A LIMIT THEOREM FOR RANDOM INTERVAL SAMPLING OF A STOCHASTIC PROCESS

By Richard F. Gebhard

United Aircraft Corporate Systems Center

1. Introduction and summary. A time series is the realization of a stochastic process. An estimate of the process average can be taken as the equally weighted average of observed values taken from the time series. Normally, the sequence of observations consists of observations taken at equal intervals of time. In certain applications, it is more appropriate to consider a sequence of observations spaced by intervals which are randomly and independently chosen from an exponential distribution.

Let $[x_t, t \varepsilon T]$ be a real stochastic process that takes on values in a space X, T being restricted to the real line. Let Ω be a space of points ω, $F_T = \beta(x_t, t \varepsilon T)$ be the Borel field of ω-sets generated by the class of sets of the form $[x_t(\omega) \varepsilon A]$ where $t \varepsilon T$ and A is any Borel set, P be the probability measure of F_T-sets, α be the Borel field of X-sets and for every real t, $x_t(\cdot)$ be a function from Ω to X such that $[x_t(\omega) \varepsilon A]$ is an F_T-set for $A \varepsilon \alpha$. Assume that the process satisfies, for all $t \varepsilon T$, the conditions

$$E\{x_t\} = \bar{x} \quad \text{exists and is independent of } t \tag{1.1}$$

$$E\{(x_t - \bar{x}^2\} = \text{var } x \quad \text{exists and is independent of } t \tag{1.2}$$

$$E\{(x_t - \bar{x})(x_{t+\tau} - \bar{x})\} = r(\tau) \text{ var } x \quad \text{depends only on } \tau \text{ and } r(\tau) \text{ is harmonizable.} \tag{1.3}$$

A stochastic process satisfying Conditions (1.1) to (1.3) is often referred to as being stationary in the wide sense [2].

$$\lim_{\tau \to \infty} \tau r(\tau) \to 0. \tag{1.4}$$

Let $\{\xi\}$ be a measurable space of mutually independent random variables ξ_i identically distributed with probability p of having value 1 and $(1 - p)$ of having value 0, all independent of the x_t-process.

The sample space Ω contains all possible realizations of the stochastic process. Given a scheme for observing the values of x_t in time and some form of estimate of process mean from the observed values, expected values of such an estimate and of its square can be determined. A measure of the efficiency of the sampling scheme may be defined as the ratio of the variance of the estimator to that which would be obtained if the observations taken were independent of each other. The limit in probability of this measure of sampling efficiency for observations spaced by exponentially distributed mutually independent intervals is obtained simply in terms of the mean observation rate and the spectral density component at zero frequency. The limit theorem considered is a weak version. By

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augmenting Condition (1.4) with relations between sample size N, mean observation rate λ, length of time series T and sufficiently rapid decay of $r(\tau)$ with τ, sharper probability statements can be made.

2. A limit theorem. Let $x_t(\omega)$ be any particular realization of the stochastic process. Let a sequence of samples of the realization taken at equal intervals of time $(\Delta t, 2\Delta t, \cdots, M\Delta t)$, with time reckoned from an origin $t = 0$, be designated by the random sequence $A_1 : (x_1, x_2, \cdots, x_M)$. Consider the sequence of mutually independent random variables $A_2 : (\xi_1, \xi_2, \cdots, \xi_M)$. Form the sequence $A_3 : (\xi_1 x_1, \xi_2 x_2, \cdots, \xi_M x_M)$. From sequence A_3, a point estimate of process mean is formed as

$$\hat{m} = \left\{ \sum_{i=1}^{M} \xi_i x_i \right\} \Big/ N \tag{2.1}$$

where $N = \sum_{i=1}^{M} \xi_i$ is an integer. Clearly, the expected value of $\hat{m}$ with respect to x over the Ω-space is $\bar{x}$. Noting that $\xi_i^2 = \xi_i$ for all i and that the process has wide-sense stationarity, the variance of $\hat{m}$ with respect to x over the Ω-space is simply

$$\operatorname{var}_\Omega(\hat{m}) = (\operatorname{var} x)/N + 2 (\operatorname{var} x) \sum_{k=1}^{M} \gamma_k r_{1,k+1}/N^2 \quad \text{with} \quad \gamma_k = \sum_{j=1}^{M-k} \xi_j \xi_{j+k} . \tag{2.2}$$

Defining $R_N(\hat{m})$ to be the ratio of $\operatorname{var}_\Omega(\hat{m})$ to the variance which would be obtained for mutually uncorrelated x_j's, we obtain

$$R_N(\hat{m}) = 1 + 2 \sum_{k=1}^{M} \gamma_k r_{1,k+1}/N. \tag{2.3}$$

This is a measure of the efficiency of the sampling scheme in forming the estimate of the mean of a stochastic process in the manner described by Equation (2.1).

THEOREM. *Given a real stochastic process* $\{x_t, t \varepsilon T\}$ *satisfying conditions* (1.1) *through* (1.4) *and a sequence of observations spaced by time intervals chosen randomly and independently from an exponential distribution, then* $\lim_{N\to\infty} E_\xi\{R_N(\hat{m})\} - \{1 + 2\lambda\psi(0)\} \to 0$ *in probability where* $\psi(0) = \int_0^\infty r(\tau)\, d\tau$ *and* λ *is the mean observation rate.*

PROOF. Setting the p associated with the ξ-space equal to $\lambda\Delta t$ and allowing Δt to approach zero transforms the sequence A_3 into the desired sequence of randomly spaced observations. For fixed N, taking expectations over the ξ-space yields

$$E_\xi(\gamma_k) = (M - k)p^2 \tag{2.4}$$

and further

$$E_\xi\{R_N(\hat{m})\} = 1 + (2\lambda(Mp)/N) \sum_{k=1}^{M} r(k\Delta t)\Delta t - (2\lambda^2/N) \sum_{k=1}^{M} r(k\Delta t)(k\Delta t)\Delta t. \tag{2.5}$$

M is distributed according to the negative binomial as the conditional distribution on the hypothesis of N. Invoking the law of large numbers,

(2.6) $$\lim_{N\to\infty} (Mp/N - 1) \to 0 \quad \text{in probability.}$$

Considering the three terms of (2.5) upon taking limits as $N \to \infty$ and $\Delta t \to 0$, we note that the first term remains intact, the second term reduces to $2\lambda \int_0^\infty r(\tau)\, d\tau$ and the third term reduces to zero by virtue of Condition (1.4) being satisfied. By the Wiener-Khintchine Theorem [1], [3], the spectral density of the $(x_t - \bar{x})$-process evaluated at zero frequency, $x(0)$ is equal to $\int_0^\infty r(\tau)\, d\tau$. Since N and Δt are not related, the order of taking limits is immaterial.

REFERENCES

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