# PSEUDO-INVERSES IN THE ANALYSIS OF VARIANCE

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**1. Summary.** The normal equations in the analysis of variance with suitable side conditions give a unique set of estimates, $\tilde{\beta}_i$, of the parameters $\beta_i$. These estimates are unique linear forms in the actual observations. In the solutions to the normal equations they appear as linear forms in the treatment and block totals; these totals are not independent and so the forms in them are not unique. Thus the normal equations, while giving a unique solution vector, admit an infinite number of pseudo-inverses of the matrix $X'X$. In this paper the relationship between the two most common pseudo-inverses is discussed.

**2. The general case.** Let $\mathbf{X}$ be the design matrix. $\mathbf{X}$ has $n$ rows and $p$ columns with rank $(p - m)$. There exists a matrix $\mathbf{D}$ of order $p \times m$ with rank $m$ such that $\mathbf{XD} = \mathbf{0}$. The normal equations $\mathbf{X'X}\tilde{\boldsymbol{\beta}} = \mathbf{X'Y}$ are consistent. If $\tilde{\boldsymbol{\beta}} = \mathbf{PX'Y}$ is any solution vector, the matrix $\mathbf{P}$ is called a pseudo-inverse or generalized inverse ($g$-inverse) of $X'X$ (Rao, 1962). When the normal equations are solved subject to the set of linear constraints $\mathbf{H}\tilde{\boldsymbol{\beta}} = \mathbf{0}$, where $\mathbf{H}$ is a matrix of order $m \times p$ such that $\mathbf{HD}$ has rank $m$, the solution vector $\tilde{\boldsymbol{\beta}}$ is unique, but the $g$-inverse $\mathbf{P}$ is not. The relationship $\mathbf{D'X'Y} = \mathbf{0}$ allows different estimates of the same parameter $\beta_i$ to be identical numerically but to differ in form by some linear combination of the rows of $\mathbf{D'X'Y}$. Thus, if $\mathbf{P}^*$ is any other $g$-inverse of $\mathbf{X'X}$, subject to $\mathbf{H}\tilde{\boldsymbol{\beta}} = 0$,

$$\mathbf{P}^* = \mathbf{P} + \mathbf{ED}', \tag{1}$$

where $\mathbf{E}$ is a matrix of order $p \times m$. If only symmetric $g$-inverses are considered, (1) becomes $\mathbf{P}^* = \mathbf{P} + \mathbf{DCD}'$, where $\mathbf{C}$ is a symmetric matrix of order $m$.

There are two standard methods of obtaining a $g$-inverse of $\mathbf{X'X}$. In one method (Graybill [1], p. 292, and Kempthorne [2], p. 79), the matrix $X'X$ is augmented to

$$\begin{pmatrix} \mathbf{X'X} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{pmatrix} = B^{-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}^{-1}$$

Then $\mathbf{B}_{11}$ is a $g$-inverse of $\mathbf{X'X}$.

In the second method (Plackett [3], p. 41, and Scheffé [5], p. 19), the $g$-inverse is $(\mathbf{X'X} + \mathbf{H'H})^{-1}$. These two $g$-inverses are not identical. Plackett has shown that

$$\operatorname{cov}(\tilde{\boldsymbol{\beta}}) = (\mathbf{I} - \mathbf{D}(\mathbf{HD})^{-1}\mathbf{H})(\mathbf{X'X} + \mathbf{H'H})^{-1}\sigma^2.$$

With this result, $\mathbf{B}_{11}$ can be obtained from $(\mathbf{X'X} + \mathbf{H'H})^{-1}$, but

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$(\mathbf{I} - \mathbf{D}(\mathbf{HD})^{-1}\mathbf{H})^{-1}$ does not exist, since $\mathbf{I} - \mathbf{D}(\mathbf{HD})^{-1}\mathbf{H}$ is idempotent and thus not of full rank.

In this note a different relationship (2) between these $g$-inverses is derived by which each may be obtained from the other. Multiplying out $\mathbf{BB}^{-1} = \mathbf{I}$ gives the four equations

$$\mathbf{X}'\mathbf{XB}_{11} + \mathbf{H}'\mathbf{B}_{21} = \mathbf{I}_p, \qquad \mathbf{X}'\mathbf{XB}_{12} + \mathbf{H}'\mathbf{B}_{22} = \mathbf{0},$$
$$\mathbf{HB}_{11} = \mathbf{0}, \qquad \mathbf{HB}_{12} = \mathbf{I}_m.$$

After multiplying the first two equations on the left by $\mathbf{D}'$ and simplifying, we have $\mathbf{B}_{22} = \mathbf{0}$, $\mathbf{B}_{12} = \mathbf{D}(\mathbf{HD})^{-1}$ and

$$\mathbf{X}'\mathbf{XB}_{11} = \mathbf{I} - \mathbf{H}'\mathbf{B}_{21} = \mathbf{I} - \mathbf{H}'(\mathbf{D}'\mathbf{H}')^{-1}\mathbf{D}'.$$

Then $\mathbf{X}'\mathbf{XB}_{11}\mathbf{X}'\mathbf{Y} = \mathbf{X}'\mathbf{Y} - \mathbf{H}'(\mathbf{D}'\mathbf{H}')^{-1}\mathbf{D}'\mathbf{X}'\mathbf{Y} = \mathbf{X}'\mathbf{Y}$ and $\tilde{\boldsymbol{\beta}} = \mathbf{B}_{11}\mathbf{X}'\mathbf{Y}$ is a solution. The variance covariance matrix of the estimates is $\operatorname{cov}(\tilde{\boldsymbol{\beta}}) = \mathbf{B}_{11}\mathbf{X}'\mathbf{XB}_{11}\sigma^2 = \mathbf{B}_{11}\sigma^2$. If $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is estimable, $\boldsymbol{\lambda}'\mathbf{D} = 0$ and $\operatorname{var}(\boldsymbol{\lambda}'\tilde{\boldsymbol{\beta}}) = \boldsymbol{\lambda}'\mathbf{B}_{11}\lambda = \boldsymbol{\lambda}'\mathbf{P}^*\boldsymbol{\lambda}$ so that, for computing the variance of any estimable function, any pseudo-inverse may be used as the variance covariance matrix.

Writing $(\mathbf{X}'\mathbf{X} + \mathbf{H}'\mathbf{H})^{-1} = \mathbf{B}_{11} + \mathbf{DCD}'$, we have

$$\mathbf{I} = (\mathbf{X}'\mathbf{X} + \mathbf{H}'\mathbf{H})(\mathbf{B}_{11} + \mathbf{DCD}') = \mathbf{X}'\mathbf{XB}_{11} + \mathbf{H}'\mathbf{HDCD}'$$
$$= \mathbf{I} - \mathbf{H}'(\mathbf{D}'\mathbf{H}')^{-1}\mathbf{D}' + \mathbf{H}'\mathbf{HDCD}'.$$

Then $\mathbf{H}'(\mathbf{D}'\mathbf{H}')^{-1}\mathbf{D}' = \mathbf{H}'\mathbf{HDCD}'$, and multiplying both sides on the left by $(\mathbf{D}'\mathbf{H}'^{-1}\mathbf{D}'$ and on the right by $\mathbf{H}'(\mathbf{D}'\mathbf{H}')^{-1}$ gives $(\mathbf{D}'\mathbf{H}')^{-1} = \mathbf{HDC}$, whence

$$\mathbf{C} = (\mathbf{HD})^{-1}(\mathbf{D}'\mathbf{H}')^{-1} = (\mathbf{D}'\mathbf{H}'\mathbf{HD})^{-1},$$

so that the desired relationship is

$$(2) \qquad (\mathbf{X}'\mathbf{X} + \mathbf{H}'\mathbf{H})^{-1} = \mathbf{B}_{11} + \mathbf{D}(\mathbf{D}'\mathbf{H}'\mathbf{HD})^{-1}\mathbf{D}'.$$

**3. Application to incomplete block designs.** Let $\mathbf{A}\hat{\boldsymbol{\tau}} = \mathbf{Q}$ be the adjusted intrablock normal equations for an incomplete block design, where $\mathbf{Q}$ is the vector of adjusted treatment totals, $\hat{\boldsymbol{\tau}}$ the vector of the estimates of the $v$ treatment effects, and $\mathbf{A}$ is a symmetric matrix of order $v$ and rank $(v - 1)$. Then $\mathbf{A1} = \mathbf{0}$ and $\mathbf{1}'\mathbf{Q} = 0$ where $\mathbf{1}$ is a vector with each element unity. We solve the equations subject to the single side condition $\mathbf{H}\hat{\boldsymbol{\tau}} = \sum_i h_i\hat{\tau}_i = 0$. Substituting $\mathbf{A}$ for $\mathbf{X}'\mathbf{X}$ and $\mathbf{1}$ for $\mathbf{D}$ in (2) above gives

$$(\mathbf{A} + \mathbf{H}'\mathbf{H})^{-1} = \mathbf{B}_{11} + \mathbf{1}(\mathbf{1}'\mathbf{H}'\mathbf{H1})^{-1}\mathbf{1}' = \mathbf{B}_{11} + \mathbf{11}'/(\sum_i h_i)^2.$$

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