ASYMPTOTIC BEHAVIOR OF EXPECTED SAMPLE SIZE IN CERTAIN ONE SIDED TESTS[1]

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0. Summary. Let R be the set of real numbers, $\mathcal{B}_1$ the set of Borel sets of R, and μ a σ-finite nonnegative measure on $\mathcal{B}_1$. Let Ω be an open real number interval (which may be infinite). Throughout we consider a Koopman-Darmois family

(1) $$\{h(\theta) \exp(\theta x), \theta \varepsilon \Omega\}$$

of generalized probability density functions on the measure space $(R, \mathcal{B}_1, \mu)$.

We consider one sided tests T of the hypothesis $\theta < 0$ against the alternative $\theta > 0$. In general, in this paper, T will be a sequential procedure. Associated with T is a stopping variable N (mention of the dependence of N on T is usually omitted). $N \geqq 0$. $N = n$ means that sampling stopped after n observations and a decision was made. In this context we consider ∞ to be an integer, and $N = \infty$ means that sampling does not stop.

In the discussion of Section 1 we will assume that if $\theta \varepsilon \Omega$ and $\theta \neq 0$ then $P_\theta(N < \infty) = 1$, that is, sampling stops with probability one. The reason for the exclusion of $\theta = 0$ will become apparent in Section 1.

We will be concerned with two events, decide $\theta < 0$, and, decide $\theta > 0$. The main result of this paper may be stated as follows.

THEOREM 1. *Suppose* $(R, \mathcal{B}_1, \mu)$, Ω, *and* $\{h(\theta) \exp(\theta x), \theta \varepsilon \Omega\}$ *are as described above. Define*

(2) $$\mu_\theta = \int_{-\infty}^{\infty} h(\theta) x \exp(\theta x) \mu(dx), \qquad \sigma^2 = \int_{-\infty}^{\infty} h(0) x^2 \mu(dx),$$

and assume $\mu_0 = 0$. *Suppose* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *and*

(3) $$\sup_{\theta > 0} P_\theta (\text{decide } \theta < 0) \leqq \beta; \qquad \sup_{\theta < 0} P_\theta (\text{decide } \theta > 0) \leqq \alpha.$$

Then

(4) $$\limsup_{\theta \to 0+} \mu_\theta^2 |\log|\log|\mu_\theta|||^{-1} E_\theta N \geqq 2\sigma^2 P_0(N = \infty); \qquad \limsup_{\theta \to 0-} \mu_\theta^2 |\log|\log|\mu_\theta|||^{-1} E_\theta N \geqq 2\sigma^2 P_0(N = \infty).$$

Received 8 June 1961; revised 13 April 1963.

[1] This research was supported by the United States Navy under Contract No. NR 401(03) monitored by the Office of Naval Research.

If $\alpha + \beta < 1$ there is a generalized sequential probability ratio test T with stopping variable N such that for the test T,

$$P_0(N = \infty) = 1 - (\alpha + \beta); \quad (3) \text{ holds;} \tag{5}$$

for the test T,

$$\lim_{\theta\to 0} \mu_\theta^2 |\log|\log|\mu_\theta|||^{-1} E_\theta N = 2\sigma^2 P_0(N = \infty). \tag{6}$$

For all tests T, if $P_0(N = \infty) > 0$ then $\lim_{\theta\to 0} \theta^2 E_\theta N = \infty$.

In Section 1, (7) and (8), it is shown that $P_0(N = \infty) \geqq 1 - \alpha - \beta$. Consequently the relations (4) and (5) of Theorem 1 are not vacuous.

We were led to formulate Theorem 1 by a problem of constructing bounded length confidence intervals. The relationship is explained in Section 2. The proof of Theorem 1 is given in Section 3.

1. Introduction. If $\theta > 0$, P_θ (decide $\theta < 0$) is sometimes called the probability of an error of Type II; likewise, if $\theta < 0$, P_θ (decide $\theta > 0$) is the probability of an error of Type I. We refer here to the description of the testing problem given in Section 0. The main hypothesis of Theorem 1 is that these probabilities should have bounds β and α respectively.

When, in a testing problem, one asks for such upper bounds on the probabilities of wrong decisions, one has a problem which has been termed "Distinguishability of sets of distributions" by Hoeffding and Wolfowitz [7]. In the problem of Theorem 1 one might expect $\lim_{\theta\to 0} E_\theta N = \infty$ since if θ is near zero in value it will be difficult to distinguish on the basis of the observations whether $\theta > 0$ or $\theta < 0$. It is the purpose of Theorem 1 to give a measure of how difficult it is to distinguish between $\theta > 0$ and $\theta < 0$ by giving an asymptotic inequality on $E_\theta N$ as θ tends to zero. To show that the inequality obtained is best in some sense, we show that the lower bound is attained for certain generalized sequential probability ratio tests.

To obtain some feeling of the asymptotic order of magnitude one might expect it is of interest to apply the lower bounds for $E_\theta N$ developed by Wald [10]. If we consider α and β fixed and for the family of generalized density functions (1) test the hypothesis $\theta = t$ against the hypothesis $\theta = -t$ then as $t \to 0+$ one finds an asymptotic lower bound for $E_t N$ which is $c/(\mu_t^2)$, c a constant. The lower bound of Theorem 1 differs by a magnitude of $|\log|\log|\mu_t|||$.

If one tries to prove an analogue of Theorem 1 for other one sided testing problems one quickly discovers that the "nice" results of Theorem 1 are very dependent on the special form of the Koopman-Darmois generalized density functions (1). In fact, suppose $\theta(\cdot)$ is a continuous and strictly increasing function of $\omega \varepsilon \Omega$, $\theta(0) = 0$. We may generalize Theorem 1 by allowing

$$\{h(\theta(\omega)) \exp (\theta(\omega)x), \omega \varepsilon \Omega\}$$

to be the family of generalized density functions. It is not difficult to show that (see (2)) $\lim_{t\to 0} \mu_t/t = \sigma^2$. Theorem 1 implies

$$\limsup_{\omega\to 0} (\theta(\omega))^2 |\log|\log|\theta(\omega)|||^{-1} E_\omega N \geqq (2/\sigma^2) P_0(N = \infty).$$

This lower bound is achieved by a generalized sequential probability test. Therefore by a suitable choice of $\theta(\cdot)$, $E_{(\cdot)}N$ may be made to have as large or small order of magnitude as desired as $\omega \to 0$.

Suppose $f \geqq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. We consider briefly the translation parameter problem of testing the hypothesis $\theta < 0$ against the hypothesis $\theta > 0$ for the family $\{f(\cdot - \theta), -\infty < \theta < \infty\}$ of density functions. We will suppose $\beta > 0$ and $\alpha > 0$ given and discuss tests satisfying (3). Let $g(\cdot)$ be defined by $g(x) = \int_{-\infty}^{x} f(y)\,dy$. It is a standard result that the derivative $g'(x)$ exists for almost all x (Lebesgue measure), and $g'(x) = f(x)$ for almost all x (Lebesgue measure). Let a be a real number such that $g'(a) = f(a) > 0$. Then

$$\lim_{\theta\to 0}(1/\theta) \int_a^{a+\theta} f(y)\,dy = f(a).$$

Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables each having $f(\cdot - \theta)$ as density function. Let χ be the characteristic function of $(-\infty, a]$. Then $\{\chi(X_n), n \geqq 1\}$ is a sequence of independently and identically distributed Bernoulli random variables which may be used to test the hypothesis $\theta < 0$ against the alternative $\theta > 0$. By definition

$$P_\theta(\chi(X_1) = 1) = \int_{-\infty}^{a} f(x - \theta)\,dx = g(a - \theta).$$

By Theorem 1 there exists a test for the problem stated such that

$$\lim_{\theta\to 0} [g(a - \theta) - g(a)]^2 |\log|\log|g(a - \theta) - g(a)|||^{-1} E_\theta N = 2g(a)(1 - g(a))(1 - \alpha - \beta),$$

or,

$$\lim_{\theta\to 0} \theta^2 |\log|\log|\theta|||^{-1} E_\theta N = (1 - \alpha - \beta)(2g(a)(1 - g(a))/f(a)).$$

The test which gives these limiting results will, of course, be a generalized sequential probability ratio test in terms of the Bernoulli random variables constructed above.

Thus, for one sided tests of a location parameter it will always be possible to obtain an order of magnitude $\theta^{-2}|\log|\log|\theta|||$ for $E_\theta N$. In some cases it is possible to do better than this. It may happen that $g'(x) = f(x)$ for every x and that $f(a) = \infty$ for some value of a. If in the argument of the preceding paragraph we use such a value of a it follows that

$$\lim_{\theta\to 0} \theta^2 |\log|\log|\theta|||^{-1} E_\theta N = 0.$$

These considerations give no information about lower bounds to $E_\theta N$. Suppose $f = g_1 * g_2$, the convolution of density functions g_1 and g_2, and that g_2 is a normal density function. If $\{X_n, n \geqq 1\}$ are independent random variables with common density function g_2, and if $\{Y_n, n \geqq 1\}$ are independent random variables with common density function g_1 and if $\{X_n, Y_m, n \geqq 1, m \geqq 1\}$ are mutually independent then $\{X_n + Y_n, n \geqq 1\}$ are independent random variables with common density function $f(\cdot - \theta)$. Consequently a one sided test of $\theta < 0$ against $\theta > 0$ for the family $\{f(\cdot - \theta), -\infty < \theta < \infty\}$ can be interpreted as a one sided test for the family $\{g_2(\cdot - \theta), -\infty < \theta < \infty\}$ using $\{X_n, n \geqq 1\}$. By Theorem 1 there is a constant $d > 0$ such that

$$\limsup_{\theta\to 0} \theta^2 |\log|\log|\theta|||^{-1} E_\theta N \geqq d.$$

(See the remark following the statement of Theorem 1; we assume here that $\alpha + \beta < 1$.) That is to say, in testing whether a location parameter is positive or negative, the smoother is the density function the more difficult is the testing problem.

We close this introduction with a few remarks. It follows from standard results on the Laplace transform that for a Koopman-Darmois family (1), $h(\cdot)$ is an analytic function of θ such that the real part $\theta \varepsilon \Omega$. Further it is easily seen that expressions like $\sum_{i=0}^n (h(\cdot))^{-i} P_{(\cdot)}(N = i, \text{decide } \theta > 0)$ are convex functions of $\theta \varepsilon \Omega$. Therefore such functions are continuous functions of $\theta \varepsilon \Omega$.

Suppose there is a number $M > 0$ such that for the test T, $P_\theta(N \leqq M) = 1$, $\theta \varepsilon \Omega$. From the remarks of the preceding paragraph the power function of T is a continuous function of $\theta \varepsilon \Omega$. A test T which satisfies (3) cannot have a continuous power function if $\alpha + \beta < 1$. Therefore if $\alpha + \beta < 1$ and a test T satisfies (3) then T must have the property that for every $n \geqq 1$ there is a $\theta_n \varepsilon \Omega$ such that $P_{\theta_n}(N \geqq n) > 0$. There is a function χ on Euclidian n-space to $[0, 1]$ such that

$$P_\theta(N \geqq n + 1) = \int \cdots \int \chi(x_1, \cdots, x_n)(h(\theta))^n \cdot \exp\left(\theta \sum_{i=1}^n x_i\right) \mu(dx_1) \cdots \mu(dx_n).$$

It follows at once that if $P_\theta(N \geqq n + 1) = 0$ for some $\theta \varepsilon \Omega$, then $P_\theta(N \geqq n + 1) = 0$ for all $\theta \varepsilon \Omega$. Since $(h(\cdot))^{-n} P_{(\cdot)}(N \geqq n + 1)$ is a convex function of $\theta \varepsilon \Omega$ it is a continuous function of $\theta \varepsilon \Omega$. Therefore it follows that if T is a test satisfying (3) and if $\alpha + \beta < 1$ and if $n \geqq 1$, and $C \subset \Omega$ is a compact set, then

$$\inf_{\theta \varepsilon C} P_\theta(N \geqq n) > 0.$$

As was observed above, $P_{(\cdot)}(N = n, \text{decide } \theta > 0)$ is a continuous function of $\theta \varepsilon \Omega$. By Fatou's lemma,

$$\begin{aligned} P_0(N < \infty, \text{decide } \theta > 0) &= \sum_{n=0}^{\infty} P_0(N = n, \text{decide } \theta > 0) \\ &\leqq \liminf_{\theta\to 0} \sum_{n=0}^{\infty} P_\theta(N = n \text{ decide } \theta > 0) \qquad (7) \\ &\leqq \liminf_{\theta\to 0-} P_\theta(N < \infty, \text{decide } \theta > 0) \\ &\leqq \limsup_{\theta\to 0-} P_\theta(N < \infty, \text{decide } \theta > 0) \leqq \alpha. \end{aligned}$$

Similarly,

(8) $P_0(N < \infty, \text{decide } \theta < 0) \leqq \limsup_{\theta \to 0+} P_\theta(N < \infty, \text{decide } \theta < 0) \leqq \beta.$

In particular,

$$P_0(N = \infty) \geqq 1 - \alpha - \beta.$$

Throughout we will use the abbreviation "GSPRT" for "generalized sequential probability ratio test." In the context of this paper GSPRT's take the following form. There are real number sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$. Let $\{X_n, n \geqq 1\}$ be a sequence of independently and identically distributed random variables with the common generalized density function $h(\theta) \exp(\theta x)$. If $n \geqq 1$ let $S_n = X_1 + \cdots + X_n$. If N is the stopping variable for the GSPRT T, and if $N \geqq n \geqq 1$ then if $1 \leqq i \leqq n - 1$, $a_i \leqq S_i \leqq b_i$. If $N = n$ and $\theta > 0$ is the decision taken, then $S_n \geqq b_n$. If $N = n$ and $\theta < 0$ is the decision taken, then $S_n \leqq a_n$.

If the test T is a GSPRT it follows by arguments similar to those used by Lehmann [8] that $P_{(\cdot)}(N < \infty, \text{decide } \theta > 0)$ is a nondecreasing function of θ. Since $P_\theta(N < \infty, \text{decide } \theta > 0) = \lim_{n \to \infty} P_\theta(N \leqq n, \text{decide } \theta > 0)$ it follows that $P_{(\cdot)}(N < \infty, \text{decide } \theta > 0)$ is a lower semi-continuous function. From lower semi-continuity it follows that

$$\begin{aligned} P_{\theta_0}(N < \infty, \text{decide } \theta > 0) &\leqq \liminf_{\theta \to \theta_0} P_\theta(N < \infty, \text{decide } \theta > 0) \\ &\leqq \limsup_{\theta \to \theta_0 -} P_\theta(N < \infty, \text{decide } \theta > 0) \\ &\leqq P_{\theta_0}(N < \infty, \text{decide } \theta > 0). \end{aligned}$$

The last inequality follows since $P_{(\cdot)}(N < \infty, \text{decide } \theta > 0)$ is a nondecreasing function. It follows that if the test T is a GSPRT then $P_{(\cdot)}(N < \infty, \text{decide } \theta > 0)$ is left continuous. Similarly, for a GSPRT it may be shown that $P_{(\cdot)}(N < \infty, \text{decide } \theta < 0)$ is a nonincreasing and right continuous function of $\theta \in \Omega$.

Of particular interest in the sequel is the following. If the test T is a GSPRT and if α and β are as in (9) and (10) then

(9)
$$\begin{aligned} \alpha = \sup_{\theta < 0} P_\theta(N < \infty, \text{decide } \theta > 0) \\ = P_0(N < \infty, \text{decide } \theta > 0) \\ = \lim_{\theta \to 0-} P_\theta(N < \infty, \text{decide } \theta > 0). \end{aligned}$$

Similarly

(10) $$\beta = P_0(N < \infty, \text{decide } \theta < 0).$$

It follows that for GSPRT's if $\alpha + \beta \leqq 1$ then

(11) $$\alpha + \beta = P_0(N < \infty).$$

Throughout we suppose $0 \varepsilon \Omega$ and that we are testing the hypothesis $\theta < 0$ against the alternative $\theta > 0$. We have assumed in the statement of Theorem 1 that

$$0 = \mu_0 = \int_{-\infty}^{\infty} x\mu(dx).$$

These assumptions are removable and are made to simplify the subsequent notation and computations. Suppose $\theta_0 \varepsilon \Omega$. We consider one sided tests of the hypothesis $\theta < \theta_0$ against the alternative $\theta > \theta_0$. It can easily be seen that by making a translation of the parameter space and defining $h_1(\cdot)$ and $\mu_1(\cdot)$ by $h_1(\theta) = h(\theta + \theta_0)$, $\mu_1(A) = \int_A \exp(\theta_0 x)\mu(dx)$, the above problem is equivalent to testing the hypothesis $\theta < 0$ against the alternative $\theta > 0$ for the Koopman-Darmois family $\{h_1(\theta) \exp(\theta x), \theta \varepsilon \Omega - \theta_0\}$. Define a new measure $\mu_2(\cdot)$ by, if $A \varepsilon \mathcal{B}_1$, then $\mu_2(A) = \mu_1(c + A)$. The constant $c = h_1(0) \int x\mu_1(dx)$. Then since $1 = h_1(0) \int \mu_1(dx)$ it follows that

$$0 = h_1(0) \int (x - c)\mu_1(dx) = h_1(0) \int x\mu_2(dx).$$

Further if we define $h_2(\cdot)$ by $h_2(\theta) = h_1(\theta) \exp(c\theta)$ then

$$1 = h_1(\theta) \int \exp(\theta x)\mu_1(dx) = h_1(\theta) \exp(c\theta) \int \exp(\theta(x - c))\mu_1(dx)$$

$$= h_2(\theta) \int \exp(\theta x)\mu_2(dx).$$

Consequently, without loss of generality, we assume in the sequel that $0 \varepsilon \Omega$, $h_2(0) \int x\mu_2(dx) = 0$, and that the hypotheses being tested are $\theta < 0$ against $\theta > 0$. We will in the sequel write "h" and "μ" instead of "h_2" and "μ_2".

Finally, suppose X is a random variable having the generalized probability density function $h(\theta) \exp(\theta x)$. In the statement of Theorem 1, $\mu_\theta = E_\theta X$ and $\sigma^2 = E_0 X^2$. The argument of the preceding paragraph says that in the general case, in which we are testing the hypothesis $\theta < \theta_0$ against the alternative $\theta > \theta_0$, we may simply change to the random variable $Y = X - \mu_{\theta_0}$ and the parameter $\theta - \theta_0$ in order to apply Theorem 1. These comments will help understand the application made in Section 2.

2. Bounded length confidence interval procedures. Let D_p be the set of all distribution functions such that if $F \varepsilon D_p$ then there is a real number $\gamma_{p,F}$ (called the p-point of F) satisfying, if $\epsilon > 0$ then $F(\gamma_{p,F} + \epsilon) > p$ and $F(\gamma_{p,F} - \epsilon) < p$. Suppose numbers $\alpha > 0$ and $L < 0$ are given and a confidence interval procedure is given. The procedure specifies three sequences of functions $\{\delta_n, n \geqq 1\}$, $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ such that if $\{Z_n, n \geqq 1\}$ is a sequence of independent random variables with common distribution function $F \varepsilon D_p$, then $\delta_n(Z_1, \cdots, Z_n)$ is the conditional probability given $Z_1, \cdots, Z_n$ that sampling stops on the nth observation. If N is the stopping variable for the procedure then $P_F(N = n)$

$= E_F\delta_n(Z_1, \cdots, Z_n)$. We assume that if $F \varepsilon D_p$ then $\sum_{n=0}^{\infty} P_F(N = n) = 1$. If sampling stops when $N = n$, a confidence interval $[b_n(Z_1, \cdots, Z_n), a_n(Z_1, \cdots, Z_n)]$ (i.e., the closed real number interval with the specified end points) is formed. In the sequel we write $a_N = a_N(Z_1, \cdots, Z_N)$ and $b_N = b_N(Z_1, \cdots, Z_N)$. The confidence interval procedure is to have the properties that

$$\text{(12)} \qquad \begin{aligned} &\text{if } F \varepsilon D_p \text{ then with probability one, } a_N - b_N \leqq L, \\ &\text{and } P_F(\gamma_{p,F} \varepsilon [b_N, a_N]) \geqq 1 - \alpha. \end{aligned}$$

We will further suppose that if $F \varepsilon D_p$ then $E_F N < \infty$, where N is the stopping variable of the confidence interval procedure.

One may expect, for a given confidence interval procedure, that if F if "flat" about its p-point $\gamma_{p,F}$ then $E_F N$ may be "large". We now make these ideas precise. Suppose $0 < \rho < 1$. We define a measure of flatness by

$$\epsilon_F = \sup\nolimits_{0<\rho<1} \min\left(F(\gamma_{p,F} + \rho L) - p,\ p - F(\gamma_{p,F} - (1 - \rho)L)\right).$$

It is easily verified that if F is continuous then there is a number ρ_F satisfying $0 < \rho_F < 1$ and

$$\epsilon_F = F(\gamma_{p,F} + \rho_F L) - p = p - F(\gamma_{p,F} - (1 - \rho_F)L).$$

In the following we will say that a density function $f(\cdot)$ is bimodal if there are numbers d_0, d_1, and d_2 such that $f(\cdot)$ is a nondecreasing function if $x \varepsilon (-\infty, d_0]$ or if $x \varepsilon [d_1, d_2]$, and $f(\cdot)$ is a nonincreasing function if $x \varepsilon [d_0, d_1]$ or if $x \varepsilon [d_2, \infty)$.

We now state and prove the following theorem.

THEOREM 2. *Let $D_p^* \subset D_p$ and suppose D_p^* contains all F in D_p having bimodal density functions which are continuous and everywhere positive. Suppose $\alpha > 0$ and $L > 0$ and a confidence interval procedure are given such that if $F \varepsilon D_p^*$ the confidence interval procedure gives an interval $[b_N, a_N]$ such that* (12) *holds. Then,*

$$\text{(13)} \quad \limsup\nolimits_{\epsilon_F \to 0, F \varepsilon D_p^*} (\epsilon_F)^2 |\log|\log|\epsilon_F|||^{-1} E_F N \geqq 2(1 - 2\alpha)p(1 - p).$$

PROOF. We will suppose in the following that $L \geqq 1$. It will be clear how to modify the argument for smaller values of L. We will show below that the class D_p^* may be extended to include other distribution functions F to which the confidence interval procedure may be applied without violation of (12) or the other assumptions above. Call the extended class D_p^{**}. Among those $F \varepsilon D_p^{**}$ will be all F having density functions $f_q(\cdot)$, $0 < q < 1$, $q \neq p$, defined by

$$\begin{aligned} f_q(x) &= q \quad \text{if} \quad 0 \leqq x \leqq 1, \\ f_q(x) &= 1 - q \quad \text{if} \quad 2L \leqq x \leqq 2L + 1, \\ f_q(x) &= 0 \quad \text{otherwise}. \end{aligned}$$

Let F_q be the distribution function corresponding to the density function f_q, $0 < q < 1$, $q \neq p$. We will first show that Theorem 1 gives information about $E_{F_q} N$ as $q \to p$. We then discuss the extension of D_p^* to D_p^{**}.

If $\{Z_n, n \geqq 1\}$ are independent random variables which have the common density function $f_q(\cdot)$, we apply the confidence interval procedure. Sampling stops with probability one and an interval $[b_N, a_N]$ of length $\leqq L$ is specified. If $q > p$ the p-point of $F_q(\cdot)$ is $\gamma_{p,F_q} = p/q$, while if $q < p$ the p-point of $F_q(\cdot)$ is $\gamma_{p,F_q} = 2L + (p - q)/(1 - q)$. Since the interval $[b_N, a_N]$ covers the p-point of $F_q(\cdot)$ with probability $\geqq 1 - \alpha$ we are able to construct a one sided test of the hypothesis $q < p$ against the alternative $q > p$ such that

$$\sup_{q<p} P_q(\text{decide } q > p) \leqq \alpha; \qquad \sup_{q>p} P_q(\text{decide } q < p) \leqq \alpha.$$

The terminal decision rule of the test is as follows. We decide $q > p$ if $[b_N, a_N]$ has points in common with $[0, 1]$. Otherwise we decide $q < p$.

The test just described is really a test about Bernoulli random variables. To see this, suppose $\{Z_{1,n}, n \geqq 1\}$ is a sequence of independent Bernoulli random variables having the common probability $P(Z_{1,n} = 1) = q$, $n \geqq 1$. Suppose $\{Z_{2,n}, n \geqq 1\}$ is a sequence of independent random variables, independent of the sequence $\{Z_{1,n}, n \geqq 1\}$, each uniformly distributed on $[0, 1]$. If $n \geqq 1$, let $Z_n = Z_{1,n}Z_{2,n} + (1 - Z_{1,n})(2L + Z_{2,n})$. Then $\{Z_n, n \geqq 1\}$ is a sequence of independent random variables with common density function $f_q(\cdot)$.

It follows that from the given confidence interval procedure we may construct for the family of Bernoulli distributions a one sided test of $q < p$ against $q > p$ which has the form described in Section 1 with $\alpha = \beta$. From Theorem 1 we find

$$\limsup_{q\to p} |q - p|^2 |\log|\log|q - p|||^{-1} E_q N \geqq 2(1 - 2\alpha)p(1 - p).$$

It is easily shown that $\epsilon_{F_q} = |q - p|$. It follows that if the distribution functions $F_q(\cdot)$ were in D_p^* then Theorem 2 would hold.

We now consider the problem of extending the class D_p^* to a class D_p^{**} which contains the distribution functions $F_q(\cdot)$, $0 < q < 1$, $q \neq p$. The density functions $f_q(\cdot)$ are upper semi-continuous, bimodal, and are zero outside the compact set $[0, 1] \cup [2L, 2L + 1]$. Let $0 < q < 1$ and $q \neq p$. It follows that in D_p^* there is a sequence $\{F_n, n \geqq 1\}$ such that if $n \geqq 1$, F_n has a continuous density function f_n and there is a real number sequence $\{d_n, n \geqq 1\}$ for which

$$\text{(14)} \qquad \begin{aligned} &\text{if } n \geqq 1, -\infty < x < \infty, \text{ then } d_n f_n(x) \geqq d_{n+1} f_{n+1}(x) \geqq f_q(x), \\ &\text{and } \lim_{n\to\infty} d_n f_n(x) = f_q(x). \end{aligned}$$

Since $f_q(\cdot)$ and $f_n(\cdot)$ are probability density functions it follows that if $n \geqq 1$ then $d_n \geqq 1$. From the monotone convergence theorem it follows that $\lim_{n\to\infty} d_n = 1$. Therefore if $-\infty < x < \infty$, $\lim_{n\to\infty} f_n(x) = f_q(x)$. Let F_n be the distribution function corresponding to the density function f_n. From the observations just made it follows that $f_n \to f_q$ in $L_1(-\infty, \infty)$ as $n \to \infty$. This implies that $F_n(\cdot) \to F_q(\cdot)$ uniformly as $n \to \infty$. We therefore conclude that

$$\text{(15)} \qquad \lim_{n\to\infty} \epsilon_{F_n} = \epsilon_{F_q}; \qquad \lim_{n\to\infty} \gamma_{p,F_n} = \gamma_{p,F_q}.$$

We now examine $E_{F_q}N$. From (15) it follows at once that $\inf_{n\geqq 1}\epsilon_{F_n} > 0$. Consequently if for the given confidence interval procedure it were true that $\sup_{n\geqq 1} E_{F_n}N = \infty$ then Theorem 2 would follow trivially for the class D_p^*. Suppose $\sup_{n\geqq 1} E_{F_n}N < \infty$. By Fatou's lemma,

$$\infty > \liminf_{n\to\infty} E_{F_n} N = \liminf_{n\to\infty} E_{F_n} \sum_{m=1}^{\infty} m\delta_m$$

$$\geqq \sum_{m=1}^{\infty} m \int \liminf_{n\to\infty} \delta_m(x_1, \cdots, x_m) \prod_{i=1}^{m} f_n(x_i) \prod_{i=1}^{m} dx_i \tag{16}$$

$$= \sum_{m=1}^{\infty} m \int \delta_m(x_1, \cdots, x_m) \prod_{i=1}^{m} f_q(x_i) \prod_{i=1}^{m} dx_i = E_{F_q} N.$$

Then $P_{F_q}(N = \infty) = 0$ and $E_{F_q}N < \infty$.

$f_q(\cdot)$ is continuous except at four points. If $\{\rho_n, n \geqq 1\}$ is a real number sequence and $\lim_{n\to\infty} \rho_n = 0$ then if $-\infty < x < \infty$,

$$\liminf_{n\to\infty} d_n f_n(x + \rho_n) \geqq \liminf_{n\to\infty} f_q(x + \rho_n) = f_q(x) \tag{17}$$

except at the four points of discontinuity of $f_q(\cdot)$. We make the explicit choice,

$$\text{if } n \geqq 1,\ \rho_n = \gamma_{p,F_n} - \gamma_{p,F_q}. \tag{18}$$

From (15) it follows that $\lim_{n\to\infty} \rho_n = 0$, as is required for (17) to hold. A change of variable shows that

$$p = \int_{-\infty}^{\gamma_{p,F_q}} f_n(x + \rho_n)\, dx.$$

In the next steps of the argument we work with the density functions $\{f_n(\cdot + \rho_n), n \geqq 1\}$ which have a common p-point γ_{p,F_q}.

Let $\{\delta_n^*, n \geqq 1\}$ be defined as follows. If $n \geqq 1$,

$$\delta_n^*(x_1, \cdots, x_n) = 1 \quad \text{if} \quad \gamma_{p,F_q} \notin [b_n(x_1, \cdots, x_n), a_n(x_1, \cdots, x_n)],$$

$$\delta_n^*(x_1, \cdots, x_n) = 0 \quad \text{otherwise}.$$

Then, if $d^* = P(\gamma_{p,F_q} \notin [b_0, a_0])$,

$$P_{F_q}(\gamma_{p,F_q} \notin [b_N, a_N]) = d^* + \sum_{m=1}^{\infty} \int \delta_m \delta_m^* \prod_{i=1}^{m} f_q(x_i) \prod_{i=1}^{m} dx_i$$

$$\leqq d^* + \sum_{m=1}^{\infty} \int \liminf_{n\to\infty} \delta_m \delta_m^* \prod_{i=1}^{m} f_n(x_i + \rho_n) \prod_{i=1}^{m} dx_i$$

$$\leqq d^* + \sum_{m=1}^{\infty} \liminf_{n\to\infty} \int \delta_m \delta_m^* \prod_{i=1}^{m} f_n(x_i + \rho_n) \prod_{i=1}^{m} dx_i$$

$$\leqq d^* + \liminf_{n\to\infty} \sum_{m=1}^{\infty} \int \delta_m \delta_m^* \prod_{i=1}^{m} f_n(x_i + \rho_n) \prod_{i=1}^{m} dx_i \leqq \alpha.$$

In a similar fashion one may verify that $P_{F_q}(a_N - b_N > L) \leqq 0$.

From (16) and the conclusions just drawn it follows that the given confidence interval procedure may be validly applied to the distributions F_q, $0 < q < 1$, $q \neq p$. Further, from (15) and (16) it follows that if $0 < \rho < \frac{1}{4}$ then for n sufficiently large, $E_{F_n}N \geqq (1 - \rho)E_{F_q}N$ and $(1 + \rho) \geqq \epsilon_{F_n}/\epsilon_{F_q} \geqq (1 - \rho)$. From this and the earlier remarks about construction of tests from confidence interval procedures Theorem 2 now follows.

A minor modification of the proof of Theorem 2 will show that the Theorem remains valid if the decision procedure gives a randomized confidence interval after sampling stops.

In a subsequent paper we will show the construction of confidence interval procedures having confidence level $\geqq 1 - \alpha$ and giving confidence intervals of length $\leqq L$ for the p-point of distribution functions. These procedures are completely nonparametric in the sense that they may be validly applied to any distribution function F such that $\epsilon_F > 0$ even if $F \not\in D_p$. In fact, confidence interval procedures may be constructed in such a way that if $\delta > 0$ then $\sup \{E_F N \mid \epsilon_F \geqq \delta\} < \infty$. Further these procedures may be constructed in such a way that

$$\limsup_{\epsilon_F \to 0} \epsilon_F^{2}|\log|\log|\epsilon_F|||^{-1} E_F N \leqq 4(1 - (\alpha/2))(p(1 - p)).$$

This work is presently contained in Farrell [4].

Tests of the type discussed in Section 1 have been constructed and applied to a different type of confidence interval problem in Farrell [5]. Similar tests have been studied by Fabian [3].

Weiss [13] has shown that, so long as one considers only unimodal density functions, then it is possible to construct confidence interval procedures having confidence level $\geqq 1 - \alpha$ and giving confidence intervals of length $\leqq L$ for the p-point of the distribution functions, the constructed procedures using only two stages of sampling. Although it is not completely clear from his results we suspect that our Theorem 2 is false in Weiss's context.

3. Proof of Theorem 1. In the sequel we examine the behavior of $E_\theta N$ for $\theta > 0$. It will be apparent that the results proven below have corresponding dual results stated about $\theta < 0$. Consequently the proof of statements about $\theta < 0$ in Theorem 1 are omitted. The reader should observe that if $\mu_1(\cdot)$ is defined by $\mu_1(A) = \mu(-A)$ for sets $A \varepsilon \mathcal{B}_1$ then $\{h(-\theta) \exp(\theta x), -\theta \varepsilon \Omega\}$ is again a family of generalized probability density functions relative to $(R, \mathcal{B}_1, \mu_1)$ and this device can be used to prove the remainder of Theorem 1.

Section 3 is divided into three subsections. In Section 3.1 we prove the existence of a test S having an upper boundary (see below) which is better (again see below) than the given test T. In Section 3.2 a series of lemmas are proven about tests having an upper boundary. Section 3.3 takes the results of Sections 3.1 and 3.2 and gives a proof of Theorem 1.

Before beginning the subsections we mention some facts about the function $h(\cdot)$ (see (1)) and the expected value $\mu_{(\cdot)}$. It was observed in Section 1 that $h(\cdot)$ is an analytic function of θ such that the real part $\theta \varepsilon \Omega$. Below we use derivatives and power series expansions of $h(\cdot)$. The function $\mu_{(\cdot)}$ was defined by (2). Since

$$(h(\theta))^{-1} = \int \exp(\theta x)\mu(dx),$$

(it is well known that we may differentiate under the integral and in the sequel this is done without comment) taking a derivative we find

$$\mu_\theta = -h'(\theta)/h(\theta), \tag{19}$$

where $h'(\cdot)$ is the derivative of $h(\cdot)$. Therefore

$$\begin{aligned} \frac{d}{d\theta}\mu_\theta = \frac{d}{d\theta} h(\theta) \int x \exp(\theta x)\mu(dx) &= (h'(\theta)/h(\theta))h(\theta) \\ \cdot \int x \exp(\theta x)\mu(dx) &+ h(\theta) \int x^2 \exp(\theta x)\mu(dx) \\ &= -(\mu_\theta)^2 + E_\theta X^2 = \operatorname{Var}_\theta X > 0. \end{aligned} \tag{20}$$

In particular it follows that $\mu_{(\cdot)}$ is a strictly increasing function of $\theta \varepsilon \Omega$. Since we assume $\mu_0 = 0$ it follows that if $\theta \geqq 0$ then $\mu_\theta \geqq 0$. Also, from l'Hospital's rule it follows that $\lim_{\theta\to 0} \mu_\theta/\theta = \sigma^2$, a fact referred to in an earlier section.

From (19) follows

$$h'(0) = 0. \tag{21}$$

From (19) we may obtain the formula

$$(d/d\theta)\mu_\theta = -h''(\theta)/h(\theta) + (h'(\theta)/h(\theta))^2. \tag{22}$$

Evaluation at zero then gives $h''(0) = -\sigma^2 h(0)$. We may then write

$$h(\theta) = h(0) \exp\left(-\tfrac{1}{2}(\sigma^2 + O(\theta))\theta^2\right). \tag{23}$$

By $O(f(\theta))$ we mean a function of θ such that $\limsup_{\theta\to 0} |O(f(\theta))/f(\theta)| < \infty$.

3.1 *The test S*. In this section we construct a test S from T. The test S, which is a test with upper boundary in the sense described below, will be used to prove Theorem 1. We will give the construction of S using the type of argument used by Weiss [12] and will give the construction of S in a context somewhat more general than needed to prove Theorem 1.

In this section tests will be described in terms of Borel subsets of Euclidian spaces. The decisions made are determined by the Borel sets containing the values of observed random variables. We suppose R is the set of real numbers, $\mathcal{B}_1$ the set of Borel subsets of R, and μ a nonnegative σ-finite measure defined on $\mathcal{B}_1$. Ω will be an open real number interval. $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ will be a family of generalized probability density functions on the measure space $(R, \mathcal{B}_1, \mu)$. $\{X_n, n \geqq 1\}$ will be a sequence of independent random variables each having $p_\theta(\cdot)$ as generalized density function. $\{Z_n, n \geqq 0\}$ will be a sequence of random variables such that $\{Z_n, n \geqq 0, X_m, m \geqq 1\}$ are independent and if $n \geqq 0$ the joint distribution of $Z_0, \cdots, Z_n$ is specified by the description of the test (T or S as the case may be) but does not depend on the parameter $\theta \varepsilon \Omega$. The random variables $\{Z_n, n \geqq 0\}$ are used for the purpose of randomization.

In order to describe the decision rules of a test we need additional notation. Let $R^{(n)}$ be Euclidian n-space, $\mathcal{B}_n$ be the set of Borel subsets of $R^{(n)}$. A test T specifies sequences of sets $\{T{:}A_0(n), n \geqq 0\}$ and $\{T{:}A_1(n), n \geqq 0\}$. A sequence $\{V_T^{(n)}, n \geqq 0\}$ is formed by defining

$$V_T^{(0)} = Z_0 ;$$
$$V_T^{(n)} = \{Z_0, X_1, Z_1, \cdots, X_n, Z_n\}, \qquad n \geqq 1.$$

We assume that if $n \geqq 0, i = 0, 1$, then $T{:}A_i(n) \varepsilon \mathcal{B}_{2n+1}$.

We will now describe the decision rules in terms of two abstract decisions d_0 and d_1. At the start Z_0 is observed and a randomized decision made as follows. If $Z_0 = 2$, take an observation on the pair X_1, Z_1; if $Z_0 = 0$ take no observations and decide d_0; if $Z_0 = 1$ take no observations and decide d_1. In case observations are taken sampling continues until for some n,

$$V_T^{(n)} \varepsilon T{:}A_0(n) \cup T{:}A_1(n).$$

If $V_T^{(n)} \varepsilon T{:}A_0(n)$ when sampling stops then decide d_0; if $V_T^{(n)} \varepsilon T{:}A_1(n)$ when sampling stops then decide d_1.

In the sequel "$\times$" will mean Cartesian product. Implied in the description of the preceding paragraph is the assumption that if $n \geqq 1$ and $m \geqq 1$ then $(T{:}A_0(n) \cup T{:}A_1(n)) \times R^{(2m)}$ is, as a set, disjoint from the set $T{:}A_0(n + m) \cup T{:}A_1(n + m)$.

The following additional notation will be helpful in the analysis which follows. The event that *more than* n observations are taken will be notated by $V_T^{(n)} \varepsilon T{:}C(n)$. Formally set wise we define $T{:}C(0) = \{2\}$ and if $n \geqq 0$,

$$T{:}C(n + 1) = (T{:}C(n)) \times R^{(2)} - (\bigcup_{i=0}^{1} T{:}A_i(n + 1)).$$

Then $T{:}C(n + 1) \varepsilon \mathcal{B}_{2n+3}$. In the following we let $V_T = \{Z_0, X_1, Z_1, \cdots, X_n, Z_n, \cdots\}$ which is a random variable taking values in infinite dimensional Euclidian space $R^{(\infty)}$. If $m \geqq 1$ and $A \varepsilon \mathcal{B}_m$ then $A \times R^{(\infty)}$ is then a cylinder Borel subset of $R^{(\infty)}$. If $i = 0, 1$, define

$$T{:}A_i = \bigcup_{j=0}^{\infty} ((T{:}A_i(j)) \times R^{(\infty)})$$

so that if $i = 0, 1$, the events $V_T \varepsilon T{:}A_i$ are the events sampling stops and decision d_i is made. Finally, we let N_T be the stopping variable for T.

We now state our hypotheses on the probability density functions $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$. We suppose there are given sequences of functions $\{t_n, n \geqq 1\}$, $\{h_n, n \geqq 1\}$ and $\{q_{\theta,n}, n \geqq 1, \theta \varepsilon \Omega\}$ such that if $n \geqq 1$ then t_n, h_n, and $q_{\theta,n}$ are measurable functions,

$$t_n{:}R^{(n)} \rightarrow R, \qquad h_n{:}R^{(n)} \rightarrow R, \quad \text{and} \quad q_{\theta,n}{:}R \rightarrow R.$$

We suppose for all $\theta \varepsilon \Omega$, $n \geqq 1$ and $(x_1, \cdots, x_n) \varepsilon R^{(n)}$ that

$$\prod_{i=1}^{n} p_\theta(x_i) = q_{\theta,n}(t_n(x_1, \cdots, x_n))h_n(x_1, \cdots, x_n).$$

Besides assuming that the t_n, $n \geqq 1$, are sufficient statistics, we make an assumption about monotone likelihood ratios. We suppose that if $\theta_1 \geqq \theta_2$, θ_1, $\theta_2 \varepsilon \Omega$, and $a_1 \geqq a_2$ then

$$q_{\theta_1,n}(a_1)q_{\theta_2,n}(a_2) \geqq q_{\theta_1,n}(a_2)q_{\theta_2,n}(a_1), \qquad n \geqq 1.$$

Finally, we require the family $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ to be a homogeneous family of density functions. That is, if $A \varepsilon \mathcal{B}_1$, $\theta_0 \varepsilon \Omega$, and $0 = \int_A p_{\theta_0}(x)\mu(dx)$ then for all $\theta \varepsilon \Omega$, $0 = \int_A p_\theta(x)\mu(dx)$.

In terms of the hypotheses just made, we will call T a test with upper boundary $\{b_n, n \geqq 1\}$ if $\{b_n, n \geqq 1\}$ is a real number sequence such that if

$$v_n = (z_0, x_1, z_1, \cdots, x_n, z_n) \varepsilon T{:}A_1(n)$$

then $t_n = t_n(x_1, \cdots, x_n) \geqq b_n$ while if $v_n \varepsilon T{:}C(n) \cup T{:}A_0(n)$ then $t_n \leqq b_n$. We use the notations "v_n" and "t_n" throughout this section.

THEOREM 3. *Suppose* $0 \varepsilon \Omega$ *and that* T *is a test of the hypothesis* $\theta < 0$ *against the alternative* $\theta > 0$. *We assume that if* $\theta \neq 0$ *then* $P_\theta(N_T < \infty) = 1$. *There exists a test* S *with upper boundary such that*

(a) *if* $0 \leqq \theta$, $n \geqq 0$, $P_\theta(V_S^{(n)} \varepsilon S{:}C(n)) \leqq P_\theta(V_T^{(n)} \varepsilon T{:}C(n))$.

(b) *if* $\theta \leqq 0$ *then* $P_\theta(V_S \varepsilon S{:}A_0) \geqq P_\theta(V_T \varepsilon T{:}A_0)$; *if* $\theta \geqq 0$ *then* $P_\theta(V_S \varepsilon S{:}A_0) \leqq P_\theta(V_T \varepsilon T{:}A_0)$.

(c) *if* $\theta \geqq 0$ *then* $P_\theta(V_S \varepsilon S{:}A_1) \geqq P_\theta(V_T \varepsilon T{:}A_1)$; *if* $\theta \leqq 0$ *then* $P_\theta(V_S \varepsilon S{:}A_1) \leqq P_\theta(V_T \varepsilon T{:}A_1)$.

This theorem says nothing about $P_\theta(N_S < \infty)$ if $\theta < 0$. It follows by (a) that if $\theta > 0$ then $P_\theta(N_S < \infty) = 1$ and by (b), (c) that $P_0(N_T < \infty) = P_0(N_S < \infty)$. The test S is constructed using the following lemma.

LEMMA. *Suppose* T *is a test of the hypothesis* $\theta < 0$ *against the alternative* $\theta > 0$ *such that if* $\theta \varepsilon \Omega$ *and* $\theta \neq 0$ *then* $P_\theta(N_T < \infty) = 1$. *Suppose* $\theta_0 \varepsilon \Omega$. *There exisst a sequence of tests* $\{T_n, n \geqq 0\}$ *having the properties described below. If* $n \geqq 0$ *let* $\{Z_{i,n}, i \geqq 0\}$ *be the sequence of random variables associated with* T_n *(used for randomization). Then* $T_0 = T$. *If* $0 \leqq m \leqq n$, $i = 0, 1$, $0 \leqq j \leqq m$ *and* $n \geqq 0$ *then* $Z_{j,m} = Z_{j,n}$, *and* $T_m{:}A_i(j) = T_n{:}A_i(j)$. *In addition the following hold. Let* $n \geqq 0$.

(a_1) *If* $\theta_0 \leqq \theta$ *and* $m \geqq 1$ *then*

$$P_\theta(m \leqq N_{T_n} < \infty) \geqq P_\theta(m \leqq N_{T_{n+1}} < \infty);$$

if $\theta \leqq \theta_0$ *the reverse inequality holds.*

(b_1) *If* $\theta \leqq \theta_0$ *and* $m \geqq 0$ *then*

$$P_\theta(V_{T_n}^{(m)} \varepsilon T_n{:}A_0(m)) \leqq P_\theta(V_{T_{n+1}}^{(m)} \varepsilon T_{n+1}{:}A_0(m));$$

if $\theta \geqq \theta_0$ *and* $m \geqq 0$ *the reverse inequality holds.*

(c_1) *If* $\theta \geqq \theta_0$ *then*

$$P_\theta(V_{T_n} \varepsilon T_n{:}A_1) \leqq P_\theta(V_{T_{n+1}} \varepsilon T_{n+1}{:}A_1);$$

if $\theta \leqq \theta_0$ *the reverse inequality holds.*

(d_1) *If* $\theta \varepsilon \Omega$, $\theta \neq 0$, *then* $P_\theta(N_{T_n} < \infty) = 1$.

(e_1) *There is a real number sequence* $\{b_n, n \geqq 1\}$ *such that if* $n \geqq 1$ *and* $\{2, x_1, z_1, \cdots, x_n, z_n\} \varepsilon T_n{:}A_1(n)$ *then*

$$t_n = t_n(x_1, \cdots, x_n) \geqq b_n;$$

if $\{2, x_1, z_1, \cdots, x_n, z_n\} \varepsilon T_n{:}C(n) \cup T_n{:}A_0(n)$ *then* $t_n \leqq b_n$.

PROOF OF THEOREM 3 FROM THE LEMMA. Using this lemma the test S is defined by $S{:}A_i(m) = T_m{:}A_i(m)$, $i = 0, 1$, $m \geqq 0$, and $\{Z_{n,n}, n \geqq 0\}$ the random variable sequence for S. We take $\theta_0 = 0$ in applying the lemma. If the test T satisfies the hypotheses of Theorem 3 then the following hold for S. It follows immediately that

(a_2) if $0 \leqq \theta$ and $m \geqq 0$ then

$$P_\theta(m \leqq N_S < \infty) \leqq P_\theta(m \leqq N_T < \infty).$$

(b_2) if $\theta \leqq 0$ and $m \geqq 0$ then

$$P_\theta(V_S^{(m)} \varepsilon S{:}A_0(m)) \geqq P_\theta(V_T^{(m)} \varepsilon T{:}A_0(m));$$

if $\theta \geqq 0$ and $m \geqq 0$ the reverse inequality holds.

From (a_1) and hypothesis on T it follows that if $\theta \geqq 0$ then $P_\theta(N_S < \infty) = P_\theta(N_T < \infty)$. Since $P_\theta(N_T < \infty) = P_\theta(V_T \varepsilon T{:}A_0) + P_\theta(V_T \varepsilon T{:}A_1)$, using ($b_2$) we obtain, if $\theta \geqq 0$, $P_\theta(V_S \varepsilon S{:}A_1) \geqq P_\theta(V_T \varepsilon T{:}A_1)$. If $\theta \leqq 0$ then by Fatou's lemma, and by (c_1),

$$\begin{aligned} P_\theta(V_S \varepsilon S{:}A_1) &= \sum_{m=0}^{\infty} \liminf_{n\to\infty} P_\theta(V_{T_n}^{(m)} \varepsilon T_n{:}A_1(m)) \\ &\leqq \liminf_{n\to\infty} P_\theta(V_{T_n} \varepsilon T_n{:}A_1) \leqq P_\theta(V_T \varepsilon T{:}A_1). \end{aligned}$$

Therefore the test S has the properties (a), (b) and (c) of Theorem 3.

PROOF OF THE LEMMA. T_0 has been defined. We suppose T_{n-1} defined and show the construction of T_n. To carry out the construction we use the fact that there is a sequence $\{\phi_n, n \geqq 1\}$ of functions such that if $n \geqq 1$ then $\phi_n{:}R^{(n)} \to R$ is a 1-1 onto measurable function such that ϕ_n^{-1} is also measurable. See for example Halmos [6]. The mappings $\{\phi_n, n \geqq 1\}$ allow us to take random variables $W_1, \cdots, W_k$ and construct a random variable $\phi_k(W_1, \cdots, W_k)$ without loss of information. It is by such a process that the random variables $\{Z_{i,n}, i \geqq 1\}$ are defined.

As the first step of the construction we describe a test T_n'. If $0 \leqq m \leqq n - 1$, $T_{n-1}{:}C(m) = T_n'{:}C(m)$, and if $0 \leqq m < n$, $i = 0, 1$, $T_{n-1}{:}A_i(m) = T_n'{:}A_i(m)$. Since $\{q_{\theta,n}, \theta \varepsilon \Omega\}$ have monotone likelihood ratios it follows that there is a constant c_n and a number ρ such that $0 \leqq \rho \leqq 1$ and such that if we define functions $f_1(\cdot)$ and $f_2(\cdot)$ by

$$(24)\quad \begin{aligned} f_1(\theta) &= P_\theta(N_{T_{n-1}} = n, t_n(X_1, \cdots, X_n) > c_n) \\ &\qquad + \rho P_\theta(N_{T_{n-1}} = n, t_n(X_1, \cdots, X_n) = c_n) \end{aligned}$$

and $f_2(\theta) = P_\theta(V_{T_{n-1}}^{(n)} \varepsilon T_{n-1}:A_1(n))$ then,

$$(25)\qquad \text{if} \quad \theta \geqq \theta_0, f_1(\theta) \geqq f_2(\theta); \qquad \text{if} \quad \theta \leqq \theta_0, f_1(\theta) \leqq f_2(\theta).$$

Let Y_n be a random variable which is independent of $V_{T_{n-1}}^{(m)}$, $m \geqq 0$, such that $P(Y_n = 1) = \rho = 1 - P(Y_n = 0)$. Define $T_n':A_1(n)$ as those points $v_n = (z_0, x_1, z_1, \cdots, x_n, z_n)$ such that if $\phi_2^{-1}(z_n) = (z_n^*, y)$ then $(z_0, x_1, z_1, \cdots, x_n, z_n^*) \varepsilon T_{n-1}:A_0(n) \cup T_{n-1}:A_1(n)$; and $t_n(x_1, \cdots, x_n) = t_n > c_n$, or, $y = 1$ and $t_n = c_n$. Define $T_n':A_0(n)$ as those points v_n such that if $\phi_2^{-1}(z_n) = (z_n^*, y)$ then $(z_0, x_1, z_1, \cdots, x_n, z_n^*) \varepsilon T_{n-1}:A_0(n) \cup T_{n-1}:A_1(n)$; and $t_n < c_n$, or, $y = 0$ and $t_n = c_n$. If $m \geqq n + 1$ then $T_n':A_i(m)$ is to be the set of those v_m such that $(z_0, \cdots, x_n, z_n^*, \cdots, x_m, z_m) \varepsilon T_{n-1}:A_i(m)$, $i = 0, 1$, z_n^* as above. The random variables associated with the test T_n' are to be $Z_{0,n}, \cdots, Z_{n-1,n}, \phi_2(Z_{n,n}, Y_n), Z_{n+1,n} \cdots$. With these definitions it follows that for all $\theta \varepsilon \Omega$, if $m \geqq 0$, $m \neq n$, $i = 0, 1$ then

$$P_\theta(V_{T_{n-1}}^{(m)} \varepsilon T_{n-1}:A_i(m)) = P_\theta(V_{T_n'}^{(m)} \varepsilon T_n':A_i(m))$$

and if $m \geqq 0$ then

$$P_\theta(N_{T_{n-1}} \geqq m) = P_\theta(N_{T_n'} \geqq m).$$

Further, by (24),

$$f_1(\theta) = P_\theta(V_{T_n'}^{(n)} \varepsilon T_n':A_1(n));$$

$$f_2(\theta) = P_\theta(V_{T_{n-1}}^{(n)} \varepsilon T_{n-1}:A_1(n)).$$

From (25) we obtain the comparison of these probabilities.

We now define T_n by redefining T_n'. Let r be a real number. Let $s_I(r)$ be the set of those points v_n such that $t_n > r$, $v_n \varepsilon T_n':C(n)$, and $s_{II}(r)$ be the set of those points v_n such that $t_n < r$, $v_n \varepsilon T_n':A_1(n)$. It is necessary to consider three cases.

Case I.

$$\sup_{r \geqq c_n} P_{\theta_0}(V_{T_n'}^{(n)} \varepsilon s_{II}(r)) = 0.$$

In this case we define $b_n = \infty$; $T_n:A_0(n) = \bigcup_{i=0}^1 T_n':A_i(n)$; $T_n:A_1(n) =$ null set. If $m \geqq 0$, $m \neq n$, $i = 0, 1$, then $T_n:A_i(m) = T_n':A_i(m)$. Since by hypothesis $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ is a homogeneous family of generalized density functions, it follows that for all $\theta \varepsilon \Omega$, $\sup_{r \geqq c_n} P_\theta(V_{T_n'}^{(n)} \varepsilon s_{II}(r)) = 0$. Therefore the redefinition does not change the values of any probabilities associated with the tests.

Case II. There is a number $r \geqq c_n$ such that

$$P_{\theta_0}(V_{T_n'}^{(n)} \varepsilon s_I(r)) = 0 = P_{\theta_0}(V_{T_n'}^{(n)} \varepsilon s_{II}(r)).$$

In Case II we define $b_n = r$. T_n is defined as follows. If $i = 0, 1$, and $0 \leqq m \leqq n - 1$, then $T_n:A_i(m) = T_n':A_i(m)$. $T_n:A_1(n)$ consists of those points v_n in

$T'_n:C(n) \cup T'_n:A_1(n)$ such that $t_n > b_n$ together with those $v_n \varepsilon T'_n:A_1(n)$ such that $t_n = b_n$. $T_n:A_0(n) = T'_n:A_0(n)$. $T_n:C(n)$ is the set of those v_n such that $v_n \varepsilon T'_n:C(n) \cup T'_n:A_1(n)$ and $t_n < b_n$ together with those $v_n \varepsilon T'_n:C(n)$ with $t_n = b_n$.

$$T_n:A_1(n+1) = s_{II}(b_n) \times R^{(2)} \cup T'_n:A_1(n+1);$$
$$T_n:A_0(n+1) = T'_n:A_0(n+1).$$

If $m \geqq n + 2$ then $T_n:A_i(m) = T'_n:A_i(m)$, $i = 0, 1$. Since $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ is a homogeneous family of generalized probability density functions the redefinition of Case II does not change the values of any probabilities associated with the tests.

Case III. Define functions $g_1(\cdot)$ and $g_2(\cdot)$ on R by

$$g_1(r) = P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon s_I(r)), \quad \text{and} \quad g_2(r) = P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon s_{II}(r)).$$

In Case III there is a real number r_0 such that $g_1(r_0) > 0$ and $g_2(r_0) > 0$. Define

$$b_n = \sup\{r \mid g_1(r) \geqq g_2(r)\}.$$

It is easily verified that $b_n < \infty$. Define sets

$$D_I = \{v_n \mid v_n \varepsilon T'_n:C(n), t_n = b_n\};$$
$$D_{II} = \{v_n \mid v_n \varepsilon T'_n:A_1(n), t_n = b_n\}.$$

We distinguish two subcases.

Case IIIa. $g_1(b_n) \geqq g_2(b_n)$. Note that the functions $g_1(\cdot)$ and $g_2(\cdot)$ are monotone functions. By definition of b_n, if $r > b_n$ then $g_1(r) < g_2(r)$. Therefore

$$g_1(b_n) = \lim_{r \to b_n +} g_1(r) \leqq \lim_{r \to b_n +} g_2(r) = P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon (s_{II}(b_n) \cup D_{II})).$$

Let Y_n be a Bernoulli random variable such that Y_n is independent of $V_{T'_n}^{(m)}$, $m \geqq 0$, and such that

$$g_1(b_n) = g_2(b_n) + P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon D_{II}, Y_n = 1).$$

Let sets be defined by

$$E_I = s_I(b_n) \times \{0, 1\}; \qquad E_{II} = s_{II}(b_n) \times \{0, 1\} \cup D_{II} \times \{1\}.$$

Case IIIb. $g_1(b_n) < g_2(b_n)$. It follows that $c_n < b_n$ since $g_2(c_n) = 0$. Then

$$g_2(b_n) = \lim_{r \to b_n -} g_2(r) \leqq \lim_{r \to b_n -} g_1(r) = P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon (s_I(b_n) \cup D_I)).$$

Let Y_n be a Bernoulli random variable independent of $V_{T'_n}^{(m)}$ if $m \geqq 0$ and such that

$$g_1(b_n) + P_{\theta_0}(V_{T'_n}^{(n)} \varepsilon D_I, Y_n = 1) = g_2(b_n).$$

Define sets by

$$E_I = s_I(b_n) \times \{0, 1\} \cup D_I \times \{1\}; \qquad E_{II} = s_{II}(b_n) \times \{0, 1\}.$$

In Case III the set E_I represents the event that T'_n continues when it "should" stop, E_{II} the event T'_n stops when it "should" continue. We will in the sequel define T_n accordingly. To do this we need to define a probability measure γ on $\mathcal{B}_{2n+2}$ by, if $A \varepsilon \mathcal{B}_{2n+2}$, then

$$\gamma(A) = P_{\theta_0}((V_{T'_n}^{(n)}, Y_n) \varepsilon A \cap E_I)/P_{\theta_0}((V_{T'_n}^{(n)}, Y_n) \varepsilon E_I).$$

Let W_n be a vector valued random variable taking values in $R^{(2n+2)}$, and such that W_n induces the measure γ on $\mathcal{B}_{2n+2}$ and W_n has its range of values contained in E_I. We suppose W_n is independent of $V_{T'_n}^{(m)}$, $m \geqq 0$, and use W_n to define the continuation of T_n in Case III. W_n exists since it may be readily constructed from the random variable $(V_{T'_n}^{(n)}, Y_n)$.

We now construct the test T_n. If $\{Z'_{i,n}, i \geqq 0\}$ are the random variables associated with T'_n then in Case III define

$$Z_{i,n} = Z'_{1,n} \quad \text{if} \quad i \neq n; \qquad Z_{n,n} = \phi_{2n+4}(Z'_{n,n}, Y_n, W_n).$$

Further, if $i = 0, 1, 0 \leqq m \leqq n - 1$, $T_n:A_i(m) = T'_n:A_i(m)$. Define T_n: $A_1(n)$ as those points v_n such that

$$(v_{n-1}, x_n, \phi_{2n+4}^{-1}(z_n)) \varepsilon \{E_I \cup [(T'_n:A_1(n)) \times \{0, 1\} - E_{II}]\} \times R^{(2n+2)},$$

define $T_n:A_0(n) = T'_n:A_0(n)$, and $T_n:C(n)$ as the set of those v_n such that

$$(v_{n-1}, x_n, \phi_{2n+4}^{-1}(z_n)) \varepsilon \{E_{II} \cup [(T'_n:C(n)) \times \{0, 1\} - E_I]\} \times R^{(2n+2)}.$$

Suppose then that $m \geqq n + 1$. To establish the correspondences between $T_n:C(m)$ and $T'_n:C(m)$, $T_n:A_0(m)$ and $T'_n:A_0(m)$, $T_n:A_1(m)$ and $T'_n:A_1(m)$, suppose C' is a set in $\mathcal{B}_{2m+1}$, and C is the set constructed to correspond to C'. Then C is the set of those v_m such that if $\phi_{2n+4}^{-1}(z_n) = (z_n^*, y, w)$ then $(v_{n-1}, x_n, z_n^*, y) \varepsilon E_{II}$ and $(w, x_{n+1}, z_{n+1}, \cdots, x_m, z_m) \varepsilon C'$ together with $(v_{n-1}, x_n, z_n^*, y) \varepsilon (T'_n:C(n)) \times \{0, 1\} - E_I$ and $(v_{n-1}, x_n, z_n^*, \cdots, x_m, z_m) \varepsilon C'$. Roughly speaking, if under T'_n the observed values fall in E_{II} then observe $W_n = w$ and continue as if $v_n = w$ had been observed; otherwise T'_n and T_n are the same procedure.

As was observed in the discussion of Cases I and II the definitions of T_n in these cases do not change any probabilities associated with the tests. Therefore as T'_n satisfies the conditions of the lemma the same is true of T_n. We now determine whether in Case III T_n as defined satisfies the conditions of the lemma.

Let $m \geqq n + 1$ and let the sets $C \varepsilon \mathcal{B}_{2m+1}$ and $C' \varepsilon \mathcal{B}_{2m+1}$ be as in the preceding paragraph. Set

$$F = (T'_n:C(n)) \times \{0, 1\} - E_I.$$

By the definition of the preceding paragraph,

$$\begin{aligned} P_\theta(V_{T_n}^{(m)} \varepsilon C) &= P_\theta((V_{T'_n}^{(n)}, Y_n) \varepsilon F, V_{T'_n}^{(n)} \varepsilon C') \\ &\quad + P_\theta((V_{T'_n}^{(n)}, Y_n) \varepsilon E_{II}, (W, X_{n+1}, \cdots, Z_m) \varepsilon C'). \end{aligned} \tag{26}$$

Using assumptions made earlier of stochastic independence the last term of (26) may be written as the product of two probabilities. We then obtain

$$
\begin{aligned}
&P_\theta(V_{T_n}^{(m)} \varepsilon\, C) - P_\theta(V_{T_n'}^{(m)} \varepsilon\, C') \\
(27) \qquad &= P_\theta((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_{II})P_\theta((W, X_{n+1}, \cdots, Z_m) \varepsilon\, C') \\
&\qquad - P_\theta((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_I, V_{T_n'}^{(m)} \varepsilon\, C').
\end{aligned}
$$

Let $\omega_1(\cdot)$ be the measure $(Z_{n+1}, \cdots, Z_m)$ induces on $\mathcal{B}_{m-n}$ and write a conditional probability

$$
P_\theta(C'|\cdot) = \int \chi_{C'}(\cdot, x_{n+1}, \cdots, z_m) \prod_{i=n+1}^{m} p_\theta(x_i) \prod_{i=n+1}^{m} \mu(dx_i)\omega_1(dz_{n+1}, \cdots, dz_m),
$$

where $\chi_{C'}$ is the characteristic function of the set C'. By definition of W_n the range of W_n is contained in E_I. Then

$$
\begin{aligned}
P_\theta((W_n, X_{n+1}, \cdots, Z_m) \varepsilon\, C') &= EP_\theta(C'|W_n) \\
&= \int_{E_I} (1/\Delta)P_\theta(C'|(z_0, x_1, \cdots, z_n, y)) \prod_{i=1}^{n} p_{\theta_0}(x_i)\mu(dx_i)\omega_2(dz_0, \cdots, dy),
\end{aligned}
$$

where $\Delta = P_{\theta_0}((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_I)$ and $\omega_2(\cdot)$ is the measure on $\mathcal{B}_{n+2}$ induced by $(Z_{0,n}', \cdots, Z_{n,n}', Y_n)$. Recall that $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ is a homogeneous family of generalized probability density functions so that the ratio $q_{\theta_0,n}(x)/q_{\theta,n}(x)$ is well defined for almost all x in the range of t_n. By our hypothesis about monotone likelihood ratios, if $\theta \geqq \theta_0$, $q_{\theta_0,n}(\cdot)/q_{\theta,n}(\cdot)$ is a nonincreasing function. On the set E_I, $t_n(x_1, \cdots, x_n) = t_n \geqq b_n$. We find

$$
\begin{aligned}
(28) \qquad &P_\theta((W_n, X_{n+1}, \cdots, Z_m) \varepsilon\, C') \\
&\leqq (q_{\theta_0,n}(b_n)/q_{\theta,n}(b_n))(1/\Delta)P_\theta((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_I, V_{T_n'}^{(m)} \varepsilon\, C').
\end{aligned}
$$

If $\theta \leqq \theta_0$ the inequality is reversed. By construction if $\theta \geqq \theta_0$ then

$$
\begin{aligned}
(29) \qquad \Delta = P_{\theta_0}((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_I) &= P_{\theta_0}((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_{II}) \\
&\geqq (q_{\theta_0,n}(b_n)/q_{\theta,n}(b_n))P_\theta((V_{T_n'}^{(n)}, Y_n) \varepsilon\, E_{II});
\end{aligned}
$$

the inequality is reversed if $\theta \leqq \theta_0$. From (27), (28) and (29) it follows that if $m \geqq n + 1$,

$$
(30) \qquad \begin{aligned}
&\text{if } \theta \geqq \theta_0 \text{ then } P\,(V_{T_n}^{(m)} \varepsilon\, C) \leqq P_\theta(V_{T_n'}^{(m)} \varepsilon\, C'); \\
&\text{if } \theta \leqq \theta_0 \text{ the inequality is reversed.}
\end{aligned}
$$

We specialize (30) to the cases of interest and obtain

$$
\begin{aligned}
&\text{if } m \geqq n + 1 \text{ and } \theta \geqq \theta_0 \text{ then} \\
(31) \qquad &P_\theta(V_{T_n}^{(m)} \varepsilon\, T_n{:}C(m)) \leqq P_\theta(V_{T_n'}^{(m)} \varepsilon\, T_n'{:}C(m)) \\
&\qquad = P_\theta(V_{T_{n-1}}^{(m)} \varepsilon\, T_{n-1}{:}C(m)).
\end{aligned}
$$

Similarly,

if $m \geqq n + 1$ and $\theta \geqq \theta_0$ then

$$(32)\qquad \begin{aligned} P_\theta(V_{T_n}^{(m)} \varepsilon\, T_n{:}A_0(m)) &\leqq P_\theta(V_{T_n'}^{(m)} \varepsilon\, T_n'{:}A_0(m)) \\ &= P_\theta(V_{T_{n-1}}^{(m)} \varepsilon\, T_{n-1}{:}A_0(m)). \end{aligned}$$

If $\theta \leqq \theta_0$ then inequalities (31) and (32) reverse.

By construction, if $\theta \,\varepsilon\, \Omega$,

$$P_\theta(V_{T_n}^{(n)} \varepsilon\, T_n{:}A_0(n)) = P_\theta(V_{T_n'}^{(n)} \varepsilon\, T_n'{:}A_0(n)).$$

Then if $\theta \geqq \theta_0$,

$$P_\theta(V_{T_n'}^{(n)} \varepsilon\, T_n'{:}A_0(n)) \leqq P_\theta(V_{T_{n-1}}^{(n)} \varepsilon\, T_{n-1}{:}A_0(n)).$$

The inequality reverses if $\theta_0 \geqq \theta$. If $\theta \,\varepsilon\, \Omega$ then

$$\begin{aligned} P_\theta(V_{T_n}^{(n)} \varepsilon\, T_n{:}C(n)) - P_\theta(V_{T_n'}^{(n)} \varepsilon\, T_n'{:}C(n)) \\ = P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_{II}) - P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_I). \end{aligned}$$

By an argument like the one used to obtain (28) and (29) it may be shown that this difference is $\leqq 0$ if $\theta \geqq \theta_0$ and is $\geqq 0$ if $\theta \leqq \theta_0$. Since if $\theta \,\varepsilon\, \Omega$,

$$P_\theta(V_{T_{n-1}}^{(n)} \varepsilon\, T_{n-1}{:}C(n)) = P_\theta(V_{T_n'}^{(n)} \varepsilon\, T_n'{:}C(n)),$$

it follows that (a_1) and (b_1) of the lemma are satisfied.

To verify (c_1) observe that if $\theta \geqq \theta_0$ then by (a_1)

$$(33)\qquad \begin{aligned} P_\theta(V_{T_{n-1}} \varepsilon\, T_{n-1}{:}A_0) + P_\theta(V_{T_{n-1}} \varepsilon\, T_{n-1}{:}A_1) \\ \leqq P_\theta(V_{T_n} \varepsilon\, T_n{:}A_0) + P_\theta(V_{T_n} \varepsilon\, T_n{:}A_1). \end{aligned}$$

If $\theta \geqq \theta_0$ then using (33) and (b_1) gives

$$\begin{aligned} 0 \leqq P_\theta(V_{T_{n-1}} \varepsilon\, T_{n-1}{:}A_0) - P_\theta(V_{T_n} \varepsilon\, T_n{:}A_0) \\ \leqq P_\theta(V_{T_n} \varepsilon\, T_n{:}A_1) - P_\theta(V_{T_{n-1}} \varepsilon\, T_{n-1}{:}A_1). \end{aligned}$$

The reverse inequalities hold if $\theta \leqq \theta_0$.

To verify (d_1),

$$(34)\qquad \begin{aligned} &P_\theta(N_{T_n} < \infty) - P_\theta(N_{T_n'} < \infty) \\ &\quad = P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_I) - P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_{II}) \\ &\quad + \sum_{m=n+1}^{\infty} P_\theta\Big((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_{II}, (W_n, X_{n+1}, \cdots, Z_m) \\ &\qquad\qquad \cdot\varepsilon \bigcup_{i=0}^{1} T_n'{:}A_i(m)\Big) \\ &\quad - \sum_{m=n+1}^{\infty} P_\theta\Big((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_I, V_{T_n'}^{(m)} \varepsilon \bigcup_{i=0}^{1} T_n'{:}A_i(m)\Big) \\ &\quad \geqq \sum_{m=n+1}^{\infty} P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_{II})\, P_\theta\Big((W_n, \cdots, Z_m) \,\varepsilon \bigcup_{i=0}^{1} T_n'{:}A_i(m)\Big) \\ &\quad - P_\theta((V_{T_n'}^{(n)}, Y_n) \,\varepsilon\, E_{II}). \end{aligned}$$

Define $f(\cdot)$ by the following. Take $\theta_1 \varepsilon \Omega$, $\theta_1 \neq 0$. Let

$$f(w) = \sum_{m=n+1}^{\infty} P_{\theta_1}(\bigcup_{i=0}^{1} T'_n : A_i(m) \mid w).$$

Then it is clear that $f(\cdot)$ is a measurable function on $R^{(n+2)}$ and that

$$Ef(W_n) = \sum_{m=n+1}^{\infty} P_{\theta_1}\Big((W_n, \cdots, Z_m) \varepsilon \bigcup_{i=0}^{1} T'_n : A_i(m)\Big).$$

We show that $Ef(W_n) = 1$. Then from (34) it will follow that if $\theta \neq 0$, $P_\theta(N_{T_n} < \infty) \geqq P_\theta(N_{T'_n} < \infty) = 1$. That would complete the proof of the lemma.

To show $Ef(W_n) = 1$, let χ_{E_I} be the characteristic function of the set E_I. Then

$$E_{\theta_1}(\chi_{E_I}(V_{T'_n}^{(n)}, Y_n) f(V_{T'_n}^{(n)}, Y_n))$$
$$= P_{\theta_1}((V_{T'_n}^{(n)}, Y_n) \varepsilon E_I, n + 1 \leqq N_{T'_n} < \infty) = P_{\theta_1}((V_{T'_n}^{(n)}, Y_n) \varepsilon E_I).$$

Therefore, if $\theta_1 \neq 0$ for almost all $(v_n, y) \varepsilon E_I$, $f(v_n, y) = 1$. By construction the distribution of W_n is the same as the distribution $(V_{T'_n}^{(n)}, Y_n)$ restricted to E_I and normalized when $\theta = \theta_0$. Since $\{p_\theta(\cdot), \theta \varepsilon \Omega\}$ is a homogeneous family of generalized probability density functions it follows that $f(W_n) = 1$ with probability one and therefore $Ef(W_n) = 1$.

3.2 *Tests with an upper boundary.* The family $\{h(\theta) \exp(\theta x), \theta \varepsilon \Omega\}$ of generalized probability density functions on $(R, \mathcal{B}_1, \mu)$ satisfies the hypotheses for Theorem 3. The functions $t_n(x_1, \cdots, x_n) = x_1 + \cdots + x_n$, $n \geqq 1$. Consequently given a test T of the hypothesis $\theta < 0$ against the alternative $\theta > 0$ for the problem of Section 1 there is, by the results of Section 3.1, a test T_1 with upper boundary $\{b_n, n \geqq 1\}$ such that if N_1 is the stopping variable for T_1 and if $\theta > 0$ then $E_\theta N_1 \leqq E_\theta N$. If $N_1 = n$ and $\theta > 0$ is decided then $S_n = X_1 + \cdots + X_n \geqq b_n$ while if $N_1 \geqq n$ and $\theta > 0$ is not the decision made then $S_n \leqq b_n$. We now state and prove a series of lemmas about tests T_1 with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1. $\{X_n, n \geqq 1\}$ will be a sequence of independently and identically distributed random variables each with generalized probability density function $h(\theta) \exp(\theta x)$ for some $\theta \varepsilon \Omega$. We assume $0 \varepsilon \Omega$, $\mu_0 = 0$ (see (2)) and that $\theta < 0$ against $\theta > 0$ are the hypotheses being tested. In the following we will use the notation $s_n = x_1 + \cdots + x_n$, $n \geqq 1$.

LEMMA 1. *If T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1 and if $P_0(N_1 < \infty) < 1$ then*

$$\lim_{n\to\infty} b_n/n^{\frac{1}{2}} = \infty. \tag{35}$$

PROOF. Assume the contrary. Then there is a constant $C > 0$ and an integer sequence $\{n_i, i \geqq 1\}$ such that if $i \geqq 1$ then $b_{n_i} < C(n_i)^{\frac{1}{2}}$. By the central limit theorem

$$\liminf_{i\to\infty} P_0(S_{n_i} > b_{n_i}) \geqq \liminf_{i\to\infty} P_0(S_{n_i} > C(n_i)^{\frac{1}{2}}) > 0.$$

By the Borel-Cantelli lemma,

$$P_0(S_{n_i} > b_{n_i} \text{ for infinitely many } i \geqq 1) > 0,$$

and therefore

$$P_0(S_n > b_n \text{ for infinitely many } n \geqq 1) > 0.$$

By Lévy [9], Section 45, it follows that

$$P_0(S_n > b_n \text{ infinitely often}) = 1.$$

By definition of a test with upper boundary, $N_1 = \infty$ implies if $n \geqq 1$, $S_n \leqq b_n$. Therefore $P_0(N_1 < \infty) = 1$. Contradiction. Therefore (35) follows.

Lemma 2. *Suppose T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1. If for all $\theta > 0$, $P_\theta(N_1 < \infty) = 1$ and if*

$$\begin{aligned} \alpha_2 &= \liminf_{\theta\to 0-} P_\theta(N_1 < \infty, \textit{decide } \theta > 0), \\ \beta_2 &= \liminf_{\theta\to 0+} P_\theta(N_1 < \infty, \textit{decide } \theta < 0), \end{aligned} \tag{36}$$

and if $\alpha_2 + \beta_2 < 1$, then

$$\liminf_{n\to\infty} b_n/n = 0. \tag{37}$$

Proof. It was shown in Section 1, (7) and (8), that

$$P_0(N_1 < \infty) \leqq \alpha_2 + \beta_2 < 1.$$

By Lemma 1 it follows there is an integer n_0 such that if $n \geqq n_0$ then $b_n > 0$. Therefore $\liminf_{n\to\infty} b_n/n \geqq 0$. We will show below that if $\liminf_{n\to\infty} b_n/n = \delta > 0$ then $P_{(\cdot)}(N_1 < \infty$, decide $\theta > 0)$ is a continuous function at the origin. It follows that

$$\begin{aligned} \beta_2 &= \liminf_{\theta\to 0+} P_\theta(N_1 < \infty, \text{decide } \theta < 0) \\ &= 1 - \limsup_{\theta\to 0+} P_\theta(N_1 < \infty, \text{decide } \theta > 0) = 1 - \alpha_2 . \end{aligned}$$

The first part follows since if $\theta > 0$ sampling stops with probability one and a decision is made; the second part follows from the assumed continuity. This contradiction shows that $\delta = 0$ and completes the proof.

Assume then that $\delta > 0$. Let $\delta_1 > 0$ be chosen so that $\mu_{\delta_1} < \delta$. Let $\epsilon > 0$ be given. By the strong law of large numbers there is an integer $N(\epsilon) \geqq 1$ such that

$$P_{\delta_1}(\text{some } n \geqq N(\epsilon), S_n \geqq b_n) < \epsilon/4.$$

It may be shown using the methods of Lehmann [8] that $P_{(\cdot)}$ (some $n \geqq N(\epsilon)$, $S_n \geqq b_n$) is a nondecreasing function of θ. Therefore,

$$\text{if } \theta \leqq \delta_1, P_\theta(\text{some } n \geqq N(\epsilon), S_n \geqq b_n) < \epsilon/4.$$

Since the event $N_1 = n$, decide $\theta > 0$, implies the event $S_n \geqq b_n$, it follows that

$$\text{if } \theta \in \Omega, P_\theta(\text{some } n \geqq N(\epsilon), S_n \geqq b_n) \geqq P_\theta(N_1 \geqq N(\epsilon), \text{decide } \theta > 0).$$

Let

$$d_{\theta_0}(\theta) = |P_\theta(N_1 < \infty, \text{decide } \theta > 0) - P_{\theta_0}(N_1 < \infty, \text{decide } \theta > 0)|.$$

Since $P_{(\cdot)}(N_1 < N(\epsilon)$, decide $\theta > 0)$ is a continuous function of θ, it follows that if $\theta_0 < \delta_1$ then $\limsup_{\theta \to \theta_0} d_{\theta_0}(\theta) < \epsilon/2$. This holds for all $\epsilon > 0$. Therefore continuity is proven.

LEMMA 3. *If T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1, and if $P_0(N_1 < \infty) < 1$ then*

(38) $\lim_{\theta \to 0-} P_\theta(N_1 < \infty, \text{decide } \theta > 0) = P_0(N_1 < \infty, \text{decide } \theta > 0).$

PROOF. If $m \geqq 1$, S_m is a sufficient statistic for the joint distributions of $X_1, \cdots, X_m$. It follows that there is a sequence $\{\gamma_m, m \geqq 1\}$ of Baire functions such that if $m \geqq 1$,

$$\gamma_m(S_m) = P(N_1 = m, \text{decide } \theta > 0 \mid X_1 + \cdots + X_m) \quad \text{a.e.} .$$

Since S_m is a sufficient statistic this conditional probability does not depend on θ. If $S_m < b_m$ then $\gamma_m(S_m) = 0$ a.e. Let

$$d_n(\theta) = P_\theta(N_1 = n, \quad \text{decide} \quad \theta > 0) = \int h(\theta)^n \gamma_n(s_n) \exp(\theta s_n) \prod_{i=1}^n \mu(dx_i).$$

Then

$$(39) \quad \frac{d}{d\theta} d_n(\theta) = \int ((nh'(\theta)/h(\theta)) + s_n) h(\theta)^n \gamma_n(s_n) \exp(\theta s_n) \prod_{i=1}^n \mu(dx_i).$$

By Lemma 1, there is an integer n_1 such that if $n \geqq n_1$ then $b_n > 0$. If $\theta \leqq 0$ then $h'(\theta)/h(\theta) = -\mu \geqq 0$. Except for a set of measure zero $\gamma_m(s_m) > 0$ implies $s_m \geqq b_m$. Therefore if $m \geqq n_1$, $\gamma_m(s_m) > 0$ implies $s_m > 0$ (except for a set of measure zero). If follows that if $\theta \leqq 0$ and $n \geqq n_1$ then (39) is nonnegative, and therefore that $\sum_{n=n_1}^\infty d_n(\theta)$ is a nondecreasing function of $\theta \leqq 0$, $\theta \in \Omega$. Since each function $d_n(\cdot)$ is nonnegative and continuous the sum $\sum_{n=n_1}^\infty d_n(\theta)$ is a lower semi-continuous function. Therefore

$$\sum_{n=n_1}^\infty d_n(0) \leqq \limsup_{\theta \to 0-} \sum_{n=n_1}^\infty d_n(\theta) \leqq \sum_{n=n_1}^\infty d_n(0).$$

Since $P_\theta(N = 0, \text{decide } \theta > 0) + \sum_{n=1}^{n_1-1} d_n(\theta)$ is continuous in θ the lemma follows.

LEMMA 4. *Let T_1 be a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1 such that $P_0(N_1 < \infty) < 1$. Let n_1 be the least integer $\geqq 1$ such that if $n \geqq n_1$ then $b_n > 0$. There exists a real number sequence $\{c_n, n \geqq 1\}$ satisfying,*

(40) *if $n \geqq n_1$ then $c_{n+1} \leqq c_n \leqq b_n/n\sigma^2$; $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} n^{\frac{1}{2}} c_n = \infty$.*

Let $0 < \rho < 1$. If $\{\theta_n, n \geqq 1\}$ is any real number sequence satisfying

for some n_0, if $n \geqq n_0$ then $0 < \theta_n \leqq \rho c_n$, then

(41) $\lim_{n\to\infty} P_{\theta_n}(N_1 = n, \textit{decide } \theta > 0) = 0;$

$\lim_{n\to\infty} P_{\theta_n}(N_1 < n, \textit{decide } \theta > 0) = P_0(N_1 < \infty, \textit{decide } \theta > 0).$

Let

$$\beta_1 = \limsup_{\theta \to 0+} P_\theta(N_1 < \infty, \textit{decide } \theta < 0). \tag{42}$$

Then

$$\begin{aligned} &\liminf_{n\to\infty} P_{\theta_n}(n \leqq N_1 < \infty, \textit{decide } \theta > 0) \\ &\qquad \geqq 1 - \beta_1 - P_0(N_1 < \infty, \textit{decide } \theta > 0). \end{aligned} \tag{43}$$

PROOF. By Lemma 1, $\lim_{n\to\infty} b_n = \infty$. Therefore the integer n_1 exists. Further this implies that $\liminf_{n\to\infty} b_n/n \geqq 0$. If $\liminf_{n\to\infty} b_n/n > 0$ it is trivial to write a sequence satisfying (40). If $\liminf_{n\to\infty} b_n/n = 0$ then a sequence $\{c_n, n \geqq 1\}$ may be defined as follows.

$$c_n = \min_{n_1 \leqq i \leqq n} b_i/i\sigma^2, \qquad n \geqq n_1. \tag{44}$$

It follows at once that $\{c_n, n \geqq n_1\}$ is a nonincreasing sequence that decreases to a limit of zero, and that if $n \geqq n_1$ then $c_n \leqq b_n/n\sigma^2$.

Define a function $g(\cdot)$ by if $m \geqq n_1$, $g(m) = m^{\frac{1}{2}}c_m$. In the trivial case we have at once that $\lim_{m\to\infty} g(m) = \infty$. In the case that the sequence $\{c_n, n \geqq n_1\}$ is defined by (44) we prove that $\lim_{m\to\infty} g(m) = \infty$ as follows. Let $\{m_i, i \geqq 1\}$ be the sequence such that if $i \geqq 1$ then $m_{i+1} > m_i \geqq n_1$ and $c_{m_i} = b_{m_i}/m_i\sigma^2$. Then by Lemma 1, $\lim_{i\to\infty} g(m_i) = \infty$. If $m_i \leqq m < m_{i+1}$ then $c_m = c_{m_i}$ and $g(m) = m^{\frac{1}{2}}c_{m_i} \geqq m_i^{\frac{1}{2}}c_{m_i} = g(m_i)$. Therefore $\lim_{m\to\infty} g(m) = \infty$.

Therefore sequences $\{c_n, n \geqq n_1\}$ satisfying (40) always exist. In the following $g(\cdot)$ is always defined by

$$\text{if} \quad m \geqq n_1 \quad \text{then} \quad g(m) = m^{\frac{1}{2}}c_m. \tag{45}$$

Then

$$\lim_{m\to\infty} g(m) = \infty. \tag{46}$$

To prove Lemma 4 we define a function $k(\cdot)$ on the positive integers such that $k(\cdot)$ has the following properties.

$$\begin{aligned} &\lim_{n\to\infty} c_{k(n)}/c_n = \infty; \qquad \lim_{n\to\infty} k(n)^{\frac{1}{2}}c_n = \infty; \\ &\lim_{n\to\infty} P_{\theta_n}(k(n) < N_1 \leqq n, \text{decide } \theta > 0) = 0; \\ &\lim_{n\to\infty} P_{\theta_n}(N_1 \leqq k(n), \text{decide } \theta > 0) = P_0(N_1 < \infty, \text{decide } \theta > 0). \end{aligned} \tag{47}$$

Since $\theta_n > 0$, $\rho > 0$, it follows that $P_{\theta_n}(N_1 < \infty) = 1$ and therefore that $P_{\theta_n}(N_1 < \infty, \text{ decide } \theta > 0) = 1 - P_{\theta_n}(N_1 < \infty, \text{ decide } \theta < 0)$. It follows that

$$\begin{aligned} &\liminf_{n\to\infty} P_{\theta_n}(n \leqq N_1 < \infty, \text{decide } \theta > 0) \\ &\quad = 1 - \limsup_{n\to\infty} P_{\theta_n}(N_1 < \infty, \text{decide } \theta < 0) \\ &\qquad\qquad - \limsup_{n\to\infty} P_{\theta_n}(N_1 < n, \text{decide } \theta > 0) \\ &\quad \geqq 1 - \beta_1 - P_0(N_1 < \infty, \text{decide } \theta > 0). \end{aligned} \tag{48}$$

The last step in (48) follows from the definition (42) and from (47). Thus (43) is verified.

We now define the function $k(\cdot)$. Let $\{\alpha_n, n \geqq 1\}$ be a sequence of positive real numbers satisfying if $n \geqq 1$, $\alpha_{n+1} \leqq \alpha_n$; $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \alpha_n g(n) = \infty$. Let $n \geqq n_1$ and define

$$(49) \quad \begin{aligned} k(n) &= \text{the greatest integer } m \text{ such that } c_m/c_n \geqq \alpha_m g(m), \\ k(n) &= 0 \text{ if for all } m \geqq n_1,\ c_m/c_n < \alpha_m g(m). \end{aligned}$$

Since $0 = \lim_{m\to\infty} c_m/c_n < \lim_{m\to\infty} \alpha_m g(m) = \infty$ it follows that if $n \geqq n_1$ then $k(n)$ is finite. Note that $c_m/c_{n+1} \geqq c_m/c_n$ so that $k(n+1) \geqq k(n), n \geqq n_1$. Since $\lim_{n\to\infty} c_m/c_n = \infty$, if n is large enough, $k(n) > m$. Therefore

$$(50) \quad \text{if} \quad n \geqq n_1, \quad k(n+1) \geqq k(n); \qquad \lim_{n\to\infty} k(n) = \infty.$$

By definition $c_{k(n)}/c_n \geqq \alpha_{k(n)} g(k(n))$. Since $\lim_{n\to\infty} k(n) = \infty$ it follows that $\lim_{n\to\infty} c_{k(n)}/c_n = \infty$. Again by definition,

$$\begin{aligned} c_{k(n)+1}/c_n &\leqq \alpha_{k(n)+1} g(k(n)+1) \\ &= \alpha_{k(n)+1}[k(n)+1]^{\frac{1}{2}} c_{k(n)+1}. \end{aligned}$$

Therefore

$$1/\alpha_{k(n)+1} \leqq c_n[k(n)+1]^{\frac{1}{2}}.$$

This implies $\lim_{n\to\infty} c_n[k(n)]^{\frac{1}{2}} = \infty$. Thus the first part of (47) is verified.

We now prove $\lim_{n\to\infty} P_{\rho c_n}(k(n) < N_1 \leqq n, \text{decide } \theta > 0) = 0$. We change the notation of Section 3.1 slightly and let $T_1{:}A_1(m)$ be the event $N_1 = m$, decide $\theta > 0$. Let $\{\gamma_m, m \geqq 1\}$ be the sequence of Baire functions specified in the proof of Lemma 3.

$$(51) \quad \begin{aligned} &\text{If } 0 < \rho < 1 \quad \text{then} \quad P_{\rho\theta}(T_1{:}A_1(m)) \\ &\qquad = \int h(\theta)^m \gamma_m(s_m)[h(\rho\theta)/h(\theta)]^m \\ &\qquad\qquad \cdot \exp((\rho-1)\theta s_m) \exp(\theta s_m) \prod_{i=1}^{m} \mu(dx_i). \end{aligned}$$

Using the approximation (23) to $h(\theta)$,

$$(52) \quad \begin{aligned} &(h(\rho\theta)/h(\theta))^m \exp((\rho-1)\theta s_m) \\ &\qquad = \exp\{m\theta\sigma^2(1-\rho)[\theta((\rho+1)/2 + O(\theta)) - s_m/(m\sigma^2)]\}. \end{aligned}$$

Let n_0 be as in (41). Since $O(\theta) \to 0$ as $\theta \to 0$, and since $(\rho+1)/2 < 1$, we may choose an integer $n_2 \geqq \max(n_0, n_1)$ such that if $0 < \theta < 1/n_2$ then $(\rho+1)/2 + O(\theta) \leqq \epsilon' < 1$ and if $i \geqq k(n_2)$ then $\theta_i \leqq c_i \leqq 1/n_2$. Then if $m \leqq n$ and if $m \geqq n_2$, on the event $T_1{:}A_1(m)$,

$$(53) \quad s_m/(m\sigma^2) \geqq b_m/(m\sigma^2) \geqq c_m \geqq c_n.$$

Therefore from (51) and (52)

$$(54)\quad \begin{aligned} P_{\rho c_n}(T_1:A_1(m)) &\leqq P_{c_n}(T_1:A_1(m)) \exp\{mc_n^2\sigma^2(1-\rho)(\epsilon'-1)\} \\ &= P_{c_n}(T_1:A_1(m)) \exp\{-mc_n^2\sigma^2(1-\rho)(1-\epsilon')\} \end{aligned}$$

Then from (54) it follows that

$$(55)\quad \begin{aligned} &P_{\rho c_n}(k(n) < N_1 \leqq n, \text{decide } \theta > 0) \\ &\leqq \sum_{i=k(n)+1}^{n} P_{c_n}(T_1:A_1(i)) \exp\{-ic_n^2\sigma^2(1-\rho)(1-\epsilon')\} \\ &\leqq \max_{k(n)<i\leqq n} \exp\{-ic_n^2\sigma^2(1-\rho)(1-\epsilon')\} \\ &\leqq \exp\{-k(n)c_n^2\sigma^2(1-\rho)(1-\epsilon')\}. \end{aligned}$$

By (47) it follows that $\lim_{n\to\infty} k(n)c_n^2 = \infty$. Therefore

$$(56)\quad \text{if } 0 < \rho < 1, \qquad \lim_{n\to\infty} P_{\rho c_n}(k(n) < N_1 \leqq n, \text{decide } \theta > 0) = 0.$$

We now evaluate $\lim_{n\to\infty} P_{\rho c_n}(N_1 \leqq k(n), \text{decide } \theta > 0)$. Near zero $P_{(\cdot)}(T_1: A_1(m))$ is a convex function. We show this by computing the second derivative and showing that it is positive. The second derivative is

$$(57)\quad \int L(\theta, m, s_m)h(\theta)^m\gamma_m(s_m) \exp(\theta s_m) \prod_{i=1}^{m} \mu(dx_i),$$

where

$$(58)\quad \begin{aligned} &L(\theta, m, s_m) \\ &= m(h(\theta)h''(\theta) - (h'(\theta))^2)/(h(\theta))^2 + (mh'(\theta)/h(\theta) + s_m)^2. \end{aligned}$$

A calculation using the quadratic formula shows that $L(\theta, m, s_m) \geqq 0$ if

$$(59)\quad s_m/m^{\frac{1}{2}} \geqq \{[(h'(\theta))^2 - h(\theta)h''(\theta)]/(h(\theta))^2\}^{\frac{1}{2}} - m^{\frac{1}{2}}h'(\theta)/h(\theta).$$

From the relations (19) to (23) we may compute

$$(60)\quad \begin{aligned} &\lim_{\theta\to0} (1/h(\theta))[(h'(\theta))^2 - h(\theta)h''(\theta)]^{\frac{1}{2}} = \sigma^2; \\ &\lim_{\theta\to0} - h'(\theta)/\theta h(\theta) = \sigma^2. \end{aligned}$$

Let $\epsilon > 0$ be chosen so small that $\rho(1+\epsilon) < 1$. Choose $\theta(\epsilon) > 0$ so small that

$$(61)\quad \text{if } 0 \leqq \theta < \theta(\epsilon) \text{ then } (1/h(\theta))[(h'(\theta))^2 - h(\theta)h''(\theta)]^{\frac{1}{2}} < 2\sigma^2; \qquad -h'(\theta)/h(\theta) < \sigma^2(1+\epsilon)\theta.$$

Then from (59) a sufficient condition that $L(\theta, m, s_m) \geqq 0$ is if $\theta < \theta(\epsilon)$ then $s_m/m^{\frac{1}{2}} \geqq 2\sigma^2 + m^{\frac{1}{2}}\sigma^2(1+\epsilon)\theta$.

On the events $T_1:A_1(m)$ we have

$$(62)\quad s_m/m^{\frac{1}{2}} \geqq b_m/m^{\frac{1}{2}}.$$

Therefore from Lemma 1 it follows that $\lim_{m\to\infty} s_m/m^{\frac{1}{2}} = \infty$. Choose $n_3 \geqq n_2$ such that

$$\text{(63)} \qquad \text{if} \quad m \geqq n_3 \quad \text{then} \quad 2\sigma^2 + m^{\frac{1}{2}}\sigma^2(1+\epsilon)\rho c_m \leqq m^{\frac{1}{2}}\sigma^2 c_m .$$

This is possible since by (40), $\lim_{m\to\infty} m^{\frac{1}{2}} c_m = \infty$, while by choice $\rho(1+\epsilon) < 1$. Then,

$$\text{(64)} \quad \text{if} \quad m \geqq n_3, \quad \theta \leqq \rho c_m, \quad \theta < \theta(\epsilon), \quad \text{then} \quad L(\theta, m, s_m) \geqq 0.$$

This holds since by (62) and (63)

$$s_m/m^{\frac{1}{2}} \geqq b_m/m^{\frac{1}{2}} \geqq m^{\frac{1}{2}}\sigma^2 c_m \geqq 2\sigma^2 + m\sigma^2(1+\epsilon)\rho c_m \geqq 2\sigma^2 + m\sigma^2(1+\epsilon)\theta.$$

Choose $n_4(\rho, \epsilon) \geqq n_3$ so that if $m \geqq n_4$ and if $\theta \leqq \rho c_m$ then $\theta < \theta(\epsilon)$. Then

(65) on the interval $[0, \rho c_n]$ the function $P_{(\cdot)}(n_4 \leqq N_1 \leqq n^*$, decide $\theta > 0)$ is a convex function for every n^* satisfying $n_4 \leqq n^* \leqq n$.

This assertion is correct since

$$\text{(66)} \qquad P_\theta(n_4 \leqq N_1 \leqq n^*, \text{decide } \theta > 0) = \sum_{m=n_4}^{n^*} P_\theta(T_1 : A_1(m)).$$

Since if $n_4 \leqq m \leqq n^*$, $n^* \leqq n$,

$$\text{(67)} \qquad \theta \leqq \rho c_n \leqq \rho c_{n^*} \leqq \rho c_m ,$$

and $\theta < \theta(\epsilon)$ follows. Therefore by (64) the term $P_{(\cdot)}(T_1 : A_1(m))$ is a convex function on the interval $[0, \rho c_n]$.

The first derivative of $P_{(\cdot)}(T_1 : A_1(n))$ is

$$\text{(68)} \qquad \int (nh'(\theta)/h(\theta) + s_n)h(\theta)^n \gamma_n(s_n) \exp(\theta s_n) \prod_{i=1}^{n} \mu(dx_i)$$

which is positive for those $\theta \geqq 0$ such that

$$\text{(69)} \qquad s_n/n \geqq b_n/n \geqq \sigma^2 c_n \geqq -h'(\theta)/h(\theta) = \mu_\theta \geqq 0.$$

Therefore if ϵ and $\theta(\epsilon)$ are as above so that $\rho(1+\epsilon) < 1$ then $-h'(\theta)/h(\theta) \leqq \sigma^2(1+\epsilon)\theta$.

(70) If $m \geqq n_4$, $\theta \leqq \rho c_n$, and $n \geqq m$, then

$$s_m/m \geqq \sigma^2 c_m \geqq \sigma^2(1+\epsilon)\rho c_n \geqq -h'(\theta)/h(\theta).$$

Therefore

(71) if $n_4 \leqq n^* \leqq n$ and $0 \leqq \theta \leqq \rho c_n$ then $P_{(\cdot)}(n_4 \leqq N_1 \leqq n^*$, decide $\theta > 0)$ is a nondecreasing function on the indicated interval.

We may now complete the proof of Lemma 4. Choose n_5 such that $n_5 \geqq n_4$ and if $n \geqq n_5$ then $c_n \leqq c_{k(n)}$. Let $d_{1,n}$ and $d_{2,n}$ be nonnegative numbers with $d_{1,n} + d_{2,n} = 1$ satisfying

$$\text{(72)} \qquad d_{2,n} = c_n/c_{k(n)} .$$

Then if $n \geqq n_5$ it follows that $d_{2,n} \leqq 1$. Further if $n \geqq n_5$ then $k(n) \leqq n$ and from (65) it follows that if $n_4 \leqq k(n)$

$$\begin{aligned} &P_{\rho c_n}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\quad \leqq d_{1,n} P_0(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad\qquad + d_{2,n} P_{\rho c_{k(n)}}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\quad \leqq P_0(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) + (c_n/c_{k(n)}). \end{aligned} \tag{73}$$

Therefore using (47)

$$\begin{aligned} &\limsup_{n\to\infty} P_{\rho c_n}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad\qquad \leqq P_0(n_4 \leqq N_1 < \infty, \text{decide } \theta > 0). \end{aligned} \tag{74}$$

By (71) and the choice of n_4, on the interval $[0, \rho c_{k(n)}]$ the function $P_{(\cdot)}(n_4 \leqq N_1 \leqq k(n)$, decide $\theta > 0)$ is nondecreasing. Then

$$\begin{aligned} &P_0(n_4 \leqq N_1 < \infty, \text{decide } \theta > 0) \\ &\qquad = \lim_{n\to\infty} P_0(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad \leqq \liminf_{n\to\infty} P_{\theta_n}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad \leqq \limsup_{n\to\infty} P_{\theta_n}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad \leqq \limsup_{n\to\infty} P_{\rho c_n}(n_4 \leqq N_1 \leqq k(n), \text{decide } \theta > 0) \\ &\qquad = P_0(n_4 \leqq N_1 < \infty, \text{decide } \theta > 0). \end{aligned} \tag{75}$$

Since $\lim_{n\to\infty} P_{\theta_n}(N_1 < n_4$, decide $\theta > 0) = P_0(N_1 < n_4$, decide $\theta > 0)$, it follows that $\lim_{n\to\infty} P_{\theta_n}(N_1 \leqq k(n)$, decide $\theta > 0) = P_0(N_1 < \infty$, decide $\theta > 0)$. Using (56) it follows that $\lim_{n\to\infty} P_{\rho c_n}(N_1 \leqq n$, decide $\theta > 0) = P_0(N_1 < \infty$, decide $\theta > 0)$. Using (71) and repeating the argument of (75) we find that if $n \geqq n_0$ implies $\theta_n < \rho c_n$ then $\lim_{n\to\infty} P_{\theta_n}(N_1 \leqq n$, decide $\theta > 0) = P_0(N_1 < \infty$, decide $\theta > 0)$. That completes the proof of Lemma 4.

LEMMA 5. *Suppose T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1. Suppose there is an integer n_6 such that if $n \geqq n_6$ then $b_{n+1} \geqq b_n$. If $n \geqq n_1$ (see the statement of Lemma 4) define $c_n = \min_{n_1 \leqq i \leqq n} b_i/(\sigma^2 i)$. If $P_0(N_1 < \infty) < 1$ and if $\liminf_{n\to\infty} b_n/n = 0$ then*

$$\limsup_{n\to\infty} (nc_n)/(n \log(\log n))^{\frac{1}{2}} \geqq (1/\sigma)2^{\frac{1}{2}}. \tag{76}$$

PROOF. Let $L = \limsup_{n\to\infty} (nc_n)/(n \log(\log n))^{\frac{1}{2}}$. If $L = \infty$ there is nothing to prove. We consider only the case L is finite. We will then show that denial of the conclusion of the lemma leads to the conclusion that $P_0(N_1 < \infty) = 1$. The later parts of the argument below will be very much like proofs of the law of the iterated logarithm. It will first be necessary to obtain more information about the possible spacing of values in the sequence $\{b_n, n \geqq 1\}$.

Let $m_1 = n_1$ and if $i \geqq 1$, let m_{i+1} be the least integer m such that $m > m_i$

and $\sigma^2 c_m = b_m/m$. Let $q_1 = m_1 = n_1$ and if $i > 1$, $q_i = m_i/m_{i-1}$. The first step of the proof is to show that if $L < \infty$ then

$$\lim_{i\to\infty} (q_{i+1})^{\frac{1}{2}}/\log(\log m_i) = 0. \tag{77}$$

The function $g(\cdot)$ defined in (45) satisfies

$$g(m_i)(m_i)^{\frac{1}{2}} = b_{m_i}/\sigma^2. \tag{78}$$

If $n \geqq n_6$ it follows from the definitions of n_6, of the sequence $\{c_n, n \geqq n_1\}$, and of the sequence $\{m_i, i \geqq 1\}$, that if $m_i = n \geqq n_6$

$$b_{m_{i+1}-1} \leqq b_{m_{i+1}}; \qquad b_{m_i}/m_i \leqq (b_{m_{i+1}-1})/(m_{i+1} - 1);$$

$$m_{i+1} \leqq 1 + (m_i/b_{m_i})b_{m_{i+1}-1} \leqq 1 + (m_i/b_{m_i})b_{m_{i+1}} \tag{79}$$

$$= 1 + g(m_{i+1})(m_i m_{i+1})^{\frac{1}{2}}/g(m_i).$$

Let an integer $n_7 \geqq n_6$ be so chosen that if $n \geqq n_7$ and $m_i \geqq n_7$ then

$$nc_n \leqq 2L(n \log(\log n))^{\frac{1}{2}}; \qquad m_{i+1} < 2g(m_{i+1})(m_i m_{i+1})^{\frac{1}{2}}/g(m_i). \tag{80}$$

Square the first half of (80), set $n = m_{i+1}$, and substitute the second half of (80) to obtain, if $m_i \geqq n_7$ then

$$m_{i+1}(g(m_{i+1}))^2 = m_{i+1}^2 c_{m_{i+1}}^2 \leqq 4L^2(2g(m_{i+1})(m_i m_{i+1})^{\frac{1}{2}}/g(m_i))$$
$$\cdot \log(\log[2g(m_{i+1})(m_i m_{i+1})^{\frac{1}{2}}/g(m_i)]);$$
$$\geqq \log(\log(2g(m_{i+1})(m_i m_{i+1})^{\frac{1}{2}}/g(m_i)))$$
$$= (g(m_i)g(m_{i+1})/(8L^2))(m_{i+1}/m_i)^{\frac{1}{2}}$$
$$= (g(m_i)g(m_{i+1})/8L^2)(q_{i+1})^{\frac{1}{2}}.$$

Taking exponentials of both sides and using the fact that $\log(m_i m_{i+1})^{\frac{1}{2}} = \log m_i + (\frac{1}{2}) \log q_{i+1}$ we obtain

$$\log(2g(m_{i+1})/g(m_i)) + \log m_i + \log(q_{i+1})^{\frac{1}{2}}$$
$$\geqq \exp((g(m_i)g(m_{i+1})/8L^2)(q_{i+1})^{\frac{1}{2}}).$$

Therefore

$$\log(m_i(q_{i+1})^{\frac{1}{2}}) \geqq \exp((g(m_i)g(m_{i+1})/8L^2)((q_{i+1})^{\frac{1}{2}} + o(1)));$$
$$[\log(\log(m_i(q_{i+1})^{\frac{1}{2}}))]/(q_{i+1})^{\frac{1}{2}} \geqq (g(m_i)g(m_{i+1})/8L^2)(1 + o(1)).$$

Since $\lim_{i\to\infty} g(m_i) = \infty$ it follows that (77) must hold.

It follows that there is an integer $n_8 \geqq n_7$ such that if $i \geqq n_8$ then for some $j > 0$, $i! \leqq m_j \leqq (i + 1)!/(\log i)$. For let $n_i^* = i!$ and $n_i' = (i + 1)!/(\log i)$. Then

$$\lim_{i\to\infty} (\log(\log n_i^*))^{-1}(n_i'/n_i^*)^{\frac{1}{2}} = \lim_{i\to\infty} (\log(\log i!))^{-1}[(i + 1)/(\log i)]^{\frac{1}{2}} = \infty.$$

Choose n_8 so large that if $i \geqq n_8$ then $(\log(\log n_i^*))^{-1}(n_i'/n_i^*)^{\frac{1}{2}} \geqq 2$ while if $i \geqq n_8$ then $(\log(\log m_i))^{-1}(q_{i+1})^{\frac{1}{2}} = (\log(\log m_i))^{-1}(m_{i+1}/m_i)^{\frac{1}{2}} < 1$. This is possible by virtue of (77). Then if $j \geqq n_8$ it is impossible that $m_j < i!$ and $m_{j+1} > (i+1)!(\log i)^{-1}$. We assume below that $\{m_i', i \geqq 1\}$ is a subsequence of $\{m_i, i \geqq 1\}$ such that if $i \geqq n_8$ then $i! \leqq m_i' \leqq (i+1)!(\log i)^{-1}$.

We now prove

$$\limsup_{i\to\infty} b_{m_i}(m_i \log(\log m_i))^{-\frac{1}{2}} \geqq \sigma 2^{\frac{1}{2}}.$$

Suppose to the contrary that for some $0 < \epsilon < 1$ that

$$\limsup_{i\to\infty} b_{m_i}(m_i \log(\log m_i))^{-\frac{1}{2}} < \epsilon\sigma 2^{\frac{1}{2}}.$$

We use arguments similar to those of Lévy [9] to show $P_0(S_n > b_n$ infinitely often$) = 1$. This will contradict the hypothesis that $P_0(N_1 < \infty) < 1$. It is sufficient to show that with probability one $S_{m_i'} > b_{m_i'}$ infinitely often. Since the random variable X_1 has moments of all orders the law of the iterated logarithm applies to $\{X_n, n \geqq 1\}$. If $\epsilon' > \sigma 2^{\frac{1}{2}}$ then with probability one for all large values of n, $S_n > -\epsilon'(n \log(\log n))^{\frac{1}{2}}$. Therefore it is sufficient to show that with probability one,

$$S_{m'_{i+1}} - S_{m_i'} > b_{m'_{i+1}} + \epsilon'(m_i' \log(\log m_i'))^{\frac{1}{2}}$$

for infinitely many i. Since $\lim_{i\to\infty}(m_i'/m'_{i+1}) = 0$, if $1 > \epsilon'' > \epsilon$ we may choose n_9 so large that if $i \geqq n_9$ then

$$b_{m'_{i+1}} + \epsilon'(m_i' \log(\log m_i'))^{\frac{1}{2}} < \epsilon''\sigma 2^{\frac{1}{2}}(m'_{i+1} \log(\log m'_{i+1}))^{\frac{1}{2}}.$$

Therefore it is sufficient to show that with probability one, for infinitely many i,

$$S_{m'_{i+1}} - S_{m_i'} \geqq \epsilon''\sigma 2^{\frac{1}{2}}(m'_{i+1} \log(\log m'_{i+1}))^{\frac{1}{2}}.$$

By virtue of the Borel zero-one criterion it is sufficient to show

$$\sum_{i=1}^{\infty} P_0(S_{m'_{i+1}} - S_{m_i'} \geqq \epsilon''\sigma 2^{\frac{1}{2}}(m'_{i+1} \log(\log m'_{i+1}))^{\frac{1}{2}}) \tag{81}$$

is a divergent series. Since X_1 has a finite third moment there is a constant $d > 0$ such that for every $n \geqq 1$,

$$\sup_{-\infty<\lambda<\infty}\left|P_0(S_n/(\sigma n^{\frac{1}{2}}) \leqq \lambda) - \int_{-\infty}^{\lambda} (1/(2\pi)^{\frac{1}{2}}) \exp(-x^2/2)\, dx\right| \leqq d/n^{\frac{1}{2}}.$$

See for example Esseen [2]. Since $\sum_{i=1}^{\infty}(m'_{i+1} - m_i')^{-\frac{1}{2}} < \infty$, to prove the series (81) divergent it suffices to treat $\{X_n, n \geqq 1\}$ as if they were independent normal $0, \sigma^2$ random variables. Recalling that if $i \geqq n_8$ then $i! \leqq m_i' \leqq (i+1)!(\log i)^{-1}$, and that $\int_\lambda^\infty (1/(2\pi)^{\frac{1}{2}}) \exp(-x^2/2)\, dx$ is asymptotically $(1/\lambda(2\pi)^{\frac{1}{2}}) \exp(-\lambda^2/2)$ as $\lambda \to \infty$, a standard calculation shows that the series in question diverges. That completes the proof of Lemma 5.

LEMMA 6. *Suppose T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1. Let $\epsilon > 0$. There exists a test T_2 with upper boundary $\{d_n, n \geqq 1\}$ and stopping variable N_2 and an integer m such that*

(a) *if $n \geqq 1$ then $d_n \leqq b_n$;*

(b) *if $n \geqq m$ then $d_{n+1} \geqq d_n$;*

(c) *if* $N_2 \geqq m$ *and* $S_{N_2} > d_{N_2}$ *then sampling stops and the decision* $\theta > 0$ *is made.*

(d) $N_2 \leqq N_1$.

(e) *if* $\theta \varepsilon \Omega$, $P_\theta(N_2 < \infty$, *decide* $\theta < 0) \leqq P_\theta(N_1 < \infty$, *decide* $\theta < 0)$.

(f) $P_0(N_2 < \infty$, *decide* $\theta > 0) \leqq P_0(N_1 < \infty$, *decide* $\theta > 0) + \epsilon$.

PROOF. Let $\rho = \inf_{n\geqq 1} P_0(X_1 + \cdots + X_n > 0)$. Since $E_0X_1 = 0$ and $\sigma^2 > 0$, if $n \geqq 1$ then $P_0(S_n > 0) > 0$. By the central limit theorem $\lim_{n\to\infty} P_0(S_n/n^{\frac{1}{2}} > 0) = \frac{1}{2}$. Therefore $\rho > 0$. We define n_{10} to be the least integer such that $P_0(n_{10} \leqq N_1 < \infty) < \epsilon\rho$.

Define a set of integers A by $n \varepsilon A$ if and only if

(82) $$n \geqq n_{10}; \qquad b_n > 0; \qquad \text{if } m \geqq n \text{ then } b_m \geqq b_n.$$

Since $\lim_{n\to\infty} b_n = \infty$, it follows that A is an infinite set. Suppose $A = \{m_i, i \geqq 1\}$ with the enumeration so chosen that if $i \geqq 1$ then $m_{i+1} \geqq m_i$.

Define $\{d_n, n \geqq 1\}$ as follows. If $n \leqq m_1$, $d_n = b_n$; if $i \geqq 1$ and $m_i < n \leqq m_{i+1}$ then $d_n = b_{m_{i+1}}$. It follows that if $n > m_1$ then $d_{n+1} \geqq d_n$. We define $m = m_1$ to satisfy (b) of the lemma.

To verify (a) observe that if $m_i < j < m_{i+1}$ and $b_j \leqq b_{m_{i+1}}$ then there is a largest integer $j' < m_{i+1}$ such that $b_{j'} \leqq b_{m_{i+1}}$. This implies $j' \varepsilon A$ contradicting the definition of A and the enumeration $\{m_i, i \geqq 1\}$. Therefore if $m_i < j < m_{i+1}$ it follows that $b_j > b_{m_{i+1}} = d_j$. Since $b_{m_i} = d_{m_i}$, $i \geqq 1$, while if $n \leqq m$ then $b_n = d_n$, (a) now follows.

We now define the decision rules for the test T_2. Let $T_i{:}A_0(n)$ be the event that $N_i = n$ and $\theta < 0$ is decided by T_i, where N_i is the stopping variable for the test T_i, $i = 0, 1$. Let $T_i{:}A_1(n)$ be the event that $N_i = n$ and $\theta > 0$ is decided, $i = 0, 1$. If $0 \leqq n \leqq m_1$, $i = 0, 1$, then $T_2{:}A_i(n) = T_1{:}A_i(n)$. If $n > m_1$ then $T_2{:}A_1(n)$ is defined as follows. If $N_2 \geqq n$, $b_n > d_n$, and $S_n \geqq d_n$ then $N_2 = n$ and $\theta > 0$ is decided; if $N_2 \geqq n$, $b_n = d_n$, and $S_n \geqq d_n$ then sampling stops and $\theta > 0$ is decided if and only if $N_1 = n$ and using T_1, $\theta > 0$ is decided. $T_2{:}A_0(n)$ is defined as follows. If $N_2 \geqq n$, $b_n > d_n$, $S_n < d_n$ and $T_1{:}A_0(n)$ then $T_2{:}A_0(n)$; if $N_2 \geqq n$, $b_n = d_n$, $S_n \leqq d_n$ and $T_1{:}A_0(n)$ then $T_2{:}A_0(n)$. In all other cases $N_2 \geqq n + 1$.

We now verify (d) of Lemma 6. In the event $N_1 = n$ and $T_1{:}A_1(n)$ then $S_n \geqq b_n \geqq d_n$. If $N_2 \geqq n = N_1$ then by the definitions of the preceding paragraph, if $b_n > d_n$ it follows that $N_2 = n$; if $b_n = d_n$ then since $S_n \geqq d_n$ and $T_1{:}A_1(n)$ it follows $N_2 = n$. In the event $N_1 = n$ and $T_1{:}A_0(n)$ then $S_n \leqq b_n$; if $N_2 \geqq n = N_1$ and $b_n > d_n$ then it follows that $N_2 = n$; if $b_n = d_n$ then $S_n \leqq d_n$ follows and $N_2 = n$ follows. Therefore $N_1 = n$, $N_2 \geqq n$ implies $N_1 = n = N_2$. That proves (d).

We now verify (c). We have defined $m = m_1$. If $N_2 = n > m$ and $S_n > d_n$ then there are two cases. If $b_n = d_n$ then $S_n > b_n$, and since $n = N_2 \leqq N_1$ it follows that $N_1 = n$, $T_1{:}A_1(n)$ holds, and therefore from the definition of T_2 that $T_2{:}A_1(n)$ holds; if $b_n > d_n$ then it follows at once from the definitions that $T_2{:}A_1(n)$ holds.

We verify (e). If $N_2 = n$ and $\theta < 0$ is decided and $b_n > d_n$ then according to the construction $S_n < d_n$, $N_1 = n$ and $T_1{:}A_0(n)$. If $b_n = d_n$ then according to the construction $S_n \leqq d_n$, $N_1 = n$, and $T_1{:}A_0(n)$. Thus the event $T_2{:}A_0(n)$ implies the event $T_1{:}A_0(n)$.

We verify (f). Observe that

$P_0(N_2 < \infty, T_2{:}A_1) \leqq P_0(N_1 < \infty, T_1{:}A_1) + P_0(m_1 < N_2 < \infty, T_2{:}A_1)$.

If $n \varepsilon A$ then since $N_2 \leqq N_1$ it follows that

$$P_0(N_2 = n, T_2{:}A_1) \leqq P_0(N_2 = n, m_1 < N_1 < \infty). \tag{83}$$

If $n \geqq m_1$ and $n \not\varepsilon A$ let $i \geqq 1$ be the least integer such that $n + i \varepsilon A$. Then

$$\begin{aligned} \rho P_0(N_2 = n, T_2{:}A_1) &\leqq P_0(X_{n+1} + \cdots + X_{n+i} > 0)P_0(N_2 = n, T_2{:}A_1) \\ &\leqq P_0(N_2 = n, m_1 < N_1 < \infty). \end{aligned} \tag{84}$$

Summing the inequalities (83) and (84) gives

$$\begin{aligned} P_0(m < N_2 < \infty, T_2{:}A_1) &\leqq (1/\rho)P_0(m < N_1 < \infty) \\ &< (1/\rho)(\rho\epsilon) = \epsilon. \end{aligned} \tag{85}$$

That completes the proof of (f) and of Lemma 6.

LEMMA 7. *Suppose T_1 is a test with upper boundary $\{b_n, n \geqq 1\}$ and stopping variable N_1 such that $P_0(N_1 < \infty) < 1$ and $\liminf_{n\to\infty} b_n/n = 0$. Let $\{c_n, n \geqq 1\}$ be defined as for Lemma 5. Then*

$$\limsup_{n\to\infty} nc_n/(n \log(\log n))^{\frac{1}{2}} \geqq 2^{\frac{1}{2}}/\sigma. \tag{86}$$

PROOF. Let ϵ be so small that $P_0(N_1 < \infty) + \epsilon < 1$. Let T_2 be a test with upper boundary $\{d_n, n \geqq 1\}$ and stopping variable N_2 such that T_2 satisfies the conclusions of Lemma 6 in relation to T_1, ϵ, m. It follows that

$$\begin{aligned} P_0(N_2 < \infty) \leqq P_0(N_1 < \infty, \text{decide } \theta < 0) + P_0(N_1 < \infty, \\ \text{decide } \theta > 0) + \epsilon = P_0(N_1 < \infty) + \epsilon < 1. \end{aligned}$$

By Lemma 1, $\lim_{n\to\infty} d_n/n^{\frac{1}{2}} = \infty$. Therefore

$$0 \leqq \liminf_{n\to\infty} d_n/n \leqq \liminf_{n\to\infty} b_n/n = 0.$$

Let n_{11} be the least integer such that if $n \geqq n_{11}$ then $d_n > 0$. Define a number sequence by $c_n' = (1/\sigma^2) \min_{n_{11} \leqq i \leqq n} d_i/i$. By Lemma 5,

$$\limsup_{n\to\infty} (nc_n')/(n \log(\log n))^{\frac{1}{2}} \geqq 2^{\frac{1}{2}}/\sigma. \tag{87}$$

Let $n_{12} \geqq n_{11}$ be the least integer n such that $b_n/n = \sigma^2 c_n$. Then if $n \geqq n_{12}$,

$$c_n = \min_{n_{12} \leqq i \leqq n} b_i/(i\sigma^2) \geqq \min_{n_{11} \leqq i \leqq n} d_i/(i\sigma^2) = c_n'. \tag{88}$$

The proof of Lemma 7 is completed by using (87) and (88) together.

3.3 *Proof of Theorem* 1 *per se*. Let T be a test of the hypothesis $\theta < 0$ against the alternative $\theta > 0$ with stopping variable N. We define a test T_1 as follows. Let N_1 be the stopping variable for T_1. Then $N = N_1$. If $N_1 = n$ then T_1 always decides $\theta > 0$. By Theorem 3 there is a test S with upper boundary $\{b_n, n \geqq 1\}$ such that

if $\theta > 0$ then $E_\theta N_S \leqq E_\theta N_1$,

$$P_0(N_1 < \infty) = P_0(N_S < \infty) \quad \text{and } P_\theta(N_S < \infty) = 1.$$

Here, N_S is the stopping variable of S. Using the test S the decision $\theta < 0$ is never made. By Lemma 1, $\lim_{n\to\infty} b_n/n^{\frac{1}{2}} = \infty$. By Lemma 3

$$\lim_{\theta\to 0-} P_\theta(N_S < \infty, \text{decide } \theta > 0) = P_0(N_S < \infty, \text{decide } \theta > 0).$$

Since $P_\theta(N_S < \infty, \text{decide } \theta < 0) = 0$ for all $\theta \varepsilon \Omega$, and since $P_0(N_S < \infty) < 1$, we may use Lemma 2 to obtain $\liminf_{n\to\infty} b_n/n = 0$. If we define n_1 as for Lemma 4 and $\{c_n, n \geqq n_1\}$ as for Lemma 5 then by Lemma 4, if $0 < \rho < 1$,

$$\begin{aligned}\lim_{n\to\infty} P_{\rho c_n}(n < N_S < \infty) &= 1 - \lim_{n\to\infty} P_{\rho c_n}(N_S < n) \\ &= 1 - \lim_{n\to\infty} P_{\rho c_n}(N_S < n, \text{decide } \theta > 0) \\ &= 1 - P_0(N_S < \infty, \text{decide } \theta > 0) \\ &= 1 - P_0(N_S < \infty) = P_0(N_S = \infty) = P_0(N = \infty).\end{aligned}$$

By Lemma 7, $\limsup_{n\to\infty} (nc_n)/(n \log(\log n))^{\frac{1}{2}} \geqq 2^{\frac{1}{2}}/\sigma$. Let $0 < \epsilon < 2^{\frac{1}{2}}/\sigma$ and $\{m_i, i \geqq 1\}$ be an integer sequence such that

$$(m_i c_{m_i})/(m_i \log(\log m_i))^{\frac{1}{2}} \geqq (2^{\frac{1}{2}}/\sigma) - \epsilon.$$

Let $\{\theta_i, i \geqq 1\}$ be a real number sequence such that if $i \geqq 1$ then $0 < \theta_i \leqq c_{m_i}$ and

$$(m_i\theta_i)/(m_i \log(\log m_i))^{\frac{1}{2}} = (2^{\frac{1}{2}}/\sigma) - \epsilon.$$

Inversion of this equality shows that

$$m_i = ((2^{\frac{1}{2}}/\sigma) - \epsilon)^2(1 + o(1))\theta_i^{-2} \log(\log|\theta_i|^{-1})).$$

Since $0 < \theta_i \leqq c_{m_i}$, by Lemma 4 it follows that

$$\lim_{i\to\infty} P_{\rho\theta_i}(m_i < N_S < \infty) = P_0(N_S = \infty) = P_0(N = \infty).$$

By construction $E_{\rho\theta_i}N \geqq E_{\rho\theta_i}N_S \geqq m_i P_{\rho\theta_i}(m_i < N_S < \infty)$. Therefore

$$(\rho\theta_i)^2|\log|\log|\rho\theta_i|||^{-1}E_{\rho\theta_i}N \geqq \rho^2((2^{\frac{1}{2}}/\sigma) - \epsilon)^2(1 + o(1))P_{\rho\theta_i}(m_i < N_S < \infty).$$

It follows that

$$\limsup_{i\to\infty} (\rho\theta_i)^2|\log|\log|\rho\theta_i|||^{-1}E_{\rho\theta_i}N \geqq \rho^2((2^{\frac{1}{2}}/\sigma) - \epsilon)^2 P_0(N = \infty).$$

Since $\rho < 1$ and $\epsilon > 0$ are arbitrary and since $\lim_{\theta\to 0} \mu_\theta/\theta = \sigma^2$, it follows that

$$\limsup_{\theta\to 0+} \mu_\theta^2|\log|\log|\mu_\theta|||^{-1}E_\theta N \geqq 2\sigma^2 P_0(N = \infty).$$

We now prove the last statement of Theorem 1. To prove $\lim_{\theta\to 0+} \theta^2 E_\theta N = \infty$ suppose the contrary. Then there exists a sequence $\{\theta_m, m \geqq 1\}$, such that if

$m \geqq 1$, then $\theta_{m+1} \leqq \theta_m$, and such that $\lim_{m\to\infty} \theta_m = 0$, $\limsup_{m\to\infty} \theta_m^2 E_{\theta_m} N < \infty$. Let $\{m_i, i \geqq 1\}$ and $\{q_i, i \geqq 1\}$ be integer sequences such that if $i \geqq 1$ then $\rho c_{m_i} \geqq \theta_{q_i} \geqq \rho c_{m_i+1}$. It follows that $\lim_{i\to\infty} m_i^{\frac{1}{2}} \theta_{q_i} = \infty$ by the definition of $\{c_n, n \geqq 1\}$.

By Lemma 4,

$$\lim_{i\to\infty} P_{\theta_{q_i}}(m_i < N_S < \infty) = P_0(N_S = \infty) = P_0(N = \infty) > 0.$$

Also

$$E_{\theta_{q_i}} N \geqq E_{\theta_{q_i}} N_S \geqq m_i P_{\theta_{q_i}}(m_i < N_S < \infty).$$

Since $\lim_{i\to\infty} m_i \theta_{q_i}^2 = \infty$ it follows that $\lim_{i\to\infty} \theta_{q_i}^2 E_{\theta_{q_i}} N = \infty$. Contradiction.

In order to complete the proof of Theorem 1 it is necessary to show the construction of certain generalized sequential probability ratio tests. In the sequel we will use the following notation: $\log_2 x = \log(\log x)$, $\log_3 x = \log(\log(\log x))$. Let $0 < \alpha$, $0 < \beta$, and $\alpha + \beta < 1$. We will show that there is a GSPRT with stopping variable N for the problem stated in Section 1 such that

$$\lim_{\theta\to 0} \mu_\theta^2 (\log_2 |\mu_\theta|^{-1})^{-1} E_\theta N = 2\sigma^2(1 - \alpha - \beta).$$

To actually construct the test we use the law of the iterated logarithm. It has been proven by Cantelli [1] that if $\{X_n, n \geqq 1\}$ is a sequence of independently and identically distributed random variables such that $E|X_1|^3 < \infty$ and if $c > 3$ then the sequence

$$\{\sigma 2^{\frac{1}{2}}(n(\log_2(n + e) + c \log_3(n + e^e)))^{\frac{1}{2}}, n \geqq 1\}$$

is in the upper class for the sequence $\{X_n, n \geqq 1\}$. We write $n + e$ and $n + e^e$ so that the quantities will be defined if $n \geqq 0$. Given $c > 3$ we may pick an integer n_{13} so large that

$$P_0(\text{some } n \geqq n_{13} + 1, |S_n| \geqq \sigma 2^{\frac{1}{2}}(n(\log_2(n + e) + (c \log_3(n + e^e)))^{\frac{1}{2}}) \leqq \min(\alpha, \beta).$$

We may pick a_i, b_i for $1 \leqq i \leqq n_{13}$ and define if $n > n_{13}$ then $-a_n = b_n = \sigma 2^{\frac{1}{2}}(n(\log_2(n + e) + c \log_3(n + e^e)))^{\frac{1}{2}}$, and introduce appropriate randomization so that the choices be made to satisfy

$$P_0(N < \infty) = \alpha + \beta, P_0(N < \infty, \text{decide } \theta > 0) = \alpha,$$
$$P_0(N < \infty, \text{decide } \theta < 0) = \beta.$$

We will assume below that randomization is used only in the cases $X_1 = a_1$ or $X_1 = b_1$, it being possible to choose $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ in this way. We suppose then $N \geqq 2$ and $S_N \geqq b_N$ means that sampling stopped and $\theta > 0$ was decided; $N \geqq 2$ and $S_N \leqq a_N$ means that sampling stopped and $\theta < 0$ was decided. $N > 0$ and $b_i > S_i > a_i$, $1 \leqq i \leqq n - 1$ means $N \geqq n$. Finally we may suppose that $P_0(N = 0) = 0$.

It follows at once from the strong law of large numbers that if $\theta \neq 0$ then

$P_\theta(N < \infty) = 1$. Further, as noted in Section 1, $P_{(\cdot)}(N < \infty$, decide $\theta > 0)$ is a left continuous and nondecreasing function. It follows that $\sup_{\theta<0} P_\theta(N < \infty$, decide $\theta > 0) \leqq \alpha$. Similarly $P_{(\cdot)}(N < \infty$, decide $\theta < 0)$ is a right continuous and nonincreasing function, and it follows that $\sup_{\theta>0} P_\theta(N < \infty$, decide $\theta < 0) \leqq \beta$.

We will show first that for the GSPRT just defined

(89) $$\liminf_{\theta\to0} \theta^2|\log|\log|\theta|||^{-1}E_\theta N \geqq (2/\sigma^2)P_0(N = \infty).$$

To do this we use the fact that $P_{(\cdot)}(N \leqq n$, decide $\theta > 0)$ is a nondecreasing function. Consequently if $0 < \rho < 1$ and $\{\theta_n, n \geqq 1\}$ is a real number sequence such that if $n \geqq 1$ then $\rho c_n \leqq \theta_n \leqq \rho c_{n-1}$, where $\{c_n, n \geqq 1\}$ is defined as for Lemma 5, then

$$P_{\rho c_n}(N < n, \text{ decide } \theta > 0) \leqq P_{\theta_n}(N < n, \text{ decide } \theta > 0)$$
$$\leqq P_{\rho c_{n-1}}(N < n-1, \text{ decide } \theta > 0) + P_{\rho c_{n-1}}(N = n-1, \text{ decide } \theta > 0).$$

It follows from one step in the proof of Lemma 4 that

$$\lim_{n\to\infty} P_{\rho c_n}(N = n, \text{ decide } \theta > 0) = 0.$$

Therefore by Lemma 4,

(90) $$\lim_{n\to\infty} P_{\theta_n}(N < n, \text{ decide } \theta > 0) = P_0(N < \infty, \text{ decide } \theta > 0).$$

Also, since $P_{(\cdot)}(N < \infty$, decide $\theta < 0)$ is a right continuous and nonincreasing function,

(91) $$\lim_{n\to\infty} P_{\theta_n}(N < \infty, \text{ decide } \theta < 0) = P_0(N < \infty, \text{ decide } \theta < 0).$$

Since

(92) $$1 = P_{\theta_n}(N < n, \text{ decide } \theta > 0) + P_{\theta_n}(n \leqq N < \infty, \text{ decide } \theta > 0) + P_{\theta_n}(N < \infty, \text{ decide } \theta < 0),$$

using (90), and (91), and taking a limit on n in (92) gives

$$\lim_{n\to\infty} P_{\theta_n}(n \leqq N < \infty, \text{ decide } \theta > 0) = P_0(N = \infty).$$

From the definition of the upper boundary $\{b_n, n \geqq 1\}$ for the GSPRT being considered it is easily verified that there is an integer n_{14} such that if $n \geqq n_{14}$ then $c_n = b_n/n$, that is,

(93) $$nc_n = (2^{\frac12}/\sigma)(n(\log_2(n + e) + c\log_3(n + e^e)))^{\frac12}.$$

As observed above inversion of the equality (93) gives

(94) $$n = ((2/\sigma^2) + o(1))c_n^{-2}\log_2|c_n|^{-1}.$$

If $\rho c_n \leqq \theta_n \leqq \rho c_{n-1}$ then by considering only the event $N \geqq n$, decide $\theta > 0$, which means the upper boundary is crossed, we find from (94)

(95) $$E_{\theta_n}N \geqq (2/\sigma^2 + o(1))c_n^{-2}\log_2|c_n|^{-1}P_{\theta_n}(n \leqq N < \infty, \text{ decide } \theta > 0).$$

It follows from the assumption that $\theta_n \geqq \rho c_n$ together with (95) that

$$\text{(96)} \qquad \liminf_{\theta\to 0+} \theta^2(\log_2 |\theta|^{-1})^{-1} E_\theta N \geqq (2/\sigma^2)\rho^2 P_0(N = \infty).$$

Since $\rho < 1$ is now arbitrary,

$$\text{(97)} \qquad \liminf_{\theta\to 0+} \theta^2(\log_2 |\theta|^{-1})^{-1} E_\theta N \geqq (2/\sigma^2) P_0(N = \infty).$$

The case $\theta < 0$, treated by similar arguments, together with (97) complete the proof of (89).

We will obtain an upper bound on $E_\theta N$ using an identity due to Wald [11]. According to this identity,

$$\text{(98)} \qquad \mu_\theta E_\theta N = E_\theta S_N .$$

Since $S_N = S_{N-1} + X_N$ and since $S_{N-1} < b_{N-1}$, we will obtain the required upper bound by finding an upper bound to $E_\theta|X_N|$, and an upper bound to $E_\theta S_{N-1}$. If we apply the identity due to Wald [11] mentioned above to $X_1^2 + \cdots + X_n^2$ then we find

$$E_\theta|X_N| \leqq (E_\theta(X_1^2 + \cdots + X_N^2))^{\frac{1}{2}} = (E_\theta X_1^2 E_\theta N)^{\frac{1}{2}}.$$

We may therefore find a constant K such that if $-1 < \theta < 1$ then

$$\text{(99)} \qquad E_\theta|X_N| \leqq K(E_\theta N)^{\frac{1}{2}}.$$

Let $f(\cdot)$ be a positive real valued function satisfying

$$\text{(100)} \qquad \lim_{\theta\to 0} f(\theta) = \infty, \quad \text{and} \quad \lim_{\theta\to 0} \theta^2 f(\theta) = 0.$$

From (98) we obtain

$$\mu_\theta E_\theta N \leqq E_\theta|X_N| + P_\theta(N \leqq f(\theta))E_\theta(S_{N-1} \mid N \leqq f(\theta)) + P_\theta(N > f(\theta))E_\theta(S_{N-1} \mid N > f(\theta)).$$

Since $P_0(N = \infty) > 0$ it follows from (89) that, using the upper bound (99), and the fact that $\lim_{\theta\to 0} \mu_\theta/\theta = \sigma^2$,

$$\text{(101)} \qquad E_\theta|X_N| = o(\mu_\theta E_\theta N).$$

Similarly, since if $N = n$ then $|S_{n-1}| \leqq \max(a_n, b_n) = b_n$ when $n \geqq n_{13}$, it follows that $E_\theta(|S_{N-1}||N \leqq f(\theta)) = o(\mu_\theta E_\theta N)$. Therefore

$$\text{(102)} \qquad \mu_\theta E_\theta N \leqq o(\mu_\theta E_\theta N) + P_\theta(N > f(\theta))E_\theta(g(N) \mid N > f(\theta)).$$

Here we define the function $g(\cdot)$ by

$$g(x) = \sigma 2^{\frac{1}{2}}(x(\log_2 (x + e) + c \log_3 (x + e^e)))^{\frac{1}{2}}.$$

There is an integer n_{15} such that on the interval $[n_{15}, \infty)$ the function $g(\cdot)$ is concave. Using Jensen's inequality on (102) and setting $y(\theta) = E_\theta(N \mid N > f(\theta))$ we find

$$\text{(103)} \qquad \mu_\theta E_\theta N \leqq o(\mu_\theta E_\theta N) + P_\theta(N > f(\theta))g(y(\theta)).$$

But $E_\theta N \geqq y(\theta)P_\theta(N > f(\theta))$ so that from (103) follows

(104) $$(1 + o(1))\mu_\theta y(\theta) \leqq g(y(\theta)).$$

Inversion of this inequality gives

(105) $$y(\theta) \leqq (2\sigma^2 + o(1))\mu_\theta^{-2} \log_2 |\mu_\theta|^{-1}.$$

From (100) it follows that

(106) $$\lim_{\theta\to 0} \mu_\theta^2 E_\theta(N \mid N \leqq f(\theta)) \leqq \lim_{\theta\to 0} (\mu_\theta/\theta)^2\theta^2 f(\theta) = 0.$$

From (105) and (106) it follows that

(107) $$\limsup_{\theta\to 0+} \mu_\theta^2(\log_2 |\mu_\theta|^{-1})^{-1}E_\theta N \leqq 2\sigma^2 \lim_{\theta\to 0+} P_\theta(N > f(\theta)).$$

It follows from Lemma 4 that

(108) $$\lim_{\theta\to 0+} P_\theta(N > f(\theta)) = P_0(N = \infty).$$

To see this let n_θ = the least integer $\geqq f(\theta)$. We may without loss of generality suppose $f(\cdot)$ is a strictly increasing function and therefore have that $\theta \leqq f^{-1}(n_\theta)$. Since $\lim_{\theta\to 0} \theta^2 f(\theta) = 0$ it follows that $\lim_{x\to\infty} x^{\frac{1}{2}}f^{-1}(x) = 0$. We may therefore find an integer n_{16} such that if $n_\theta \geqq n_{16}$ then $f^{-1}(n_\theta) \leqq \rho c_{n_\theta}$. It follows that there is a $\delta > 0$ such that if $0 < \theta < \delta$ then

(109) $$0 < \theta \leqq f^{-1}(n_\theta) \leqq \rho c_{n_\theta}.$$

Then

$$P_0(N < n_\theta, \text{ decide } \theta > 0) \leqq P_\theta(N < n_\theta, \text{ decide } \theta > 0) \leqq P_{\rho c_{n_\theta}}(N < n_\theta, \text{ decide } \theta > 0).$$

By Lemma 4 it follows that

(110) $$\lim_{\theta\to 0+} P_\theta(N < n_\theta, \text{ decide } \theta > 0) = P_0(N < \infty, \text{ decide } \theta > 0).$$

Also if $n_\theta \geqq n$ then

$$P_0(N < n_\theta, \text{ decide } \theta < 0) \geqq P_\theta(N < n_\theta, \text{ decide } \theta < 0) \geqq P_\theta(N < n, \text{ decide } \theta < 0).$$

Therefore taking a limit on θ,

$$P_0(N < \infty, \text{ decide } \theta < 0) \geqq \limsup_{\theta\to 0+} P_\theta(N < n_\theta, \text{ decide } \theta < 0)$$
$$\geqq \liminf_{\theta\to 0+} P_\theta(N < n_\theta, \text{ decide } \theta < 0) \geqq P_0(N < n, \text{ decide } \theta < 0).$$

Since this holds for every $n \geqq 1$ it follows that

(111) $$\lim_{\theta\to 0+} P_\theta(N < n_\theta, \text{ decide } \theta < 0) = P_\theta(N < \infty, \text{ decide } \theta < 0).$$

(110) and (111) together imply that (108) must hold.

The proof is completed by a similar argument for $\theta < 0$.

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