SIGNIFICANCE PROBABILITY BOUNDS FOR RANK ORDERINGS

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1. Summary and introduction. The problem of two-sample rank order tests is examined from the point of view of I. R. Savage (1956, 1957, 1959, 1962). Under suitable restriction of the class of alternatives some rank orderings are always more probable than (dominated by) others. Hence, if the rejection region of a test contains the ordering c it must also contain all orderings dominated by c if the test is to be admissible. Thus, by counting the number of orderings dominated by c we arrive at a lower bound for the size of an admissible test which rejects when c is observed. Similar reasoning leads to an upper bound. The counting is achieved by putting all orderings in one-one correspondence with paths on a grid; all paths lying below or along the observed path correspond to the orderings dominated by the observed ordering. An expression for this number of paths is obtained. This expression is used to compute significance bounds for a pair of illustrative examples. Finally, the main result for the two-sample problem is extended to obtain upper and lower bounds for a one-sample problem.

2. Two-sample case. Let $X_1, \cdots, X_m$ be a sample from the distribution with density $f(x)$, and let $Y_1, \cdots, Y_n$ be an independent sample from the distribution with density $g(y)$. We wish to test the hypothesis $f = g$ against alternatives for which the likelihood ratio $g(u)/f(u)$ is monotone, say increasing.

Now, let the vector of ranks of the X's in the pooled sample be denoted by $c = (c_1, \cdots, c_m)$ whose elements are in increasing order. If $c' = (c_1', \cdots, c_m')$ is any vector in the sample space of c, we will say that c dominates c' (written $c' < c$) provided $c_j' \leqq c_j$ for all j. For example, in the case $m = 3$, $c = (134)$, the c' dominated by c are [(134), (124), (123)]. I. R. Savage (1956) has shown that if $c' < c$ ($c' \neq c$), then $P(c') > P(c)$ under alternatives with increasing likelihood ratio.

Let Ω be a rejection region for a non-randomized admissible test such that $c \, \varepsilon \, \Omega$. Since Ω is admissible, $c' \, \varepsilon \, \Omega$ whenever $P(c') > P(c)$, hence $c' \, \varepsilon \, \Omega$ whenever $c' < c$. Therefore, the size of Ω cannot be less than $T_m(c)/T = L(c)$, where $T_m(c)$ is the number of $c' < c$, and $T = \binom{m+n}{n}$. In other words, if c is the observed ordering then no admissible test can reject the hypothesis at levels less than $L(c)$. Alternatively, if *any* admissible test has size less than $L(c)$, then c is not in the rejection region for that test.

The number $T_m(c)$ may be computed by representing c as a path on a grid from the point $(0, 0)$ to the point $(m, c_m - m)$, thus: Starting from $(0, 0)$ take the jth step to the right if j is an element of c, otherwise take a step up. Then

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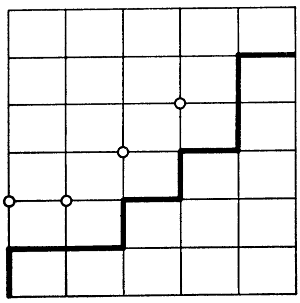

FIG. 1.

the number of $c' < c$ may be found by counting the paths which lie entirely below or along the path representing c. Now we observe that if a path does not satisfy this condition, then it must pass through at least one of the points $q_j = (j - 1, c_j - j + 1), j = 1, \cdots, m - 1$. The case $m = 5, n = 6, c = (2, 3, 5, 7, 10)$ is illustrated in Figure 1. The heavy line is the path corresponding to the observed ordering c, and the circled points are the q_j's.

Therefore, the number of distinct paths from $(0, 0)$ to $(m, c_m - m)$ which cross the path c is equal to $\sum_{j=1}^{m-1} f_j h_j$, where

$$h_j = \text{total number of paths from } q_j \text{ to } (m, c_m - m) = \binom{c_m - c_j}{m - j + 1}, \quad \text{and}$$

$$\begin{aligned} f_j &= \text{number of paths from } (0, 0) \text{ to } q_j \text{ which do not pass through } q_k, \\ &\qquad\qquad k = 1, \cdots, j - 1 \\ &= (\text{total number of paths from } (0, 0) \text{ to } q_j) \\ &\quad - \sum_{k=1}^{j-1} (\text{number of paths from } (0,0) \text{ to } q_{j-k}) \cdot (\text{number of paths from } q_{j-k} \text{ to } q_j). \end{aligned}$$

That is

$$f_j = \binom{c_j}{j - 1} - \sum_{k=1}^{j-1} f_{j-k} \binom{c_j - c_{j-k}}{k}, \qquad \text{with } f_1 = 1. \tag{1}$$

Hence, $T_m(c)$ = number of paths which do *not* cross path c

$$= \binom{c_m}{m} - \sum_{j=1}^{m-1} \binom{c_m - c_j}{m - j + 1} \cdot f_j, \tag{2}$$

where the f_j are given by (1). Finally, $T_m(c)/T$ gives the desired lower bound, $L(c)$.

Similarly, we can find an upper bound $U(c)$ such that every nonrandomized admissible test of size greater than $U(c)$ will have the ordering c in its rejection region. This can be obtained by interchanging the roles of the X's and Y's, thus: Let $\bar{c}$ denote the vector of Y-ranks corresponding to the vector of X-ranks,

c. For example, if $m = 3$, $n = 4$, $c = (134)$, then $\bar{c} = (2567)$. Then the desired upper bound is $U(c) = 1 - T_n(\bar{c})/T$.

3. Examples. Suppose we have observed 7 X's and 10 Y's and we are interested in significance levels of 10 % or less.

(a) The ordered ranks of the X's in the pooled sample are $c = (2, 5, 6, 9, 10, 11, 16)$. Then, using (1) we get

$$f_1 = 1, \qquad f_2 = 2, \qquad f_3 = 7, \qquad f_4 = 16, \qquad f_5 = 62, \qquad f_6 = 158;$$

and using (2) we get

$$T_7(c) = \binom{16}{7} - \binom{14}{7}1 - \binom{11}{6}2 - \binom{10}{5}7 - \binom{6}{3}62 - \binom{5}{2}158 = 1{,}940;$$

$$T = \binom{17}{7} = 19{,}448.$$

Hence the hypothesis will not be rejected by any admissible test which has significance level less than $L(c) = 1{,}940/19{,}448 = 10.0\%$. For this example, $U(c) = 77.3\%$.

(b) The ordered ranks of the X's in the pooled sample are $c = (1, 2, 4, 5, 6, 10, 13)$. Then $\bar{c} = (3, 7, 8, 9, 11, 12, 14, 15, 16, 17)$ and $T_{10}(\bar{c}) = 17{,}461$. Hence the hypothesis will be rejected by every admissible test which has significance level greater than $U(c) = 1 - 17{,}461/19{,}448 = 10.2\%$. For this example, $L(c) = 0.4\%$.

4. One sample case. Let $X_1, \cdots, X_N$ be a sample from a distribution with density $f(x)$. We wish to test the hypothesis that $f(x)$ is symmetric about zero. Now let the observations be ranked according to magnitude and let the vector of ranks of the positive observations be denoted by $c = (c_1, \cdots, c_m)$, $m \leqq N$. Further, let ${}_kc = (c_{m-k+1}, \cdots, c_m)$ be the rank vector of the k largest positive observations, $k = 0, 1, \cdots, m$.

If $c' = (c_1', \cdots, c_r')$, $r \leqq N$, is any vector in the range of c, we will say that $c' \prec c$ provided $r \leqq m$ and $c' < {}_rc$, as defined in Section 2. For example, in the case $N = 6$, $c = (134)$, the $c' \prec c$ are [(134), (124), (123), (34), (24), (14), (23), (13), (12), (4), (3), (2), (1), (none)].

It is easy to show that if $c' \prec c$ $(c \neq c')$, then $P(c') > P(c)$ provided we consider only those alternatives for which, whenever $x > 0$, $f(-x)/f(x)$ is greater than one and increases with x. [In this regard we have Theorems 2.1 and 3.1 of Savage (1959).] Within this context, the significance level of an admissible test which rejects when the observed vector is c cannot be less than (number of $c' \prec c)/2^N$.

However, we have that (number of $c' \prec c$) $= \sum_{k=0}^{m}$ (number of $c' < {}_kc$) $= \sum_{k=0}^{m} T_k({}_kc)$, where T_k is given by (2). Hence the desired lower bound is $\sum_{k=0}^{m} T_k({}_kc)/2^N$. The corresponding upper bound is easily found to be $1 - \sum_{k=0}^{N-m} T_k({}_k\bar{c})/2^N$.

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REFERENCES

SAVAGE, I. R. (1956). Contributions to the theory of rank order statistics: the two-sample case. *Ann. Math. Statist.* **27** 590–615.

SAVAGE, I. R. (1957). Contributions to the theory of rank order statistics: the "trend" case. *Ann. Math. Statist.* **28** 968–977.

SAVAGE, I. R. (1959). Contributions to the theory of rank order statistics: the one-sample case. *Ann. Math. Statist.* **30** 1018–1023.

SAVAGE, I. R. (1962). Contributions to the theory of rank order statistics: applications of lattice theory. Technical Report No. 15, Univ. of Minnesota.