B. V. GNEDENKO, *The Theory of Probability*, (translated from the Russian by B. D. Seckler). Chelsea Publishing Company, New York, 1962. $8.75.

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The number of introductory text-books in probability theory seems to increase at a threatening pace. Most of these books resemble each other strongly, and it is only seldom that one appears with unique characteristics that merit special attention. Feller's book was one; the present one by Gnedenko is another.

Gnedenko's book is not new. Its first Russian edition was published already in 1950 and a German edition appeared in 1957. It has been used as a textbook in many European countries. The English translation follows the original closely except for a few additions.

After a preliminary discussion of the concept of probability the author mentions some alternative ways of constructing a theory of randomness based on symmetry considerations or starting from frequencies. An outline is given of Kolmogorov's axiomatic system, but no knowledge is assumed of measure theory nor does the author attempt to present a mathematically complete treatment of the elements of this theory. This results in certain gaps in the exposition; the reader is made aware of them and referred elsewhere for detailed proofs. In this way the author gives the reader a good idea of how probability theory can be deduced from a few axioms; he does this using comparatively little space and without obscuring the main body of results by measure-theoretic manipulations. This seems to be a good compromise for an elementary text-book. All the time the results are illustrated by concrete and well-chosen examples.

Much attention is given to probabilistic limit theorems. The theorem of de Moivre-Laplace, for example, is given both in its local and integrated form, and both in the univariate and the multivariate case. The error of approximation is also studied numerically. Finite Markov chains have a chapter of their own. The convergence of the transition probabilities in the positive case is obtained using Markov's own very pretty proof. An item seldom found in introductory books is the generalization of the theorem of de Moivre-Laplace to the case of a Markov chain with two possible states.

The usual difficulties arise when stochastic variables are introduced. Again Gnedenko avoids the more troublesome measure theoretic difficulties but points them out to the reader. To the reviewer this seems an economic and honest way of solving the didactic problem. Some readers may object to it and it is certainly not the orthodox thing to do among contemporary writers of text-books in mathematics.

One of the best chapters in the book is that dealing with the law of large numbers, as could be expected from the author's own interests. The derivations are concise, often elegant, and the results are sometimes a good deal stronger or

more complete than one usually finds in elementary books. For example, the law of large numbers is proved in Khinchin's formulation, but without using Fourier analysis. The strong law is derived from an inequality of Hájek-Rényi, which includes as a particular case the famous inequality of Kolmogorov.

Characteristic functions are discussed in detail in a separate chapter. In a lucid manner the author proves the uniqueness and continuity theorems. An unusual topic for an introductory book is the section on positive definite functions, which is not only relevant to the study of characteristic functions but is also useful for a reader interested in stationary stochastic processes.

The central limit theorem is studied in various forms, under the Lindeberg condition, in the formulation of Liapunov, and locally for a lattice distribution. The scope is widened by considering more general limit theorems leading to the infinitely divisible distributions. The latter ones are studied and characterized by Kolmogorov's canonical representation. With this knowledge the reader gets acquainted with the basic limit theorems for sums, including the special limit theorems belonging to the normal and Poisson distributions. For the probabilistic reader this is indispensible, but also the theoretical statistician should be familiar with these beautiful results. It is high time that this subject should be included in the curriculum of the theoretical statistician; at present this is far from always the case.

The same is true a fortiori concerning stochastic processes. Gnedenko treats some of the basic processes. The Poisson process is derived in Khinchin's simple and attractive manner. Proceeding to Markov processes, the author uses of course the concept of conditional probability. It is surprising that he does not at least sketch the modern way of introducing this via Radon-Nikodym derivatives. The purely continuous as well as the purely discontinuous Markov processes are presented and their functional equations are derived and studied. In a very short space the author manages to convey a lot of valuable information to the reader about these topics as well as about stochastic processes with independent increments. Stationary processes are defined although the author does not choose to include a description of the construction of the whole probability measure. These sections are limited to second order properties such as the Bochner-Khinchin representation of the covariance function and the corresponding spectral representation of the process due to Cramér. The author goes outside the second order framework by proving the strong ergodic theorem for stationary sequences.

The last chapter of the book is called "Elements of Statistics," but its contents are not the conventional ones. It is true that confidence intervals and statistical tests are discussed briefly, but classical estimation is scarcely mentioned and the standard procedures are not described except in a few cases. Instead there is a very elegant treatment of Kolmogorov's and Smirnov's results concerning empirical distribution functions. The section on Bayes estimation will appeal to many readers; the asymptotic behavior of the a posteriori distribution is obtained

in the normal case. These results certainly deserve to be better known, but this chapter leaves one with a feeling that a better balance could have been achieved.

This translation of Gnedenko's book is a welcome addition to the textbook literature in the English language. The least satisfactory part of it is perhaps when the author discusses the philosophy of probability. This discussion seems a bit nebulous and is certainly at variance with the conventional way of dealing with these questions in Western text-books. It should be stressed, though, that this concerns only a minor part of the book and is more than compensated by its merits. Among these I would mention especially the emphasis on concrete mathematical results, the elegance of the presentation, and the lucid style of the author.