. Some of these are found in [1], [24], [40], and [41]. According to Pearson,

$$H(R^2/\sigma^2;\, n,\, r^2/\sigma^2) \cong \Pr[w^2 \leqq M] \tag{1.15}$$

where

$$M = \left[\frac{R^2}{\sigma^2} + \frac{r^4/\sigma^4}{n + 3(r^2/\sigma^2)}\right] \Big/ \frac{n + 3r^2/\sigma^2}{n + 2r^2/\sigma^2}$$

and w^2 is distributed as central chi-square with fractional degrees of freedom $v' = (n + 2r^2/\sigma^2)^3/(n + 3r^2/\sigma^2)^2$. To evaluate (1.15), again we may use tables of chi-square, chi-square divided by degrees of freedom, or the incomplete gamma function. The approximation seems to be reasonably good even though interpolation is required for fractional degrees of freedom. The Haynam and Leone tables make it possible to investigate the accuracy of the various approximations.

1.3 *Region centered at the origin, variances unequal.* For this case (1.1) and (1.2) reduce to

$$P(\cdot) = \underset{\sum_{i=1}^{n} x_i^2 \leqq R^2}{\int \cdots \int} \left[(2\pi)^{\frac{1}{2}n} \prod_{i=1}^{n} \sigma_i\right]^{-1} \exp\left[-\frac{1}{2}\sum_{i=1}^{n} (x_i/\sigma_i)^2\right] \tag{1.16}$$

$$= \Pr\left[\sum_{i=1}^{n} y_i^2/(R/\sigma_i)^2 \leqq 1\right]. \tag{1.17}$$

Some discussion of the n-dimensional case will be found in Section 1.5.

1.3.1 *Two dimensional case.* The integral (1.16) is expressed in various forms more adaptable to calculation in [6], [9], [10], [16], [22], and [58] (computing schemes for numerical integration are discussed in the first five of these references). The simplest of these, due primarily to Fettis, is obtained by letting

$$x_1 = \sigma_1\rho\cos\theta, \qquad x_2 = \sigma_1\rho\sin\theta, \qquad R = K\sigma_1, \qquad c = \sigma_2/\sigma_1, \tag{1.18}$$

where $0 < c \leqq 1$, followed by replacing 2θ by ϕ. This changes (1.16) to

$$P(K, c) = \frac{1}{2\pi c}\int_0^K \int_0^{2\pi} \rho \exp\{-(\rho^2/4c^2)[(1 + c^2) - (1 - c^2)\cos\phi]\}\, d\phi\, d\rho \tag{1.19}$$

$$= \frac{1}{c}\int_0^K \rho \exp\left[-\frac{\rho^2}{2}\left(\frac{1 + c^2}{2c^2}\right)\right] I_0\left(\frac{1 - c^2}{4c^2}\rho^2\right) d\rho. \tag{1.20}$$

For his numerical integration scheme, Harter [22] changes (1.19) to

$$P(K, c) = \frac{2c}{\pi}\int_0^{\pi} \frac{1 - \exp\{-(K^2/4c^2)[(1 + c^2) - (1 - c^2)\cos\phi]\}}{(1 + c^2) - (1 - c^2)\cos\phi}\, d\phi$$

by letting $Z = \rho^2/4c^2$ and integrating with respect to Z. Another interesting form of $P(K, c)$, used by Esperti [9], is obtained from (1.20) by making the substitution $s = \rho^2(1 - c^2)/4c^2$, replacing $I_0(s)$ by $\sum_{m=0}^{\infty} (s/2)^{2m}/(m!)^2$ and integrating term by term. Esperti's result is

$$P(K, c) = \frac{2c}{1 - c^2}\sum_{m=0}^{\infty} \frac{(2m)!}{(2^m m!)^2}\left(\frac{1 - c^2}{1 + c^2}\right)^{2m+1}\left[1 - e^{-\mu}\sum_{v=0}^{2m} \frac{\mu^v}{v!}\right] \tag{1.21}$$

where $\mu = \frac{1}{4}K^2(1/c^2 + 1)$. Finally, letting

$$\frac{\rho^2}{2}[(1 + c^2)/2c^2] = x, \qquad c = (c_2/c_1)^{\frac{1}{2}}, \qquad K = (tc_2)^{\frac{1}{2}}$$

and putting $c_1 + c_2 = c_1c_2$, (1.20) becomes

$$\frac{2}{(c_1 + c_2)^{\frac{1}{2}}} \int_0^{\frac{1}{2}(c_1+c_2)t} e^{-x} I_0 \left[\left(\frac{1}{c_1} - \frac{1}{c_2}\right) x\right] dx,$$

the result of Grad and Solomon. In terms of the original parameters we have

$$c_1 = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}, \qquad c_2 = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2}, \qquad t = \frac{R^2}{\sigma_1^2 + \sigma_2^2}.$$

Harter has tabulated $P(K, c)$ to seven decimal places for $K = 0.1(0.1)6.0$, $c = 0.0(0.1)1.0$. The Esperti table gives $P(K, c)$ to six decimal places for $K = 0.00(0.01)4.99$, $c = 0.0(0.1)1.0$. Hence the main difference between this table and the one prepared by Harter is the finer intervals for K. The Esperti table contains about ten times as many entries as Harter's and also includes three graphs that are of interest. These display curves of

$P(K, c)$ versus K, for $c = 0.0(0.1)1.0$

$P(K, c)$ versus c, for $K = 0.0(0.1)3.0$

K versus c, for $P(K, c) = 0.0(0.1)0.9$, 0.99, 0.999, 0.9999, 0.99999.

We note that when $n = 2$ (1.17) is $\Pr[y_1^2 + c^2y_2^2 \leqq K^2]$. Grad and Solomon [16] have tabulated

$$\Pr[a_1y_1^2 + a_2y_2^2 \leqq t] = \Pr[(a_1/a_2)y_1^2 + y_2^2 \leqq t/a_2] = \Pr[y_1^2 + (a_1/a_2)y_2^2 \leqq t/a_2]$$

where $a_1 \leqq a_2$ and $a_1 + a_2 = 1$. Thus to compare these tables with those of Harter, set $c = (a_1/a_2)^{\frac{1}{2}}$ and $K = (t/a_2)^{\frac{1}{2}}$. Solomon [54] has extended the original tables of [16] to include twelve pairs of (a_2, a_1). These are (.5, .5), (.6, .4), (2/3, 1/3), (.7, .3), (.75, .25), (.8, .2), (.875, .125), (.9, .1), (.95, .05), (.99, .01), and (1, 0). Entries are given to eight decimal places for these values of (a_2, a_1) and $t = .005(.005).10(.01)1.00(.02)2.50(.025)3.50(.05)5.00(.25)6.00(.50)7.00(1.00)10.00$. In comparing this table to the one prepared by Harter, we find that not only is the latter table more compact but also it is much more adaptable to coverage problems. Solomon has also included an inverse table in [54]. The entry is t which is given for $\Pr[a_1y_1^2 + a_2y_2^2 \leqq t] = .05(.05).95$ and the same twelve pairs of (a_2, a_1) used in his other table. Usefulness is limited by the small number of values of (a_2, a_1) for which t appears. Parts of Solomon's tables are given in Sections 8.3 and 8.4 of [38].

DiDonato and Jarnagin [6] have tabulated inverse tables in terms of K for a given $P(K, c)$ and a given c. Values of K are given to seven significant figures for $c = 0, .10(.05)1.00$, $P(K, c) = .99(.0005).9990(.0001).9999(.00001).99999(.000001).999999$. It is easier to evaluate $P(K, c)$ from this inverse table than from Solomon's direct table if it is necessary to interpolate between pairs (a_2, a_1).

Another good inverse table has been compiled by Marsaglia [29]. He has tabulated to four significant figures the r required to make $\Pr[y_1^2 + s^2y_2^2 \leqq r^2] = \Pr[s^2y_1^2 + y_2^2 \leqq r^2] = .01(.01).99$ for $s = 1.0(.1)5.0(.2)10.0$. In order to identify his notation with that which we have used previously, we note that for $n = 2$ (1.17) is $\Pr[(\sigma_1^2/\sigma_2^2)y_1^2 + y_2^2 \leqq R^2/\sigma_2^2]$. Hence $s = \sigma_1/\sigma_2$, the ratio of the larger standard deviation to the smaller, and $r = R/\sigma_2$, the ratio of the radius of the circle to the smaller standard deviation. In addition to the table graphs are included. Curves are given for $\Pr[s^2y_1^2 + y_2^2 \leqq r^2] = .01, .05(.05).95, .99$ graphed over the ranges $1 \leqq s \leqq 10$, $0 \leqq r \leqq 15$. Marsaglia refers to [29] as the preliminary form of his publication and he does not present any computational procedures.

Two abbreviated inverse tables appear in the literature. These are included in the articles by Weingarten and DiDonato [58] and by Harter [22]. The former is more extensive giving K for $c = .05(.05)1.0$, $P(K, c) = .05(.05).95(.01).99$ while in the Harter article K appears for $c = 0(.1)1.0$, $P(K, c) = .5000, .7500, .9000, .9500, .9750, .9900, .9950, .9975, .9990$.

Oberg [37] gives an approximation for R given $P(K, c)$, σ_1, and σ_2. With all the tables now available there should be little demand for this result.

If some of the above tables are not available, or if the needed value of $P(K, c)$ does not appear for a particular choice of K and c, Equation (1.22) may be useful. It is known that

$$P(K, c) = q\left[\frac{K}{2}\left(\frac{1}{c} - 1\right), \frac{K}{2}\left(\frac{1}{c} + 1\right)\right] - q\left[\frac{K}{2}\left(\frac{1}{c} + 1\right), \frac{K}{2}\left(\frac{1}{c} - 1\right)\right], \tag{1.22}$$

so that the Marcum table [28] can be used for its evaluation. Since (1.22) can be written in the form

$$P(K, c) = \left\{1 - q\left[\frac{K}{2}\left(\frac{1}{c} + 1\right), \frac{K}{2}\left(\frac{1}{c} - 1\right)\right]\right\} - \left\{1 - q\left[\frac{K}{2}\left(\frac{1}{c} - 1\right), \frac{K}{2}\left(\frac{1}{c} + 1\right)\right]\right\},$$

the Bell table [3] may also be used. The origin of this result is somewhat obscure (see p. 349 of [7] for historical background) but the following proof is due to Fettis [10]. Write

$$\begin{aligned} q(R, r) &= 1 - e^{-\frac{1}{2}r^2} \int_0^R te^{-\frac{1}{2}t^2} I_0(rt)\, dt \\ &= e^{-\frac{1}{2}r^2} \int_R^\infty te^{-\frac{1}{2}t^2} I_0(rt)\, dt. \end{aligned} \tag{1.23}$$

Integrating by parts, (1.23) becomes

$$q(R, r) = e^{-\frac{1}{2}(R^2+r^2)} I_0(Rr) + re^{-\frac{1}{2}r^2} \int_R^\infty e^{-\frac{1}{2}t^2} I_1(rt)\, dt. \tag{1.24}$$

Next differentiate (1.24) with respect to r. After some simplification it is found that

$$\partial q(R, r)/\partial r = Re^{-\frac{1}{2}(R^2+r^2)} I_1(Rr).$$

Differentiating (1.24) with respect to R yields

$$\partial q(R, r)/\partial R = -Re^{-\frac{1}{2}(R^2+r^2)} I_0(Rr).$$

Letting $R = \alpha K$, $r = \beta K$, one gets

$$\begin{aligned}\partial q(R, r)/\partial K &= \alpha(\partial q/\partial R) + \beta(\partial q/\partial r)\\ &= -Ke^{-\frac{1}{2}K^2(\alpha^2+\beta^2)}[\alpha^2 I_0(\alpha\beta K^2) - \alpha\beta I_1(\alpha\beta K^2)].\end{aligned}$$

Now

$$\int_0^K \frac{\partial q(\alpha K, \beta K)}{\partial K}\, dK = \int_0^K - Ke^{-\frac{1}{2}K^2(\alpha^2+\beta^2)}[\alpha^2 I_0(\alpha\beta K^2) - \alpha\beta I_1(\alpha\beta K^2)]\, dK,$$

or

$$q(\alpha K, \beta K) - q(0, 0) = -\int_0^K \rho e^{-\frac{1}{2}\rho^2(\alpha^2+\beta^2)}[\alpha^2 I_0(\alpha\beta\rho^2) - \alpha\beta I_1(\alpha\beta\rho^2)]\, d\rho,$$

so that

$$q(\alpha K, \beta K) = 1 - \int_0^K \rho e^{-\frac{1}{2}\rho^2(\alpha^2+\beta^2)}[\alpha^2 I_0(\alpha\beta\rho^2) - \alpha\beta I_1(\alpha\beta\rho^2)]\, d\rho.$$

Hence

$$(1.25) \qquad q(\beta K, \alpha K) - q(\alpha K, \beta K) = (\alpha^2 - \beta^2) \int_0^K \rho e^{-\frac{1}{2}\rho^2(\alpha^2+\beta^2)} I_0(\alpha\beta\rho^2)\, d\rho.$$

The integral (1.25) is the same as (1.20) if $\alpha = \frac{1}{2}(1/c + 1)$, $\beta = \frac{1}{2}(1/c - 1)$. Thus (1.25) reduces to (1.22). Kleinecke [25] has published a geometrical proof of (1.22).

1.3.2. *Three dimensional case.* The most extensive table available, one well adapted to coverage problems, has been prepared by Marsaglia [29]. He has tabulated the value of r to four decimal places for which $\Pr[y_1^2 + s^2 y_2^2 + v^2 y_3^2 \leqq r^2] = .01(.01).99$ where

$$s = 1.0(.1)3.0, \qquad v = s(.1)3.0$$

and

$$s = 1.0(.5)7.0, \qquad v = s(.5)7.0.$$

In terms of R, σ_1, σ_2, σ_3, we have $s = \sigma_2/\sigma_1$, $v = \sigma_3/\sigma_1$, $r = R/\sigma_1$ where $\sigma_1 \leqq \sigma_2 \leqq \sigma_3$. The table is 62 pages long and has approximately 30,000 entries. He has a procedure for approximating $\Pr[y_1^2 + s^2 y_2^2 + v^2 y_3^2 \leqq r^2]$ for values of s and v beyond those of the table. However, neither it nor any other computational information is included in the preliminary edition.

Grad and Solomon [16] have tabulated $\Pr[a_1y_1^2 + a_2y_2^2 + a_3y_3^2 \leqq t]$ where $a_3 \geqq a_2 \geqq a_1$ and $a_1 + a_2 + a_3 = 1$. Solomon [54] later extended the original tables. Entries are given to eight decimal places (only four place accuracy is guaranteed if the a's are different, six places if one pair of a's are equal, and eight places if all three a's are equal) for ten sets of (a_3, a_2, a_1), $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, (.4, .3, .3), (.4, .4, .2), (.5, .3, .2), (.6, .2, .2), (.5, .4, .1), (.6, .3, .1), (.7, .2, .1), (.8, .1, .1), (.9, .05, .05) and for $t = .01(.01)2.00(.05)5.00(.50)10.00$. A small inverse table for the same ten sets is included in the report. The entry is t for $\Pr[a_1y_1^2 + a_2y_2^2 + a_3y_3^2 \leqq t] = .05(.05).95$. The same statement regarding accuracy given above still applies. A detailed account of the computing scheme is included in the report. Needless to say, the use of the tables is somewhat restricted by the small number of sets of (a_3, a_2, a_1). A difficult, if not impossible, interpolation problem arises if the a's do not coincide with one of the ten sets. Parts of Solomon's tables are given in Sections 8.7 and 8.8 of [38].

1.4 *Region not centered at origin, variances unequal* ($n = 2$). For this situation we seek an evaluation of

$$P(\cdot) = \Pr\left[\frac{(y_1 - b_1/\sigma_1)^2}{(R/\sigma_1)^2} + \frac{(y_2 - b_2/\sigma_2)^2}{(R/\sigma_2)^2} \leqq 1\right]. \tag{1.26}$$

Since (1.26) depends upon four parameters, a fairly complete tabulation of this function would be extremely bulky. Four tables currently available have been been prepared by Germond [12], Di Donato and Jarnagin [4], Lowe [26], and Rosenthal and Rodden [47]. We will describe each of these.

To use the Germond table, one needs $h = b_1/\sigma_1$, $k = b_2/\sigma_2$, $a = R/\sigma_1$, $b = R/\sigma_2$. The entry is $P(\cdot) = P(h, k, a, b)$ given to four place accuracy for $h = 0.0(0.5)3.0$, $k = 0.0(.05)3.0$, $a = 0.5(0.5)3.0$, $b = 0.5(0.5)3.0$. The computational procedure is described in the report.

The Lowe table has over twice as many entries as Germond's. The probability $P(\cdot)$ is given to three decimal places for $R/\sigma_2 = b = 1, 2, 4, 8, 16, 32, 64$, $\sigma_1/\sigma_2 = b/a = 1, 2, 4, 8$ with eight values for each $h = b_1/\sigma_1$ and $k = b_2/\sigma_2$ so chosen "to cover the b_1, b_2 region in which the variation of $P(\cdot)$ is appreciable."

A much more extensive table is the one prepared by Di Donato and Jarnagin. The entry is R for a given $P(\cdot)$, b_1, b_2, and $\sigma_y = \sigma_2/\sigma_1 \geqq 1$. Obviously we can choose $\sigma_1 = 1$ with no loss of generality. Values of R are given to five significant figures for

$$\begin{aligned} P(\cdot) &= .05, .20, .50, .70, .90, .95 \\ b_1 &= 0.00, 0.25, 0.50, 0.75, 1, 2, 3, 4, 5, 6, 8, 10, 20, 50 \\ b_2 &= 0.00, 0.50, 1, 2, 3, 4, 5, 6, 8, 10, 20, 30, 50, 80, 120 \\ \sigma_y &= 1, 2, 3, 4, 5, 6, 7, 8, 10. \end{aligned}$$

A detailed discussion concerning the computing method is contained in this report and in [5].

Another extensive table is the one prepared by Rosenthal and Rodden. It contains about fifty per cent more entries than the Di Donato and Jarnagin table. The table was designed to give probabilities of being in a circle centered at the origin when aiming errors have a normal distribution with unequal variances and a mean not at the origin. They have evaluated $P(\cdot)$ to five significant figures for $\sigma_y = \sigma_2/\sigma_1 = .2(.2)1$, $b_1/\sigma_1 = 0(.5)3.0$, $b_2/\sigma_1 = 0(.5)3.0$ and over values of R/σ_1 in intervals of .05 from 0 up to those values which make the probability nearly 1. The authors describe their numerical integration scheme and state that they believe that the overall accuracy is better than ± 1 in the third decimal place.

Gilliland [15] has derived a series solution that can be used to evaluate (1.26). His result is obtained by making the substitution (1.18) in

$$\int_{x_1^2+x_2^2 \leq R^2} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2}\sum_{i=1}^{2}(x_i - b_i)^2/\sigma_i^2\right] dx_1\, dx_2,$$

replacing exponential terms by infinite series expansions, and reversing the order of integration and summation. His result is

$$P(\cdot) = \{\exp[\tfrac{1}{2}(b_1^2/\sigma_1^2 + b_2^2/\sigma_2^2)]/2\pi c\}\sum_{m=0}^{\infty} B_m P_m(\mu) \tag{1.27}$$

where μ has the same value as in (1.21),

$$P_m(\mu) = 1 - e^{-\mu}\sum_{v=0}^{m} \mu^v/v!,$$

$$B_m = \tfrac{1}{2}m!(4c^2/(1+c^2))^{m+1}\sum_{i=0}^{m} D_{m,i},$$

$$D_{m,i} = \left(\frac{1-c^2}{4c^2}\right)^i \frac{1}{i!}\sum_{j=0}^{m-i} \frac{(b_1^2/\sigma_1^2)^j(b_2^2/c^2\sigma_2^2)^{m-i-j}}{(2j)!(2m-2i-2j)!} G(i, 2j, 2m-2i-2j),$$

and

$$G(p, q, t) = \int_0^{2\pi} \cos^p 2\theta \cos^q \theta \sin^t \theta\, d\theta.$$

In the special case $b_1 = b_2 = 0$ his series reduces to (1.21), the Esperti result. When $\sigma_1 = \sigma_2 = \sigma$ he gets

$$e^{-r^2/2\sigma^2}\sum_{m=0}^{\infty} [(r^2/\sigma^2)^m/m!2^m]P_m(R^2/2\sigma^2),$$

the correctness of which can be verified by integrating (1.7) from 0 to R^2/σ^2 with $n = 2$. Equation (1.27) appears to be well adapted to use with a desk calculator. An error analysis is included in the paper.

1.5 *Some remarks on general theory.* A number of papers devoted to expressing $\Pr[\sum_{i=1}^{n} a_i(y_i - b_i)^2 \leq t]$, $a_i > 0$, $i = 1, 2, \cdots, n$, in various forms have appeared in the literature in recent years. We will indicate briefly some of the results. Most of the references mentioned in this section list further references of interest.

Robbins [45] showed that

$$\Pr\left[\frac{1}{2}\sum_{i=1}^{n} a_i y_i^2 \leqq t\right] = \frac{t^{n/2}}{D^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{c_k(-t)^k}{\Gamma(\frac{1}{2}n + k + 1)}$$

where $D = \prod_{i=1}^{n} a_i$ and the c_k are constants easily computed from rather complicated recursive formulas. Pachares [39] improved on this result slightly by showing

$$\Pr\left[\frac{1}{2}\sum_{i=1}^{n} a_i y_i^2 \leqq t\right] = \frac{t^{n/2}}{D^{1/2}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{E(Q_n^*)^k}{\Gamma(\frac{1}{2}n + k + 1)}$$

where $Q_n^* = \frac{1}{2}\sum_{i=1}^{n} a_i^{-1} y_i^2$. The moments of Q_n^* are easily obtained from the rth cumulant of Q_n^* which is $\frac{1}{2}(r-1)!\sum_{i=1}^{n} a_i^{-r}$. Shah and Khatri [52] generalized the Pachares result showing that

$$\Pr\left[\frac{1}{2}\sum_{i=1}^{n} a_i(y_i - b_i)^2 \leqq t\right]$$

$$= \frac{t^{n/2}}{D^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} b_i^2\right) \sum_{j,k=0}^{\infty} \frac{(-1)^j 2^k t^{j+k} E(Q^{*j} L^{*2k})}{j!(2k)!\Gamma(\frac{1}{2}n + j + k + 1)}$$

where $Q^* = \frac{1}{2}\sum_{i=1}^{n} a_i^{-1}(y_i - b_i)^2$, $L^* = \sum_{i=1}^{n} (2a_i)^{-\frac{1}{2}}(y_i - b_i)b_i$. The moments are obtained from the (i, j) cumulant of (Q^*, L^*). Denoting this by $K_{i,j}$ they have found

$$K_{r,0} = (r-1)!\tfrac{1}{2}\textstyle\sum_{i=1}^{n} a_i^{-r}, \qquad r = 1, 2, \cdots$$

$$K_{r,2} = r!\textstyle\sum_{i=1}^{n} b_i^2/(2a_i^{r+1}), \qquad r = 0, 1, 2, \cdots$$

$$K_{r,j} = 0, \qquad j = 1, 3, 4, 5, \cdots, \quad r = 0, 1, 2, \cdots.$$

All three papers give exact expressions for computing bounds on the exact probability when it has been approximated by a finite number of terms.

A number of investigators have obtained $\Pr[\sum_{i=1}^{n} a_i y_i^2 \leqq t]$ by inverting the characteristic function of the quadratic form and using numerical procedures. This method was used by Grad and Solomon [16]. Friberger and Jones [11] give a brief review of some of the numerical methods used by others and present another of their own. Imhof [23] uses the same procedure to evaluate $\Pr[\sum_{i=1}^{n} a_i(y_i - b_i)^2 > t]$.

In another group of papers the distribution function of the quadratic form is expressed as a linear combination of an infinite number of chi-square or noncentral chi-square distribution functions. Robbins [45] showed that

$$\Pr[\sum_{i=1}^{n} a_i y_i^2 \leqq t] = \sum_{k=0}^{\infty} (-1)^k d_k F_{n+2k}(t/a)$$

where $a = (a_1 \cdot a_2 \cdots a_n)^{1/n}$,

$$F_n(t) = \int_0^t \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} u^{\frac{1}{2}n-1} e^{-\frac{1}{2}u}\, du,$$

and the d_k are obtained from the c_k mentioned previously in connection with his infinite series. Robbins and Pitman [46] obtained the slightly different result

$$\Pr[\sum_{i=1}^{n} a_i y_i^2 \leqq t] = \sum_{j=0}^{\infty} c_j F_{n+2j}(t/a_1)$$

where $c_j \geqq 0, j = 0, 1, 2, \cdots, \sum_{j=0}^{\infty} c_j = 1, a_1 = \min_i a_i$, and the c_j are obtained stepwise. Ruben [49] has extended these results, proving

$$\Pr\left[\sum_{i=0}^{n} a_i(y_i - b_i)^2 \leqq t\right] = \sum_{j=0}^{\infty} c_j F_{n+2j}(t/p) = \sum_{j=0}^{\infty} d_j G_{n+2j;\kappa}(t/p),$$

where his c_j and d_j ($j = 0, 1, 2, \cdots$) can be computed several ways, the most convenient being recursively. Here $G_{n;\kappa}(t/p) = H(t^2/p; n, \kappa^2)$ in our notation, that is, it is the integral of the non-central chi-square p.d.f. A discussion is included on the appropriate choice of p. All three papers give bounds on the error after k terms.

Imhof [23] has included another interesting result in his paper. He extends the Pearson approximation getting

$$\Pr[\sum_{i=1}^{n} a_i(y_i - b_i)^2 > t] \cong \Pr[w^2 > t']$$

where w^2 has a chi-square distribution with fractional degrees of freedom $h' = c_2^3/c_3^2$. Here $t' = (t - c_1)(h'/c_2)^{\frac{1}{2}} + h'$ and

$$c_j = \sum_{i=1}^{n} a_i^j(1 + jb_i^2), \qquad j = 1, 2, 3.$$

2. Zero-one damage function, X' probability not concentrated at one point.

2.0 *Introduction.* In Section 0 we observed that when the damage function is equal to unity inside or on C_1 and to zero elsewhere, then (0.1) simplifies to

$$P(\cdot) = \int_{-\infty}^{\infty} \int_{C_1} dF(X)\, dG(X'). \tag{2.1}$$

In general C_1 will depend upon both X' and X. For the cases to be reviewed X will have the density function (0.4).

When $\sigma_i = \sigma$, $i = 1, 2, \cdots, n$ and C_1 is the region $\sum_{i=1}^{n} (x_i - x_i')^2 \leqq R^2$, then $P(\cdot)$ can be simplified somewhat. In Section 1.2 we have already observed that for this situation

$$\int_{C_1} dF(X) = H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right)$$

where $r^2 = \sum_{i=1}^{n} x_i'^2$. Hence in these cases (2.1) becomes

$$P(\cdot) = \int_{-\infty}^{\infty} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) dG(X'). \tag{2.2}$$

From $G(X')$ we can find the distribution function of r/σ, say $Q(r/\sigma)$. Thus (2.2) can be replaced by the single integral

$$P(\cdot) = \int_0^\infty H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) dQ\left(\frac{r}{\sigma}\right). \tag{2.3}$$

The first six cases of this section fall into this category and are further characterized by the following conditions:

Case I. The distribution of X' gives equal weight to each point on $\sum_{i=1}^n x_i'^2 = D^2$, no weight elsewhere.

Case II. X' has density function

$$g(X') = V^{-1}, \sum_{i=1}^n x_i'^2 \leqq D^2$$
$$= 0, \quad \text{elsewhere}$$

where V is the volume of the sphere $\sum_{i=1}^n x_i'^2 \leqq D^2$.

Case III. X' has a density function $g(X')$ taking on the form (in spherical coordinates)

$$p(r, \alpha_1, \cdots, \alpha_{n-1}) = (2D\pi^{n-1})^{-1}, \qquad 0 \leqq r \leqq D$$
$$0 \leqq \alpha_i \leqq \pi, \quad i = 1, \cdots, n-2$$
$$0 \leqq \alpha_{n-1} \leqq 2\pi$$
$$= 0, \quad \text{elsewhere.}$$

Case IV. r/σ has a gamma distribution.

Case V. r^2/σ^2 has a gamma distribution.

Case VI. r/σ has a beta distribution.

In the last two cases considered the variances are not equal and the following conditions hold:

Case VII. C_1 is the region $\sum_{i=1}^n (x_i - x_i')^2 \leqq R^2$ and the density of X' is

$$g(x_1', \cdots, x_n') = \left[(2\pi)^{n/2} \prod_{i=1}^n \sigma_i'\right]^{-1} \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i'/\sigma_i')^2\right].$$

Case VIII. $n = 2$

$$C_1 \text{ is the region } x_1' - \beta_1' < x_1 < x_1' + \beta_1', \quad x_2' - \beta_2' < x_2 < x_2' + \beta_2'$$
$$g(x_1', x_2') = (4\beta_1\beta_2)^{-1}, \ -\beta_1 < x_1' < \beta_1, \ -\beta_2 < x_2' < \beta_2$$
$$= 0, \quad \text{elsewhere.}$$

In Cases I, II, and VIII where X' is uniform over some region, say C_2, $P(\cdot)$ may also be interpreted as the expected overlap of regions C_1 and C_2.

2.1 *Case* I. Here $P(\cdot)$ can be interpreted as the probability that a sphere S_1 of radius R captures a point selected at random on the surface of a sphere S_2 of radius D if the center of S_1 is aimed at the center of S_2 with aiming errors being spherically normally distributed.

Since $r^2 = D^2$, (2.2) reduces to

$$(2.4)\qquad P(\cdot) = \int_{\sum_{i=1}^n x_i'^2 = D^2} H\left(\frac{R^2}{\sigma^2}; n, \frac{D^2}{\sigma^2}\right) dG(X') = H\left(\frac{R^2}{\sigma^2}; n, \frac{D^2}{\sigma^2}\right).$$

Consequently all the tables and methods described in Section 1.2 are available to evaluate this probability.

2.2 *Case* II. Now $P(\cdot)$ can be interpreted as the probability that a sphere S_1 of radius R captures a point selected at random within or on a sphere S_2 of radius D if the center of S_1 is aimed at the center of S_2 with aiming errors being spherically normally distributed.

The integral (2.2) can be written

$$(2.5)\qquad P(\cdot) = \int \cdots \int_{\sum_{i=1}^n x_i'^2 \leq D^2} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{1}{V}\, dx_1' \cdots dx_n'$$

where $V = \pi^{n/2}D^n/\Gamma((n+2)/2)$. To simplify (2.5), change to spherical coordinates. Thus

$$\begin{aligned} x_1' &= r\cos\alpha_1, \\ x_2' &= r\sin\alpha_1\cos\alpha_2, \\ &\vdots \\ x_{n-1}' &= r\sin\alpha_1\cdots\sin\alpha_{n-1}\cos\alpha_{n-1}, \\ x_n' &= r\sin\alpha_1\cdots\cdots\cdots\sin\alpha_{n-1}, \\ |J| &= r^{n-1}(\sin\alpha_1)^{n-2}(\sin\alpha_2)^{n-3}\cdots\sin\alpha_{n-2}, \end{aligned}$$

so that (2.5) becomes

$$P(\cdot) = 2^n \int_0^D \int_0^{\frac{1}{2}\pi} \cdots \int_0^{\frac{1}{2}\pi} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{|J|}{V}\, d\alpha_1 \cdots d\alpha_{n-1}\, dr.$$

Integrating out the α's yields

$$(2.6)\qquad \begin{aligned} P(\cdot) &= \int_0^D H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{nr^{n-1}}{D^n}\, dr \\ &= \int_0^{D/\sigma} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{n(r/\sigma)^{n-1}}{(D/\sigma)^n}\, d(r/\sigma). \end{aligned}$$

Now integrate (2.6) by parts with

$$dv = [n(r/\sigma)^{n-1}/(D/\sigma)^n]\, d(r/\sigma)$$

$$u = H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) = \int_0^{R^2/\sigma^2} \left(\frac{W}{r/\sigma}\right)^{\frac{1}{2}(n-2)} e^{-\frac{1}{2}[W^2+(r/\sigma)^2]} I_{(n-2)/2}\left(\frac{rW}{\sigma}\right) dW^2.$$

After a few routine reductions it is found that

$$du = -\frac{(R/\sigma)^{n/2}}{(r/\sigma)^{(n-2)/2}} I_{n/2}\left(\frac{Rr}{\sigma^2}\right) \exp\left[-(R^2+r^2)/2\sigma^2\right] d(r/\sigma).$$

Thus the integral (2.6) can be changed to

$$(2.7)\quad H\left(\frac{R^2}{\sigma^2}; n, \frac{D^2}{\sigma^2}\right) + \frac{(R/\sigma)^{n/2}}{(D/\sigma)^n} \int_0^{D/\sigma} (r/\sigma)^{(n+2)/2} \cdot \exp[-(R^2 + r^2)/2\sigma^2] I_{n/2}\left(\frac{Rr}{\sigma^2}\right) d(r/\sigma).$$

Another integration by parts with

$$dv = (r/\sigma)\exp[-(R^2 + r^2)/2\sigma^2]\, d(r/\sigma) \quad \text{and}$$
$$u = [(R/\sigma)^{n/2}/(D/\sigma)^n](r/\sigma)^{n/2} I_{n/2}(Rr/\sigma^2),$$

converts (2.7) to

$$H\left(\frac{R^2}{\sigma^2}; n, \frac{D^2}{\sigma^2}\right) - \left(\frac{R/\sigma}{D/\sigma}\right)^{n/2} \exp[-(R^2 + D^2)/2\sigma^2] I_{n/2}\left(\frac{RD}{\sigma^2}\right)$$
$$+ \left(\frac{R/\sigma}{D/\sigma}\right)^n \int_0^{D/\sigma} \frac{1}{2}\left(\frac{r/\sigma}{R/\sigma}\right)^{(n-2)/2} \exp[-(R^2 + r^2)/2\sigma^2] I_{(n-2)/2}\left(\frac{rR}{\sigma^2}\right) 2\left(\frac{r}{\sigma}\right) d\left(\frac{r}{\sigma}\right).$$

The last integral is equivalent to

$$\left(\frac{R/\sigma}{D/\sigma}\right)^n \int_0^{D^2/\sigma^2} \frac{1}{2}\left(\frac{r/\sigma}{R/\sigma}\right)^{(n-2)/2} \exp[-(R^2 + r^2)/2\sigma^2] I_{(n-2)/2}\left(\frac{rR}{\sigma^2}\right) d\left(\frac{r}{\sigma}\right)^2$$
$$= \left(\frac{R/\sigma}{D/\sigma}\right)^n H\left(\frac{D^2}{\sigma^2}; n, \frac{R^2}{\sigma^2}\right).$$

Consequently we may replace (2.5) by

$$(2.8)\quad P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) = H\left(\frac{R^2}{\sigma^2}; n, \frac{D^2}{\sigma^2}\right) - \left(\frac{R/\sigma}{D/\sigma}\right)^{n/2} \exp[-(R^2 + D^2)/2\sigma^2] I_{n/2}\left(\frac{RD}{\sigma^2}\right) + \left(\frac{R/\sigma}{D/\sigma}\right)^n H\left(\frac{D^2}{\sigma^2}; n, \frac{R^2}{\sigma^2}\right)$$

or, alternatively, because of (1.10),

$$(2.9)\quad P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) = H\left(\frac{R^2}{\sigma^2}; n+2, \frac{D^2}{\sigma^2}\right) + \left(\frac{R/\sigma}{D/\sigma}\right)^n H\left(\frac{D^2}{\sigma^2}; n, \frac{R^2}{\sigma^2}\right).$$

The derivation of (2.9) is due to Guenther [20].

When $n = 2$, (2.8) reduces to

$$(2.10)\quad P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) = H\left(\frac{R^2}{\sigma^2}; 2, \frac{D^2}{\sigma^2}\right) - \left(\frac{R/\sigma}{D/\sigma}\right) \exp[-(R^2 + D^2)/2\sigma^2] I_1\left(\frac{RD}{\sigma}\right) + \left(\frac{R/\sigma}{D/\sigma}\right)^2 H\left(\frac{D^2}{\sigma^2}; 2, \frac{R^2}{\sigma^2}\right),$$

a result due to Germond [14]. He has tabulated $P(R/\sigma, D/\sigma)$ to four decimal places for $D/\sigma = 0(0.1)6.5$, $R/\sigma = 0(0.5)3.0(1.0)6.0$ and for $D/\sigma - R/\sigma = -3.2(0.1)3.5$, $R/\sigma = 3(1)6(2)20$. If further values are desired, (2.10) can be evaluated from well tabulated functions. Either the Marcum [28] or the Bell [3] table can be used to evaluate $H(R^2/\sigma^2;\, 2,\, D^2/\sigma^2)$ and $H(D^2/\sigma^2;\, 2,\, R^2/\sigma^2)$. Watson [57] has included a table of $e^{-x}I_1(x)$ in his book. It is given to seven decimal places for $x = 0.00(.02)16.00$.

When $n = 3$, (2.8) reduces to

$$
\begin{aligned}
P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) &= \Phi\left(\frac{D+R}{\sigma}\right) - \Phi\left(\frac{D-R}{\sigma}\right) \\
&\quad + \frac{R^3}{D^3}\left[\Phi\left(\frac{D+R}{\sigma}\right) - \Phi\left(\frac{R-D}{\sigma}\right)\right] + \frac{1}{(D/\sigma)^3} \\
&\quad \cdot \left[\left(\frac{D^2 - RD + R^2}{\sigma^2} - 1\right)\phi\left(\frac{D+R}{\sigma}\right)\right. \\
&\qquad \left. - \left(\frac{D^2 + RD + R^2}{\sigma^2} - 1\right)\phi\left(\frac{D-R}{\sigma}\right)\right],
\end{aligned} \tag{2.11}
$$

a result found in [18] which contains an abbreviated table. The probability $P(R/\sigma, D/\sigma)$ is given to two decimal places for $R/\sigma = 0(.5)3.5$, $D/\sigma = 0(.5)3.5$. Other values can be computed using [34], [35], [36].

There are a number of other ways to evaluate $P(\cdot)$. One of these is to use the Haynam and Leone [21] tables on (2.9). A second is to use the Pearson [41] approximation twice. Thus

$$
P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) = \Pr\,[w_1^2 < M_1] + \left(\frac{R/\sigma}{D/\sigma}\right)^n \Pr\,[w_2^2 < M_2],
$$

where

$$
M_1 = \frac{\dfrac{R^2}{\sigma^2} + \dfrac{D^4/\sigma^4}{n + 2 + 3(D^2/\sigma^2)}}{\dfrac{n + 2 + 3(D^2/\sigma^2)}{n + 2 + 2(D^2/\sigma^2)}}, \qquad M_2 = \frac{\dfrac{D^2}{\sigma^2} + \dfrac{R^4/\sigma^4}{n + 3(R^2/\sigma^2)}}{\dfrac{n + 3(R^2/\sigma^2)}{n + 2(R^2/\sigma^2)}}
$$

and w_1^2 and w_2^2 are distributed as chi-square with fractional degrees of freedom

$$
\begin{aligned}
v_1' &= [n + 2 + 2(D^2/\sigma^2)]^3/[n + 2 + 3(D^2/\sigma^2)]^2, \\
v_2' &= [n + 2(R^2/\sigma^2)]^3/[n + 3(R^2/\sigma^2)]^2.
\end{aligned}
$$

A third way is to substitute the series (1.7) into (2.6), replace $e^{-r^2/2}$ by its series expansion, interchange the order of integration and summation, and obtain

$$
P(\cdot) = n \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (D^2/2\sigma^2)^{i+j}}{i!\, j!\, (n + 2i + 2j)}\, I\left[\frac{R^2/2\sigma^2}{(n/2 + j + 1)^{\frac{1}{2}}}, \frac{n}{2} + j\right] \tag{2.12}
$$

where $I(u, p)$ is the incomplete gamma integral of Pearson [42]. The series (2.12) can be evaluated with a desk calculator.

We note that if X' has a distribution that is uniform over an annulus bounded by $\sum_{i=1}^{n} x_i'^2 = D_1^2$ and $\sum_{i=1}^{n} x_i'^2 = D_2^2$, $D_2 > D_1$, the probabilities can be computed using the results of this section. In this situation the density of r/σ is

$$q\left(\frac{r}{\sigma}\right) = \frac{n(r/\sigma)^{n-1}}{(D_2/\sigma)^n - (D_1/\sigma)^n}, \qquad \frac{D_1}{\sigma} < \frac{r}{\sigma} < \frac{D_2}{\sigma}$$
$$= 0, \quad \text{elsewhere},$$

and

$$(2.13) \quad P(\cdot) = \frac{(D_2)^n}{(D_2)^n - (D_1)^n} P\left(\frac{R}{\sigma}, \frac{D_2}{\sigma}\right) - \frac{(D_1)^n}{(D_2)^n - (D_1)^n} P\left(\frac{R}{\sigma}, \frac{D_1}{\sigma}\right),$$

where $P(R/\sigma, D/\sigma)$ is defined by (2.8) or (2.9).

2.3 *Case* III. The interpretation is the same as for Case II except for the meaning of "at random." In Section 2.2 $G(X')$ was such that the point X' was as likely to be in a volume of a given size as in any other volume of the same size in C_2. Now X' is governed by a probability law under which the spherical coordinates have independent uniform distributions. It is worth pointing out that the result of this section holds whenever r and the vector $(\alpha_1, \cdots \alpha_{n-1})$ are independent, a somewhat more general result.

The integral (2.2) takes the form

$$(2.14) \quad P(\cdot) = \int_{\sum_{i=1}^n x_i'^2 \leq D^2} \cdots \int H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) g(x_1', \cdots, x_n')\, dx_1' \cdots dx_n'.$$

In spherical coordinates this is

$$P(\cdot) = \int_0^D \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{1}{2D\pi^{n-1}}\, d\alpha_1 \cdots d\alpha_{n-1}\, dr$$
$$= \int_0^D H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{1}{D}\, dr.$$

Thus we may write for (2.14)

$$(2.15) \qquad P\left(\frac{R}{\sigma}, \frac{D}{\sigma}\right) = \int_0^{D/\sigma} H\left(\frac{R^2}{\sigma^2}; n, \frac{r^2}{\sigma^2}\right) \frac{1}{D/\sigma}\, d(r/\sigma).$$

For $n = 2$, (2.15) has been tabulated by Bell [2] to four decimal places for $D/\sigma = 0(0.1)6.5$, $R/\sigma = 0(0.5)3.0(1.0)6.0$ and for $D/\sigma - R/\sigma = -3.2(0.1)3.5$, $R/\sigma = 3(1)6(2)20$.

A series similar to (2.12) can be derived by following the steps outlined there. It is

$$P(\cdot) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (D^2/2\sigma^2)^{i+j}}{i!\, j!\, (2i + 2j + 1)} I\left[\frac{R^2/2\sigma^2}{(n/2 + j + 1)^{\frac{1}{2}}}, \frac{n}{2} + j\right].$$

The preceding formulas can be used to give the probability of capturing a randomly selected point in an annulus bounded by $\sum_{i=1}^n x_i'^2 = D_1^2$ and $\sum_{i=1}^n x_i'^2 = D_2^2$, $D_2 > D_1$, where "randomly" is interpreted as in this section. For this case

$$q(r/\sigma) = 1/[(D_2/\sigma) - (D_1/\sigma)], \qquad D_1/\sigma < r/\sigma < D_2/\sigma$$

and

$$(2.16) \qquad P(\cdot) = \frac{D_2}{D_2 - D_1} P\left(\frac{R}{\sigma}, \frac{D_2}{\sigma}\right) - \frac{D_1}{D_2 - D_1} P\left(\frac{R}{\sigma}, \frac{D_1}{\sigma}\right),$$

where $P(R/\sigma, D/\sigma)$ is defined by (2.15).

2.4 *Cases* IV, V, VI. The densities governing the behavior of r are respectively

$$q_1(r/\sigma) = [1/\beta^\alpha\Gamma(\alpha)](r/\sigma)^{\alpha-1}e^{-r/\sigma\beta}, \qquad r/\sigma > 0$$

$$q_2(r^2/\sigma^2) = [1/\beta^\alpha\Gamma(\alpha)](r^2/\sigma^2)^{\alpha-1}e^{-r^2/\sigma^2\beta}, \qquad r^2/\sigma^2 > 0$$

and

$$q_3(r/\sigma) = [1/d^{a+b-1}\beta(a, b)](r/\sigma)^{a-1}(d - r/\sigma)^{b-1}, \quad 0 < r/\sigma < d$$

where α and β are known constants in the gamma distribution and d, a, and b are known constants in the beta distribution. Following the procedure used to derive (2.12), one gets for these cases

$$P_1(\cdot) = \frac{2^{\alpha/2}}{2\beta^\alpha\Gamma(\alpha)} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^i\Gamma[(\alpha + 2j + i)/2]}{i!\,j!\,\beta^i} I\left[\frac{R^2/2\sigma^2}{(n/2 + j + 1)^{\frac{1}{2}}}, \frac{n}{2} + j\right],$$

$$P_2(\cdot) = \left(\frac{2}{B}\right)^\alpha \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha + j)}{j!\,(1 + 2/B)^{\alpha+j}} I\left[\frac{R^2/2\sigma^2}{(n/2 + j + 1)^{\frac{1}{2}}}, \frac{n}{2} + j\right],$$

$$P_3(\cdot) = \frac{1}{\beta(a, b)} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^i(d^2/2)^{i+j}\beta(a + 2i + 2j, b)}{i!\,j!} \cdot I\left[\frac{R^2/2\sigma^2}{(n/2 + j + 1)^{\frac{1}{2}}}, \frac{n}{2} + j\right].$$

The three formulas for $n = 2, 3$ have been given by McNolty [30]. If α, a, and b are integers then the complete gamma functions can be evaluated from [50]. If this is not the situation then [56] can be used to evaluate $\log \Gamma(p)$, $1 < p < 2$ (see the introduction of [42] for further suggestions).

2.5 *Case* VII. The integral (2.1) takes the form

$$P(\cdot) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underset{\sum_{i=1}^{n} (x_i - x_i')^2 \leq R^2}{\int \cdots \int} \cdot \frac{1}{(2\pi)^n \prod_{i=1}^{n} \sigma_i \sigma_i'} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i^2}{\sigma_i^2} + \frac{x_i'^2}{\sigma_i'^2}\right)\right] dx_1 \cdots dx_n\, dx_1' \cdots dx_n' .$$

If we let $x_i - x_i' = Z_i$ and reverse the order of integration this becomes upon integrating out $x_1', \cdots, x_n'$

$$P(\cdot) = \int_{\sum_{i=1}^{n} Z_i^2 \leq R^2} \cdots \int \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} (\sigma_i^2 + \sigma_i'^2)^{1/2}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} \frac{Z_i^2}{\sigma_i^2 + \sigma_i'^2}\right] dZ_1 \cdots dZ_n .$$

Making the substitution $Z_i = y_i(\sigma_i^2 + \sigma_i'^2)^{\frac{1}{2}}$ we get

$$P(\cdot) = \int_{\sum_{i=1}^{n} y_i^2(\sigma_i^2+\sigma_i'^2) \leq R^2} \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} y_i^2\right] dy_1 \cdots dy_n . \tag{2.17}$$

The evaluation of (2.17) has already been discussed in Section 1.3 (and in 1.1 if $\sigma_i^2 + \sigma_i'^2 = \sigma^2$, $i = 1, 2, \cdots, n$).

2.6 *Case* VIII. In this case (2.1) is interpreted as the probability that a rectangle C_1 captures a point selected at random in a rectangle C_2 (defined by $-\beta_1 < x_1' < \beta_1$, $-\beta_2 < x_2' < \beta_2$) if the center of C_1 is aimed at the center of C_2 with aiming errors being independently normally distributed. It is assumed that C_1 lands on the plane of C_2 in such a way that the corresponding sides of the two rectangles are parallel.

The integral (2.1) reduces to

$$\begin{aligned} P(\cdot) &= \frac{1}{\beta_1 \beta_2} \int_{-\beta_1}^{\beta_1} \int_{-\beta_2}^{\beta_2} \int_{x_1'-\beta_1'}^{x_1'+\beta_1'} \int_{x_2'-\beta_2'}^{x_2'+\beta_2'} \frac{1}{2\pi\sigma_1 \sigma_2} \\ &\qquad \cdot \exp\left[-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right] dx_2\, dx_1\, dx_2'\, dx_1' \\ &= \left[\int_{-\beta_1}^{\beta_1} \int_{x_1'-\beta_1'}^{x_1'+\beta_1'} \frac{1}{2\beta_1(2\pi)^{\frac{1}{2}}\sigma_1} \exp\left(-x_1^2/2\sigma_1^2\right) dx_1\, dx_1'\right] \\ &\qquad \cdot \left[\int_{-\beta_2}^{\beta_2} \int_{x_2'-\beta_2'}^{x_2+\beta_2'} \frac{1}{2\beta_2(2\pi)^{\frac{1}{2}}\sigma_2} \exp\left(-x_2^2/2\sigma_2^2\right) dx_2\, dx_2'\right] \\ &= [A(\beta_1', \beta_1, \sigma_1)][A(\beta_2', \beta_2, \sigma_2]. \end{aligned} \tag{2.18}$$

After a routine integration by parts and some reductions it is found that

$$\begin{aligned} A(\beta_1', \beta_1, \sigma_1) &= A^*\left(\frac{\beta_1'}{\sigma_1}, \frac{\beta_1}{\sigma_1}\right) = \frac{1}{\beta_1/\sigma_1}\left\{\frac{\beta_1' + \beta_1}{\sigma_1} \Phi\left(\frac{\beta_1' + \beta_1}{\sigma_1}\right)\right. \\ &\quad \left. - \frac{\beta_1' - \beta_1}{\sigma_1} \Phi\left(\frac{\beta_1' - \beta_1}{\sigma_1}\right) - \frac{\beta_1}{\sigma_1} - \left[\phi\left(\frac{\beta_1' - \beta_1}{\sigma_1}\right) - \phi\left(\frac{\beta_1' + \beta_1}{\sigma_1}\right)\right]\right\}. \end{aligned} \tag{2.19}$$

Obviously $A(\beta_2', \beta_2, \sigma_2)$ is obtained from (2.19) by changing all subscripts from 1's to 2's. Thus (2.18) can be written

$$P(\cdot) = [A^*(\beta_1'/\sigma_1, \beta_1/\sigma_1)][A^*(\beta_2'/\sigma_2, \beta_2/\sigma_2)]. \tag{2.20}$$

Germond has given an equivalent result in [13].

Apparently there are no tables for (2.20). However, since only well tabulated functions are involved, the calculation of the probability for given values of the parameter is routinely done. Germond's report [13] includes a graph for the special case in which C_1 and C_2 are squares and $\sigma_1 = \sigma_2 = \sigma$. He has plotted $P(\cdot) = P(\beta_1'/\sigma, \beta_1/\sigma)$ against the ratio

$$d = \frac{\text{area of square } C_1}{\text{area of square } C_2} = \frac{\beta_1'^2}{\beta_1^2} \qquad \text{for} \quad 0 \leqq d \leqq 16.$$

Curves are presented for $\beta_1/\sigma = \frac{1}{2}, 1, 2$.

3. Other damage functions.

3.0 *Introduction.* In Section 0 we observed that the most general form of coverage problem as defined by (0.1) can be interpreted as the probability that a weapon aimed at the origin damages a point target if aiming errors are governed by the distribution $F(X)$, the target's position is determined by the distribution $G(X')$ and the damage function is $P_1(X, X')$. If X' has density

$$\begin{aligned} g(X') &= V^{-1}, \quad X' \,\varepsilon\, C_2 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

then (0.1) may also be interpreted as the expected proportion of a target damaged where the target has a volume (or area) V.

In Sections 1 and 2 the conditional damage function was assumed to be

$$\begin{aligned} P_1(X, X') &= 1, \quad X \,\varepsilon\, C_1 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

In other words, the point X' was either damaged or it was not. A number of other reasonable damage functions can be constructed. It would seem desirable to have $P_1(X, X')$ be a non-increasing function of the distance from X' to X. The choice

$$P_1(X, X') = \exp\left[-\sum_{i=1}^{n} (x_i' - x_i)^2/2B^2\right], \tag{3.1}$$

where B^2 is constant, has this property together with the additional advantage of producing some relatively simple results in the special cases we are about to consider. Another selection which has some merit is

$$\begin{aligned} P_1(X, X') &= 1, & \sum_{i=1}^{n} (x_i' - x_i)^2 \leqq R^2 \\ &= \exp\left\{-\left[\sum_{i=1}^{n} (x_i' - x_i)^2 - R^2\right] \Big/ 2B^2\right\}, & \sum_{i=1}^{n} (x_i' - x_i)^2 > R^2. \end{aligned} \tag{3.2}$$

Thus the probability of damaging X' is 1 if X is within a distance R and a decreasing function of the distance between the points otherwise.

Throughout this section we will assume that the damage function is of the form (3.1). The constant B^2 can be chosen in various ways depending upon one's objective. If it is known that 50 per cent damage occurs at a distance d_0 from the point of impact, then we might select B^2 so that $\exp(-d_0^2/2B^2) = .50$ or $B^2 = .721\, d_0^2$. We will also assume that X has the density (0.4) and that X' has a distribution with non-zero probability in a region C_2. Consequently (0.1) may be written

$$P(\cdot) = \int_{C_2} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i} \cdot \exp\left\{ -\frac{1}{2} \left[\sum_{i=1}^{n} \frac{(x_i' - x_i)^2}{B^2} + \sum_{i=1}^{n} \frac{x_i^2}{\sigma_i^2} \right] \right\} dx_1 \cdots dx_n \right] dG(X') \tag{3.3}$$

which reduces to

$$P(\cdot) = \int_{C_2} \frac{B^n}{\prod_{i=1}^{n} (\sigma_i^2 + B^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2}{\sigma_i^2 + B^2} \right) dG(X'). \tag{3.4}$$

The integral (3.4) will be evaluated for four cases. These are:

Case I. $\sigma_i = \sigma$, $i = 1, 2, \cdots, n$. C_2 is the region $\sum_{i=1}^{n} x_i'^2 = D^2$. X' has distribution which gives equal weight to each point on $\sum_{i=1}^{n} x_i'^2 = D^2$, no weight elsewhere.

Case II. C_2 is the region $\sum_{i=1}^{n} x_i'^2 \leqq D^2$, X' has density function

$$g(x_1', \cdots, x_n') = V^{-1}, \quad \sum_{i=1}^{n} x_i'^2 \leqq D^2$$
$$= 0, \quad \text{elsewhere}$$

where V is the volume of the sphere $\sum_{i=1}^{n} x_i'^2 \leqq D^2$.

Case III. $\sigma_i = \sigma$, $i = 1, 2, \cdots, n$. C_2 is the region $\sum_{i=1}^{n} x_i'^2 \leqq D^2$. X' has a density function taking on the form (in spherical coordinates)

$$p(r, \alpha_1, \cdots, \alpha_{n-1}) = (2D\pi^{n-1})^{-1}, \quad 0 \leqq r \leqq D$$
$$0 \leqq \alpha_i \leqq \pi, \quad i = 1, 2, \cdots, n-2$$
$$0 \leqq \alpha_{n-1} \leqq 2\pi$$
$$= 0, \quad \text{elsewhere.}$$

Case IV. $\sigma_i = \sigma$, $i = 1, 2, \cdots, n$. r^2/σ^2 has a gamma distribution.

Case V. C_2 is the entire X' space and

$$g(x_1', \cdots, x_n') = \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i'} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} (x_i'/\sigma_i')^2 \right].$$

Morgenthaler [32], [33] has suggested the possibility of using an integral of the type (0.1), particularly one of the form (3.4), to approximate the corresponding problem arising in Section 2. For example, to obtain an approximate solution for Case III of Section 2, we might use the result for Case III of this section with B^2 being chosen to make the approximation as good as possible.

3.1 *Case* I. Since $\sum_{i=1}^{n} x_i'^2 = D^2$, (3.4) reduces to

$$P(\cdot) = \frac{B^n}{(B^2 + \sigma^2)^{n/2}} \exp\left[-\frac{D^2}{2(B^2+\sigma^2)}\right] \int_{\sum_{i=1}^{n} x_i'^2 = D^2} dG(X'). \tag{3.5}$$

Because the multiple integral has the value 1, (3.5) can be written

$$P\left(\frac{B}{\sigma}, \frac{D}{\sigma}\right) = \frac{(B/\sigma)^n}{[1 + B^2/\sigma^2]^{n/2}} \exp\left\{-\frac{(D/\sigma)^2}{2[1 + B^2/\sigma^2]}\right\}, \tag{3.6}$$

The evaluation of (3.6) can be accomplished by using [34] and [35].

The integral (3.5) can be interpreted as the probability that a point selected at random on the surface of a sphere of radius D is captured or damaged by a weapon aimed at the center of the sphere if aiming errors are spherically normally distributed and the damage function is (3.1). It also represents the expected proportion of the surface area damaged if aiming errors and damage behavior are so governed.

We note that the results of this section hold no matter how $G(X')$ distributes the probability on the surface of the sphere.

3.2 *Case* II. Now $P(\cdot)$ is interpreted as the probability that a point selected "at random" within or on a sphere of radius D is captured or damaged by a weapon aimed at the center of the sphere if aiming errors have an elliptical normal distribution and damage behavior is governed by (3.1). "At random" should be interpreted as in Paragraph 2.2. In addition $P(\cdot)$ represents the expected proportion of the volume destroyed under these conditions.

The integral (3.4) becomes

$$P(\cdot) = \frac{2^{n/2} B^n \Gamma((n+2)/2)}{D^n} \left[\int \cdots \int_{\sum_{i=1}^{n} x_i'^2 \leq D^2} \cdot \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} (\sigma_i^2 + B^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{x_i'^2}{\sigma_i^2 + B^2}\right) dx_1' \cdots dx_n' \right] \tag{3.7}$$

or

$$P(\cdot) = \frac{2^{n/2} B^n \Gamma((n+2)/2)}{D^n} \left[\int \cdots \int_{\sum_{i=1}^{n} (\sigma_i^2 + B^2) y_i^2 \leq D^2} \cdot \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} y_i^2\right) dy_1 \cdots dy_n \right]. \tag{3.8}$$

The integral in the bracket has been discussed in Paragraph 1.3 (in 1.1 if $\sigma_i = \sigma$, $i = 1, 2, \cdots, n$).

If X' has a distribution that is uniform over an annulus, then $P(\cdot)$ can still be evaluated from (2.13) using (3.8) in place of $P(R/\sigma, D/\sigma)$.

3.3 *Case* III. Changing to spherical coordinates (3.4) becomes

$$P(\cdot) = \frac{B^n}{(\sigma^2 + B^2)^{n/2}} \int_0^D \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \exp\left[-\frac{r^2}{2(\sigma^2 + B^2)}\right] \cdot \frac{1}{2D\pi^{n-1}} \, d\alpha_1 \cdots d\alpha_{n-1} \, dr$$

$$(3.9) \qquad = \frac{B^n}{(\sigma^2 + B^2)^{n/2}} \int_0^D \exp\left[-\frac{r^2}{2(\sigma^2 + B^2)}\right] dr$$

$$= \frac{(2\pi)^{\frac{1}{2}} B^n}{(\sigma^2 + B^2)^{n/2}} \int_0^{D/(\sigma^2+B^2)^{\frac{1}{2}}} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) \, dt.$$

Hence (3.9) can be written

$$(3.10) \quad P\left(\frac{B}{\sigma}, \frac{D}{\sigma}\right) = \frac{(2\pi)^{\frac{1}{2}}(B/\sigma)^n}{(D/\sigma)(1 + B^2/\sigma^2)^{(n-1)/2}} \left\{\Phi\left[\frac{D/\sigma}{(1 + B^2/\sigma^2)^{\frac{1}{2}}}\right] - \frac{1}{2}\right\}.$$

As in Section 2.3 the above results hold whenever r and the vector $(\alpha_1, \cdots, \alpha_{n-1})$ are independent allowing $p(r, \alpha_1, \cdots, \alpha_{n-1})$ to take on a somewhat more general form than specified in Section 3.0.

The interpretation of (3.9) is the same as for (3.7) except for the fact that the meaning of "at random" should be the one given in Paragraph 2.3. For an annulus (2.16) still applies if $P(R/\sigma, D/\sigma)$ is replaced by the $P(B/\sigma, D/\sigma)$ of (3.10).

3.4 *Case* IV. When r^2/σ^2 has a gamma distribution (3.4) reduces to

$$P(\cdot) = \int_0^\infty \frac{B^n}{(\sigma^2 + B^2)^{n/2}} \exp\left(-\frac{1}{2}\frac{r^2}{\sigma^2 + B^2}\right) \frac{(r^2/\sigma^2)^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} \exp\left(-\frac{r^2}{\sigma^2\beta}\right) d\left(\frac{r^2}{\sigma^2}\right).$$

Letting $r^2/\sigma^2 = 2u/[(1 + B^2/\sigma^2)^{-1} + 2/\beta]$, this becomes

$$(3.11) \qquad P(\cdot) = 2^\alpha B^n \Gamma(\alpha)/(\sigma^2 + B^2)^{n/2}[\sigma^2/(\sigma^2 + B^2) + 2/\beta]^\alpha.$$

If r/σ has a gamma or beta distribution, series expansions can be obtained as in Paragraph 2.4.

3.5 *Case* V. Now (3.4) may be written

$$P(\cdot) = \left[B^n \Big/ \prod_{i=1}^n (\sigma_i^2 + B^2)^{\frac{1}{2}}\right] \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[1 \Big/ (2\pi)^{n/2} \prod_{i=1}^n \sigma_i'\right] \cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^n x_i'^2 \left[\frac{\sigma_i^2 + \sigma_i'^2 + B^2}{(\sigma_i^2 + B^2)\sigma_i'^2}\right]\right\} dx_1' \cdots dx_n'$$

which is readily simplified to

$$(3.12) \qquad P(\cdot) = B^n \Big/ \prod_{i=1}^n (\sigma_i^2 + \sigma_i'^2 + B^2)^{\frac{1}{2}}.$$

If $\sigma_i^2 + \sigma_i'^2 = \sigma^2$ (3.12) reduces to

$$P\left(\frac{B}{\sigma}\right) = \left[\frac{B^2/\sigma^2}{1 + B^2/\sigma^2}\right]^{n/2}.$$

REFERENCES

[1] ABDEL-ATY, S. H. (1954). Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution. *Biometrika* **41** 538–540.

[2] BELL, STOUGHTON (1962). Tables of some expected weighted coverage functions. Sandia Corporation Technical Memorandum No. SCTM-265-62(54), Albuquerque, New Mexico.

[3] BELL AIRCRAFT CORPORATION (1956). Tables of circular normal probabilities. Report No. 02-949-106, Operations Analysis Group, Dynamics Section, Bell Aircraft Corporation, Buffalo, New York.

[4] DIDONATO, A. R. and JARNAGIN, M. P. (1960). Integration of the general bivariate gaussian distribution over an offset ellipse. N. W. L. Report No. 1710, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

[5] DIDONATO, A. R. and JARNAGIN, M. P. (1961). Integration of the general bivariate gaussian distribution over an offset circle. *Math. Comput.* **15** 375–382.

[6] DIDONATO, A. R. and JARNAGIN, M. P. (1962). A method for computing the generalized circular error function and circular coverage function. N. W. L. Report No. 1768, U. S. Naval Weapons Laboratory, Dahlgren, Virginia.

[7] DIDONATO, A. R. and JARNAGIN, M. P. (1962). A method for computing the circular coverage function. *Math. Comput.* **16** 347–355.

[8] DIXON, W. J. and MASSEY, F. J., JR. (1957). *Introduction to Statistical Analysis*, (2nd ed.). McGraw-Hill, New York.

[9] ESPERTI, R. V. (1960). Tables of the elliptical normal probability function. Defense Systems Division, General Motors Corporation, Detroit, Michigan.

[10] FETTIS, H. E. (1957). Some mathematical identities and numerical methods relating to the bivariate normal probability for circular regions. WADC Technical Note 57-383, Wright Air Force Development Center, Wright Patterson Air Force Base, Ohio.

[11] FREIBERGER, WALTER F. and JONES, RICHARD H. (1960). Computation of the frequency function of a quadratic form in random normal variables. *J. Assoc. Comput. Mach.* **7** 245–250.

[12] GERMOND, H. H. (1949). Integration of the gaussian distribution over an offset ellipse. Rand Report No. P-94, The Rand Corporation, Santa Monica, California.

[13] GERMOND, H. H. (1949). Target coverage. Rand Report No. RM-145, The Rand Corporation, Santa Monica, California.

[14] GERMOND, H. H. (1950). The circular coverage function. Rand Report No. RM-330, The Rand Corporation, Santa Monica, California.

[15] GILLILAND, DENNIS C. (1962). Integral of the bivariate normal distribution over an offset circle. *J. Amer. Statist. Assoc.* **57** 758–767.

[16] GRAD, ARTHUR and SOLOMON, HERBERT (1955). Distribution of quadratic forms and some applications. *Ann. Math. Statist.* **26** 464–477.

[17] GRAYBILL, FRANKLIN A. (1961). *An Introduction to Linear Models*, (V.I.). McGraw-Hill, New York.

[18] GUENTHER, W. C. (1961). On the probability of capturing a randomly selected point in three dimensions. *SIAM Rev.* **3** 247–250.

[19] GUENTHER, W. C. (1962). An n-dimensional coverage problem. Unpublished. Univ. of Wyoming.

[20] GUENTHER, W. C. (1962). A generalization of the integral of the circular coverage function. To appear in *Amer. Math. Monthly*.

[21] Haynam, G. E. and Leone, F. C. (1962). Tables of cumulative non-central chi-square distribution. Unpublished. Case Institute of Technology.

[22] Harter, H. Leon (1960). Circular error probabilities. *J. Amer. Statist. Assoc.* **55** 723–731.

[23] Imhof, J. P. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika* **48** 419–426.

[24] Johnson, N. L. (1959). On an extension of the connexion between Poisson and χ^2 distributions. *Biometrika* **46** 352–363.

[25] Kleinecke, D. C. (1962). A result concerning the coverage of central ellipses. Technical Memorandum CDRP-182-104, Office of Civil Defense, Department of Defense, Washington, D. C.

[26] Lowe, J. R. (1960). A table of the integral of the bivariate normal distribution over an offset circle. *J. Roy. Statist. Soc. Ser. B* **22** 177–187.

[27] Mann, H. B. (1949). *Analysis and Design of Experiments.* Dover Publications, Inc., New York.

[28] Marcum, J. I. (1950). Table of Q-functions. Rand Report No. RM-339, The Rand Corporation, Santa Monica, California.

[29] Marsaglia, George (1960). Tables of the distribution of quadratic forms of ranks two and three. Boeing Scientific Research Laboratories Report No. D1-82-0015-1, Seattle, Washington.

[30] McNolty, Frank. (1962). Kill probability when the weapon bias is randomly distributed. *Operations Res.* **10** 693–701.

[31] Miser, Hugh J. (1951). Bombing accuracy and damage from atomic bombs: an introduction. Operations Analysis Technical Memorandum No. 25, USAF, Washington, D.C.

[32] Morgenthaler, George W. (1959). Some target coverage problems. Report No. P-59-69, The Martin Company, Denver, Colorado.

[33] Morgenthaler, George W. (1961). Some circular coverage problems. *Biometrika* **48** 313–324.

[34] National Bureau of Standards (1951). *Tables of the Exponential Function e^x.* N.B.S. Applied Mathematics Series 14, U S. Govt. Printing Office, Washington, D. C.

[35] National Bureau of Standards (1955). *Tables of the Descending Exponential.* N.B.S. Applied Mathematics Series 46, U. S. Govt. Printing Office, Washington, D. C.

[36] National Bureau of Standards (1953). *Tables of Normal Probability Functions.* N.B.S. Applied Mathematics Series 23, U. S. Govt. Printing Office, Washington, D. C.

[37] Oberg, E. N. (1947). Approximate formulas for the radii of circles which include a specified fraction of a normal bivariate distribution. *Ann. Math. Statist.* **18** 442–447.

[38] Owen, Donald B. (1962). *Handbook of Statistical Tables.* Addison-Wesley, Massachusetts.

[39] Pachares, James (1955). Note on the distribution of a definite quadratic form. *Ann. Math. Statist.* **26** 128–131.

[40] Patnaik, P. B. (1949). The non-central χ^2 and F distributions and their applications. *Biometrika* **36** 202–232.

[41] Pearson, E. S. (1959). Note on an approximation to the distribution of non-central χ^2. *Biometrika* **46** 364.

[42] Pearson, Karl (1951). *Tables of the Incomplete Gamma Function.* Cambridge Univ. Press.

[43] Quenouille, M. H. (1949). The evaluation of probabilities in a normal multivariate distribution with special reference to the correlation ratio. *Proc. Edinburgh Math. Soc.* **8** 95–100.

[44] RAND CORPORATION (1952). Offset circle probabilities. Rand Report No. R-234, Computor Sciences Department, The Rand Corporation, Santa Monica, California.

[45] ROBBINS, HERBERT (1948). The distribution of a definite quadratic form. *Ann. Math. Statist.* **19** 266–270.

[46] ROBBINS, HERBERT and PITMAN, E. T. G. (1949). Application of the method of mixtures to quadratic forms in normal variates. *Ann. Math. Statist.* **20** 552–560.

[47] ROSENTHAL, G. W. and RODDEN, I. J. (1961). Tables of the integral of the elliptical bivariate normal distribution over offset circles. Lockheed Report No. LMSD-800619, Sunnyvale, Calif.

[48] RUBEN, HAROLD (1960). Probability content of regions under spherical normal distribution, I. *Ann. Math. Statist.* **31** 598–618.

[49] RUBEN, HAROLD (1962). Probability content of regions under spherical normal distributions, IV: the distribution of homogeneous and non-homogeneous quadratic functions of normal variables. *Ann. Math. Statist.* **33** 542–570.

[50] SALZAR, HERBERT (1951). Tables of $N!$ and $\Gamma(N + \frac{1}{2})$ for the first thousand values of N. U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 16.

[51] SANDIA CORPORATION and GUPTA, SHANTI S. (1962). The coverage problem. Sandia Report SCR-443. Sandia Corporation, Albuquerque, New Mexico.

[52] SHAH, B. K. and KHATRI, C. G. (1961). Distribution of a definite quadratic form for non-central normal variates. *Ann. Math. Statist.* **32** 883–887.

[53] SOLOMON, HERBERT (1953). Distribution of the measure of a random two-dimensional set. *Ann. Math. Statist.* **24** 650–656.

[54] SOLOMON, HERBERT (1960). Distribution of quadratic forms, tables and applications. Technical Report No. 45, Applied Mathematics and Statistics Laboratories, Stanford Univ.

[55] STECK, G. P. (1958). Uses of tables of the circular coverage function. Technical Memorandum 214-58(51), Sandia Corporation, Albuquerque, New Mexico.

[56] TRACTS FOR COMPUTERS, No IV (1921), Cambridge Univ. Press.

[57] WATSON, G. N. (1952). *A Treatise on the Theory of Bessel Functions*, (2nd ed.). Cambridge Univ. Press.

[58] WEINGARTEN, G. and DIDONATO, A. R. (1961). A table of generalized circular error. *Math. Comput.* **15** 169–173.