ASYMPTOTIC DISTRIBUTION OF DISTANCES BETWEEN ORDER STATISTICS FROM BIVARIATE POPULATIONS

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1. Summary. The exact and limiting distribution of quantiles in the univariate case is well known. Mood [3] investigated the joint distribution of medians in samples from a multivariate population, showing that their distribution is asymptotically multivariate normal. Recently Siddiqui [4] considered the joint distribution of two quantiles and an auxiliary statistic and showed that asymptotically their joint distribution is trivariate normal. Further, he showed the "distances" $X'_{i+l} - X'_i$, $X'_i - X'_{i-h}$, (l and h fixed positive integers) between quantiles in the univariate case, when appropriately normalized are asymptotically independently distributed as Chi square r.v.'s with $2l$ and $2h$ d.f. respectively. In this paper the joint distribution of several quantiles from a bivariate population is obtained and it is shown that the distances between quantiles in the separate component populations are independent asymptotically.

2. Assumptions and notation. Let $F(x, y)$ be the absolutely continuous d.f. of the pair of random variables (r.v.'s) (X, Y), having joint p.d.f. $f(x, y)$ and marginal d.f.'s and p.d.f.'s $F_1(x)$, $F_2(y)$, $f_1(x)$, and $f_2(y)$ respectively. Let ζ_α, η_β be the unique real numbers satisfying $F_1(\zeta_\alpha) = \alpha$, $F_2(\eta_\beta) = \beta$, $0 < \alpha, \beta < 1$, with $f_1(\zeta_\alpha) \neq 0, f_2(\eta_\beta) \neq 0$, and put $q_1 = F(\zeta_\alpha, \eta_\beta)$, $q_2 = \beta - q_1$, $q_3 = \alpha - q_1$, $q_4 = 1 - \alpha - \beta + q_1$. Then ζ_α, η_β are quantiles of orders α and β, respectively, of F_1 and F_2. We assume that $F(x, y)$ has first and second partial derivatives continuous in a neighborhood of $(\zeta_\alpha, \eta_\beta)$. Let (X_i, Y_i), $i = 1, \cdots, n$, be a random sample drawn from $F(x, y)$, and let $Z_1^{(n)} \leqq Z_2^{(n)} \leqq \cdots \leqq Z_n^{(n)}$ be the ordered sample values of $X_1, X_2, \cdots, X_n$. Similarly, let $W_1^{(n)} \leqq W_2^{(n)} \leqq \cdots \leqq W_n^{(n)}$ be the ordered values of $Y_1, \cdots, Y_n$. Finally, $\{r^{(n)}\}$, $\{s^{(n)}\}$ will denote sequences of positive integers depending on the sample size n in such a way that $r^{(n)}/n \rightarrow \alpha$, $s^{(n)}/n \rightarrow \beta$. We shall first find an asymptotic approximation to the joint p.d.f. of the $k + l + 2$ r.v.'s $Z_{r^{(n)}+i}^{(n)}$, $W_{s^{(n)}+j}^{(n)}$, $i = 0, 1, \cdots, k$; $j = 0, 1, \cdots, l$, where k and l are fixed positive integers, and then show that the "distances"

$$d_{r^{(n)}+i,1}^{(n)} = Z_{r^{(n)}+i+1}^{(n)} - Z_{r^{(n)}+i}^{(n)}, \qquad i = 0, 1, \cdots, k \tag{1}$$

$$d'^{(n)}_{s^{(n)}+j,1} = W_{s^{(n)}+j+1}^{(n)} - W_{s^{(n)}+j}^{(n)}, \qquad j = 0, 1, \cdots, l \tag{2}$$

are asymptotically independent r.v.'s. Unless otherwise specified, the range on

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i and j will henceforth be assumed as given above. For simplicity we shall omit the superscript in $r^{(n)}$ and $s^{(n)}$ in what follows.

We shall use the following notation:

$$
\begin{aligned}
&\rho_1 = F(x_0, y_0), \qquad \rho_2 = F_2(y_0) - F(x_k, y_0), \qquad \rho_3 = F_1(x_0) - F(x_0, y_l), \\
(3)\quad &\rho_4 = 1 - F_1(x_k) - F_2(y_l) + F(x_k, y_l), \qquad \rho_i^{(1)} = \int_{-\infty}^{y_0} f(x_i, v)\, dv, \\
&\sigma_j^{(1)} = \int_{-\infty}^{x_0} f(u, y_j)\, du, \qquad \rho_i^{(2)} = \int_{y_l}^{\infty} f(x_i, v)\, dv, \qquad \sigma_j^{(2)} = \int_{x_k}^{\infty} f(u, y_j)\, du.
\end{aligned}
$$

3. Asymptotic joint distribution of $(d_{r^{(n)}+i,1}^{(n)}, d_{s^{(n)}+j,1}^{(n)})$. First we shall derive the joint density function of $Z_{r+i}^{(n)}$, $W_{s+j}^{(n)}$, $i = 0, 1, \cdots, k; j = 0, 1, \cdots, l$. We find the probability $P(A)$ of the event

$$
A = \{x_i - \tfrac{1}{2}\Delta x_i \leqq Z_{r+i}^{(n)} \leqq x_i + \tfrac{1}{2}\Delta x_i ; y_j - \tfrac{1}{2}\Delta y_j \leqq W_{s+j}^{(n)} \leqq y_j + \tfrac{1}{2}\Delta y_j, \quad i = 0, 1, \cdots, k; j = 0, 1, \cdots, l\}.
$$

Divide the whole plane into mutually disjoint rectangles $R_\mu(\mu = 1, 2, 3, 4)$, $R_i^{(1)}(i = 0, 1, \cdots, k)$, $S_j^{(1)}(j = 0, 1, \cdots, l)$, etc., as shown in Figure 1.

We consider the disjoint events A_1, A_2 where

$$A_1 = \{\text{at least one of } (Z_{r+i}^{(n)}, W_{s+j}^{(n)}) = (X_\alpha, Y_\alpha) \text{ for } \alpha = 1, 2, \cdots, n\}$$

$$A_2 = \{\text{no } Z_{r+i}^{(n)} \text{ and no } W_{s+j}^{(n)} \text{ are components of the same random vector}\}.$$

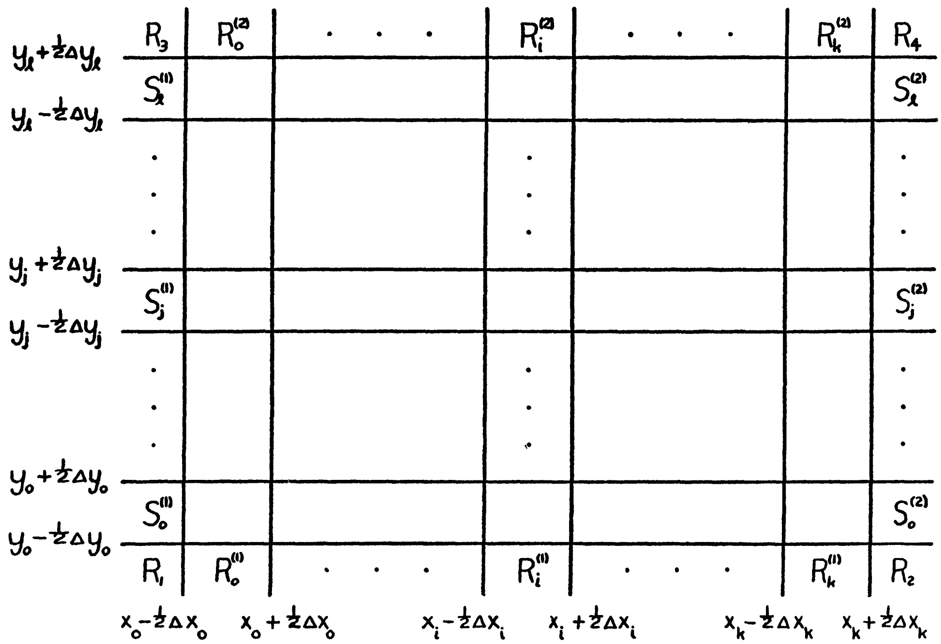

FIG. 1

Clearly, $P(A) = P(A \cap A_1) + P(A \cap A_2)$. It can be easily shown that $P(A \cap A_1) = P(A \cap A_2)k_n(x, y)/n$, where $k_n(x, y)$ tends to a finite limit as n tends to infinity. As such

$$(4) \qquad P(A) = P(A \cap A_2)[1 + O(n^{-1})].$$

In order that the event $A \cap A_2$ be satisfied, an examination of the situation reveals that points $V_h = (X_h, Y_h)$ can only fall in $R_\mu(\mu = 1, 2, 3, 4)$, along with one point in each of $R_i^{(\nu)}$, $S_j^{(\nu)}$; $\nu = 1$ or 2, $i = 0, 1, \cdots, k$ and $j = 0, 1, \cdots, l$. The distribution of V_h in the plane will be as follows:

Let n_μ be the number of V_h's in $R_\mu(\mu = 1, 2, 3, 4)$; then $\sum_{\mu=1}^4 n_\mu = n - (k + l + 2)$. The remaining $(k + l + 2)V_h$'s fall in $R_i^{(\nu)}$ and $S_j^{(\nu)}$ in all possible 2^{k+l+2} different ways arising from $i = 0, 1, \cdots, k$; $j = 0, 1, \cdots, l$ and $\nu = 1$ or 2. Thus

$$(5) \qquad P(A \cap A_2) = \sum_{\nu_i,\nu_j}' \sum_{n_1=0}^{r-\delta} \left(\prod_{\mu=1}^{4} \frac{n!}{n_\mu!} P[R_\mu^{(n_\mu)}]\right) \prod_{j=0}^{l} \prod_{i=0}^{k} P[R_i^{(\nu_i)}]P[S_j^{(\nu_j)}]$$

where

$$(6) \qquad \begin{gathered} \sum_{\mu=1}^{4} n_\mu = n - (k + l + 2), \\ n_1 + n_2 = s - 1, s - 2, \cdots, \text{or } s - k - 1, \\ n_1 + n_3 = r - 1, r - 2, \cdots, \text{or } r - l - 1, \qquad \delta = 1, 2, \cdots, l + 1 \end{gathered}$$

and the prime on summation means that the summation is to be carried out through all 2^{k+l+2} different combinations of $\nu_i = 1, 2$ and $\nu_j = 1, 2$.

Henceforth, whenever the expression (5) appears, it will be subject to the restrictions given in (6). Thus from (4) and (5), we get

$$(7) \qquad P(A) = [1 + O(n^{-1})] \sum_{\nu_i,\nu_j}' \sum_{n_1=0}^{r-\delta} \left(\prod_{\mu=1}^{4} \frac{n!}{n_\mu!} P[R_\mu^{(n_\mu)}]\right) \prod_{j=0}^{l} \prod_{i=0}^{k} P[R_i^{(\nu_i)}]P[S_j^{(\nu_j)}].$$

Dividing by $\prod_{i=0}^{k} \prod_{j=0}^{l} \Delta x_i \Delta y_j$, and taking the limit as $\Delta x_i \to 0$, $\Delta y_j \to 0$, we obtain the joint probability density function $g^{(n)}(\mathbf{x}, \mathbf{y})$ of $Z_{r+i}^{(n)}$, $W_{s+j}^{(n)}$. Now for $\mu = 1, 2, 3, 4$; $i = 0, 1, \cdots, k$; $j = 0, 1, \cdots, l$; and $\nu = 1, 2$, $\lim P[R_\mu] = \rho_\mu$, $\lim P[R_i^{(\nu)}] = \rho_i^{(\nu)}$, and $\lim P[S_j^{(\nu)}] = \sigma_j^{(\nu)}$ where all the limits are taken as $\Delta x_i \to 0$ and $\Delta y_j \to 0$. Hence from (7), we get

$$(8) \qquad g^{(n)}(\mathbf{x}, \mathbf{y}) = \sum_{\nu_i,\nu_j}' \sum_{n_1=0}^{r-\delta} \left(\prod_{\mu=1}^{4} \frac{n!}{n_\mu!} \rho_\mu^{n_\mu}\right) \prod_{j=0}^{l} \prod_{i=0}^{k} \rho_i^{(\nu_i)} \sigma_j^{(\nu_j)} [1 + O(n^{-1})],$$

$-\infty < x_0 < \infty$, $x_i < x_{i+1} < \infty$, $i = 0, 1, \cdots, k - 1$, $-\infty < y_0 < \infty$, $y_j < y_{j+1} < \infty$, $j = 0, 1, \cdots, l - 1$ where $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, \cdots, y_l)$. Let

$$(9) \qquad \begin{aligned} c_n = \rho_1 + \rho_2 + \rho_3 + \rho_4 = 1 &- [F_1(x_k) - F_1(x_0)] - [F_2(y_l) - F_2(y_0)] \\ &+ [F(x_k, y_l) - F(x_k, y_0) - F(x_0, y_l) + F(x_0, y_0)] \end{aligned}$$

and $\theta_\mu = \rho_\mu / c_n$, so that $\sum_{\mu=1}^4 \theta_\mu = 1$.

We now consider the normalized r.v.'s $U_i = nd_{r+i,1}^{(n)}$, $V_j = \text{n}d_{s+j,1}'^{(n)}$, $(i = 1, \cdots, k; j = 1, \cdots, l)$, $T_1 = n^{\frac{1}{2}}(Z_r^{(n)} - \zeta)$, and $T_2 = n^{\frac{1}{2}}(W_s^{(n)} - \eta)$. On applying the transformation $u_i = n(x_i - x_{i-1})$, $v_j = n(y_j - y_{j-1})$, $t_1 = n^{\frac{1}{2}}(x_0 - \zeta)$, and $t_2 = n^{\frac{1}{2}}(y_0 - \eta)$ to (10) [with Jacobian $n^{-(k+l+1)}$], we find that the asymptotic joint density function of $U_1, \cdots, U_k, V_1, \cdots, V_l, T_1, T_2$ is given by

$$\overline{(10)} \qquad h^{(n)}(\mathbf{u}, \mathbf{v}, t_1, t_2) = A_1^{(n)}A_2^{(n)}A_3^{(n)}[1 + O(n^{-1})]$$

where

$$(11) \qquad A_1^{(n)} = \prod_{\mu=1}^{4} C_n^{n_\mu} = C_n^{n-(k+l+2)}$$

$$(12) \qquad A_2^{(n)} = \sum_{\nu_i, \nu_j'}' \prod_{j=0}^{l} \prod_{i=0}^{k} \rho_i^{(\nu_i)} \sigma_j^{(\nu_j)}$$

$$(13) \qquad A_3^{(n)} = n \sum_{n_1=0}^{r-\delta} \prod_{\mu=1}^{4} \{[n - (k + l + 2)]!/n_\mu!\}\theta_\mu^{n_\mu},$$

with $0 \leqq u_i < \infty (i = 1, \cdots, k)$, $0 \leqq v_j < \infty (j = 1, \cdots, l)$, $-\infty < t_1, t_2 < \infty$, $\mathbf{u} = (u_1, \cdots, u_{k-1})$, $\mathbf{v} = (v_1, \cdots, v_{l-1})$; and $x_0 = \zeta + n^{-\frac{1}{2}}t_1$, $y_0 = \eta + n^{-\frac{1}{2}}t_2$. (Note that $n^{-(k+l+2)}n! \sim [n - (k + l + 2)]!$, for fixed integers k and l, as $n \to \infty$).

To find the joint asymptotic density of $Z_r^{(n)}$, $W_s^{(n)}$, $d_{r+i,1}^{(n)}$ and $d_{s+j,1}'^{(n)}$, we consider

$$C_n = \Bigg\{1 - \left[F_1\left(x_0 + \sum_{i=1}^{k} u_i \Big/ n\right) - F_1(x_0)\right] - \left[F_2\left(y_0 + \sum_{j=1}^{l} v_j \Big/ n\right) - F_2(y_0)\right] + \left[F\left(x_0 + \sum_{i=1}^{k} u_i \Big/ n, y_0 + \sum_{j=1}^{l} v_j \Big/ n\right) - F\left(x_0 + \sum_{i=1}^{k} u_i \Big/ n, y_0\right)\right] - \left[F\left(x_0, y_0 + \sum_{j=1}^{l} v_j \Big/ n\right) - F(x_0, y_0)\right]\Bigg\}_{x_0=\zeta+n^{-\frac{1}{2}}t_1, y_0=\eta+n^{-\frac{1}{2}}t_2}$$

Since $F(x, y)$, $F_1(x)$ and $F_2(y)$ have continuous derivatives with respect to x and y, we apply the law of mean, and obtain after some simplification,

$$(14) \quad c_n = \left[1 - \left(\sum_{i=1}^{k} u_i \Big/ n\right) f_1\left(\zeta + \frac{t_1}{n^{\frac{1}{2}}}\right) - \left(\sum_{j=1}^{l} v_j \Big/ n\right) f_2\left(\eta + \frac{t_2}{n^{\frac{1}{2}}}\right)\right] + o(n^{-1})$$

so that

$$(15) \qquad A_1^{(n)} \to \exp -\left[f_1(\zeta) \sum_{i=1}^{k} u_i + f_2(\eta) \sum_{j=1}^{l} v_j\right].$$

Also, it is easily seen that

$$A_2^{(n)} = \prod_{j=0}^{l} (\sigma_j^{(1)} + \sigma_j^{(2)}) \prod_{i=0}^{k} (\rho_i^{(1)} + \rho_i^{(2)}).$$

The limiting value of $A_2^{(n)}$ (evaluated at $x_0 = \zeta + n^{-\frac{1}{2}}t_1$, $y_0 = \eta + n^{-\frac{1}{2}}t_2$) is obtained as follows. Referring to (3), we see that

$$\begin{aligned}
\rho_i^{(1)} + \rho_i^{(2)} &= \left(\int_{-\infty}^{y_0} f(x_i, v)\, dv + \int_{y_l}^{\infty} f(x_i, v)\, dv\right)_0 \\
&= \left(f_1(x_i) - \int_{y_0}^{y_l} f(x_i, v)\, dv\right)_0 \\
&= \left(f_1(x_i) - n^{-1} \sum_{j=1}^{l} v_j f(x_i, \xi)\right)_0, \qquad (y_0 < \xi < y_l) \\
&= f_1(\zeta) + \frac{x_i - \zeta}{n^{\frac{1}{2}}} \left(\frac{\partial f_1(x)}{\partial x}\right)_{x=\zeta} + O(n^{-1}) \to f_1(\zeta),
\end{aligned} \tag{16}$$

where $(\ )_0$ means that $x_0 = \zeta + n^{-\frac{1}{2}}t_1$, $y_0 = \eta + n^{-\frac{1}{2}}t_2$ are to be substituted in the expression within the brackets. In these calculations we used the fact that

$$x_i = \sum_{\alpha=1}^{i} u_i/n + t_1/n^{\frac{1}{2}} + \zeta, \qquad y_l = \sum_{\beta=1}^{l} v_\beta/n + t_2/n^{\frac{1}{2}} + \eta.$$

Similarly,

$$\sigma_j^{(1)} + \sigma_j^{(2)} \to f_2(\eta). \tag{17}$$

It follows now from (16) and (17) that

$$A_2^{(n)} \to [f_1(\zeta)]^{k+1}[f_2(\eta)]^{l+1}. \tag{18}$$

Finally, the limiting expression for $h^{(n)}(\mathbf{u}, \mathbf{v}, t_1, t_2)$ will follow on evaluating $\lim A_3^{(n)}$. To do this, we proceed as follows. From (9) and (14) (and noting that $\theta_\mu = C_n^{-1}\rho_\mu$), it follows that $c_n = 1 + O(1/n)$, $\theta_\mu = \rho_\mu[1 + O(1/n)]$. Using the law of the mean and recalling the definition of q_μ, $\mu = 1, 2, 3, 4$, it is easily shown that $\rho_\mu = q_\mu[1 + O(n^{-\frac{1}{2}})]$, and hence that $\theta_\mu = q_\mu[1 + O(n^{-\frac{1}{2}})]$, $\mu = 1, 2, 3, 4$. Using the theorem in the Appendix, it readily follows that

$$\begin{aligned}
n \sum_{n_1=0}^{n} \prod_{\mu=1}^{4} [(n-k-l-2)!/n_\mu!]\, \theta_\mu^{n}\, \mu \\
= \left[k \exp\left\{-\frac{1}{2} D\Sigma^{-1}D'\right\}\right][1 + O(n^{-\frac{1}{2}})]
\end{aligned} \tag{19}$$

where $k = (2\pi\,|\Sigma|)^{-\frac{1}{2}}$, $D = [t_1 f_1(\zeta)\ t_2 f_2(\eta)]$, and

$$\Sigma = \begin{pmatrix} \alpha(1-\alpha) & \alpha\beta - q_1 \\ \alpha\beta - q_1 & \beta(1-\beta) \end{pmatrix}.$$

Thus from (10), (14), (18) and (19), we finally obtain that the limiting joint density function of $U_1, \cdots, U_k, V_1, \cdots, V_l, T_1, T_2$ is given by

$$h(\mathbf{u}, \mathbf{v}, t_1, t_2) = g(t_1, t_2)\left[\prod_{i=1}^{k} h_1(u_i)\right]\left[\prod_{j=1}^{l} h_2(v_j)\right],$$

where $g(t_1, t_2) = (2\pi|\Sigma|^{\frac{1}{2}})^{-1}[\exp\{-\frac{1}{2}D\Sigma^{-1}D'\}]f_1(\zeta)f_2(\eta)$, (agreeing with a result obtained by Siddiqui [4], p. 148) and

$$h_1(u_i) = f_1(\zeta)e^{-u_i f_1(\zeta)}, \qquad h_2(v_j) = f_2(\eta)e^{-v_j f_2(\eta)}.$$

Hence, the distances $\{d_{r+i,1}^{(n)}\}_{i=0}^{k-1}$, $\{d_{s+j,1}'^{(n)}\}_{j=0}^{l-1}$ are mutually asymptotically independently distributed, independently of $Z_r^{(n)}$ and $W_s^{(n)}$. Finally we infer the following

THEOREM. *$d_{r,k}^{(n)}$, $d_{s,l}'^{(n)}$ and $d^{(n)} = Z_r^{(n)} - W_s^{(n)}$ are asymptotically independent.*

PROOF. Since $d_{r,j}^{(n)} = \sum_{i=0}^{k-1} d_{r+i,1}^{(n)}$ and $d_{s,l}'^{(n)} = \sum_{j=0}^{l-1} d_{s+j,1}'^{(n)}$, the asserted asymptotic independence follows from the asymptotic mutual independence of the r.v.'s $d_{r+i,1}^{(n)}$ and $d_{s+j,1}'^{(n)}$.

APPENDIX

Here we obtain a theorem on which the derivation of (19) is based. We use the following trivariate normal approximation to the multinomial probability law, as given by Gnedenko [2], p. 85.

LEMMA. *Let $f_n(n_1, n_2, n_3, n_4) = n!\prod_{i=1}^{4} \lambda_i^{n_i}/n_i!$, with $0 < \lambda_i < 1$, $0 \leqq n_i \leqq n$, $i = 1, 2, 3, 4$, and $n^{-1}\sum_{i=1}^{4} n_i = \sum_{i=1}^{4} \lambda_i = 1$, and set $v_i = n^{-\frac{1}{2}}(n_i - n\lambda_i)$, so that $v_4 = -(v_1 + v_2 + v_3)$. Then uniformly in all the n_i for which the corresponding v_i lie in the arbitrary finite intervals $c_i \leqq v_i \leqq d_i$,*

$$f_n(n_1, n_2, n_3, n_4) = [e^{-Q/2}/(2n\pi)^{\frac{3}{2}}(\lambda_1\lambda_2\lambda_3\lambda_4)^{\frac{1}{2}}][1 + O(n^{-1})],$$

where $a_{ij} = \lambda_4^{-1}$ for $i \neq j$, $a_{ii} = \lambda_4^{-1} + \lambda_i^{-1}$ and $Q = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}v_iv_j$.

For $i = 1, 2, 3, 4$, we define the quantiles $p_i^{(n)}$ by $p_1^{(n)} = F(x, y)$ $p_2^{(n)} = F_2(y) - F(x, y)$, $p_3^{(n)} = F_1(x) - F(x, y)$ and $p_4^{(n)} = 1 - F_1(x) - F_2(y) + F(x, y)$, with $x = \zeta + n^{-\frac{1}{2}}t_1$, $y = \eta + n^{-\frac{1}{2}}t_2$. Clearly, $p_i^{(n)} \to q_i$ so that $p_1^{(n)} + p_3^{(n)} \to \alpha$, $p_2^{(n)} + p_4^{(n)} \to 1 - \alpha$, $p_1^{(n)} + p_2^{(n)} \to \beta$, $p_3^{(n)} + p_4^{(n)} \to 1 - \beta$, and $p_2^{(n)}p_3^{(n)} - p_1^{(n)}p_4^{(n)} \to \alpha\beta - q_1$. (Note the similarity between the $p_i^{(n)}$ here and the $\rho_i^{(n)}$ defined in Section 2). We now prove the following theorem.

THEOREM. *For $i = 1, 2, 3, 4$, let $v_i = n^{-\frac{1}{2}}(n_i - np_i^{(n)})$ with $p_i^{(n)}$ as defined above. Then*

$$n\sum_{n_1=0}^{n} f_n(n_1, n_2, n_3, n_4) \to (2\pi\,|\,\Sigma\,|^{\frac{1}{2}})^{-1}\exp\{-\tfrac{1}{2}D\Sigma^{-1}D'\},$$

where $D = [t_1f_1(\zeta)\ t_2f_2(\eta)]$, and

$$\Sigma = \begin{pmatrix} \alpha(1-\alpha) & \alpha\beta - q_1 \\ \alpha\beta - q_1 & \beta(1-\beta) \end{pmatrix}.$$

PROOF. By the lemma,

$$(20) \qquad n\sum_{n_1=0}^{n} f_n(n_1, n_2, n_3, n_4) = \left[H_n\sum_{n_1=0}^{n} n^{-\frac{1}{2}}e^{-Q_n}\right][1 + O(n^{-1})]$$

where $H_n^{-1} = (2\pi)^{+\frac{3}{2}}(p_1^{(n)}p_2^{(n)}p_3^{(n)}p_4^{(n)})^{\frac{1}{2}}$, and Q_n is Q with λ_i replaced by $p_i^{(n)}$. For v_i in the arbitrary finite intervals $c_i \leqq v_i \leqq d_i$, we find that

$$v_1 + v_2 = n^{\frac{1}{2}}[\beta - (p_1^{(n)} + p_2^{(n)}] + \epsilon_2/n^{\frac{1}{2}}$$
$$v_1 + v_3 = n^{\frac{1}{2}}[\alpha - (p_1^{(n)} + p_3^{(n)}] + \epsilon_1/n^{\frac{1}{2}}$$

where ϵ_1 and ϵ_2 depend on n in such a way that both tend to zero with increasing n. Further, let $u_1 = n^{\frac{1}{2}}[\beta - F_2(y)] + \epsilon_2/n^{\frac{1}{2}}$, $u_2 = n^{\frac{1}{2}}[\alpha - F_1(x)] + \epsilon_1/n^{\frac{1}{2}}$, so that $v_2 = u_1 - v_1$, $v_3 = u_2 - v_1$. Putting $\pi_1^{(n)} = \sum_{i=1}^{4}[p_i^{(n)}]^{-1}$, $\pi_j^{(n)} = [p_j^{(n)}]^{-1} + [p_4^{(n)}]^{-1}$, $j = 1, 2$, after algebraic simplifications we see that $Q_n = Q_n^{(1)} + Q_n^{(2)}$, where

$$Q_n^{(1)} = \pi_1^{(n)}[v_1 - (u_1\pi_2^{(n)} + u_2\pi_3^{(n)})/\pi_1^{(n)}]^2$$

and

$$Q_n^{(2)} = (\pi_1^{(n)})^{-1}\{\pi_2^{(n)}[\pi_1^{(n)} - \pi_2^{(n)}]u_1^2 + 2[\pi_1^{(n)}/p_4^{(n)} - \pi_2^{(n)}\pi_3^{(n)}]u_1 u_2 + \pi_3^{(n)}[\pi_1^{(n)} - \pi_3^{(n)}]u_2^2\}.$$

Substituting in (20), we obtain

$$n\sum_{n_1=0}^{n} f_n(n_1, n_2, n_3, n_4) = H_n L_n M_n[1 + O(n^{-1})],$$

where $L_n = \exp(-Q_n^{(2)}/2)$ and $M_n = \sum_{n_1=0}^{n} n^{-\frac{1}{2}}\exp(-Q_n^{(1)}/2)$. From the expansions $F_1(\zeta + n^{-\frac{1}{2}}t_1) = F_1(\zeta) + n^{-\frac{1}{2}}t_1f_1(\zeta) + (2n)^{-1}t_1^2f_1'(\theta_1)$, $\zeta < \theta_1 < \zeta + n^{-\frac{1}{2}}t_1$, and $F_2(\eta + n^{-\frac{1}{2}}t_2) = F_2(\eta) + n^{-\frac{1}{2}}t_2f_2(\eta) + (2n)^{-1}t_2^2f_2'(\theta_2)$, $\eta < \theta_2 < \eta + n^{-\frac{1}{2}}t_2$, we obtain easily that

$$u_1 = -t_2f_2(\eta) + O(n^{-\frac{1}{2}}) \to -t_2f_2(\eta)$$
$$u_2 = -t_1f_1(\zeta) + O(n^{-\frac{1}{2}}) \to -t_1f_1(\zeta).$$

Simple calculations now show that $L_n \to \exp(-\frac{1}{2}D\Sigma^{-1}D')$ and $H_n^{-1} \to (2\pi)^{\frac{3}{2}}(q_1q_2q_3q_4)^{\frac{1}{2}}$. Noting that M_n is a Riemann sum, and that c_i, d_i are arbitrary, it can be shown that $M_n \to (2\pi\psi_1^{-1})^{\frac{1}{2}}$, where $\psi_1 = \lim \pi_1^{(n)} = \sum_{i=1}^{4} q_i^{-1}$. Since $\psi_1\prod_{i=1}^{4} q_i = \alpha\beta(1-\alpha)(1-\beta) - (\alpha\beta - q_1)^2 = |\Sigma|$, the theorem follows.

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