AN APPLICATION OF A GENERALIZED GAMMA DISTRIBUTION

By Gerald S. Rogers

University of Arizona

1. Introduction and summary. Let $\{x_{i1}, \cdots, x_{in_i}\}$, $i = 1, 2, \cdots, k$, $k \geqq 2$, $n_i \geqq 2$, be random samples from stochastically independent Gaussian populations with unknown means μ_i and unknown variances σ_i^2. To test the hypothesis $H_0: \sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$, unknown, one may use a likelihood ratio test wherein H_0 is rejected if $\lambda < \lambda_0$,

$$\lambda = \prod_{i=1}^{k} \left[N \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 / n_i \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right]^{\frac{1}{2}n_i},$$

$N = \sum_{i=1}^{k} n_i$, $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and λ_0 is determined by the significance level. Under H_0, $Y_i = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/\sigma^2$ are stochastically independent chi-square variables with $n_i - 1$ degrees of freedom (throughout capital letters denote random variables and the corresponding lower case letters values in their ranges). A general discussion of such derivations can be found in [2], pp. 183–195. The distribution of Λ under H_0 is then obtainable from the extended form of the corollary below with $A_i = N/n_i$, $\beta_i = n_i/2$, $d_i = (n_i - 1)/2$.

The corollary is a particularization of results obtained from a generalized gamma distribution, introduced by Stacy [3], with density

$$(1) \qquad f(y_i ; a_i, d_i, p_i) = p_i y_i^{d_i - 1} \exp[-(y_i/a_i)^{p_i}]/a_i^{d_i} \Gamma(d_i/p_i)$$

for $y_i \geqq 0$, a_i, d_i, p_i all positive. (Throughout, only the nonzero portions of densities will be indicated.) Inconveniently, formulas (4) and (6) below must be evaluated by numerical methods. Since the case $k > 3$ follows by induction from that for $k = 3$, only the cases $k = 2, 3$ will be given here.

2. The case $k = 3$. $\prod$ and $\sum$ will always have indices $i = 1, 2, 3$.

Lemma 1. *Let Y_1, Y_2, Y_3 be stochastically independent with densities* (1) *and p_i all equal to p. Let $Z = Y_1 + Y_2 + Y_3$, $W_i = Y_i/Z$. Then the joint density of W_1, W_2 is $g(w_1, w_2)$ =*

$$(2) \qquad p^2 \Gamma\left(\sum d_i/p\right) \left[\prod w_i^{d_i - 1}/a_i^{d_i} \Gamma(d_i/p)\right] / \left[\sum (w_i/a_i)^p\right]^{\sum d_i/p}$$

where $w_1 + w_2 + w_3 = 1$, $0 \leqq w_i \leqq 1$.

Proof. For any $p_i > 0$, the density of W_1, W_2, Z is

$$z^{(\sum d_i - 1)} \prod \{[p_i w_i^{d_i - 1}/a_i^{d_i} \Gamma(d_i/p_i)] \exp[-(w_i z/a_i)^{p_i}]\}.$$

Since $0 \leqq z < \infty$, when the p_i are equal, a simple integration yields the form (2).

Lemma 2. *Let Y_i, W_i be as in Lemma 1 with $p = 1$, $a_1 = a_2 = a_3$. Then the Mellin transform of $W = W_1^{\beta_1} W_2^{\beta_2} W_3^{\beta_3} = W_1^{\beta_1} W_2^{\beta_2} (1 - W_1 - W_2)^{\beta_3}$ with β_1, β_2,*

Received 12 August 1963; revised 25 February 1964.

$\beta_3 > 0$ *is*

$$(3) \qquad \Gamma(\textstyle\sum d_i)\{\prod [\Gamma(\beta_i(s-1)+d_i)/\Gamma(d_i)]\}/\Gamma(\sum [\beta_i(s-1)+d_i]).$$

PROOF. $E[W^{s-1}] = \int_0^1 \int_0^{1-w_2} w^{s-1} g(w_1, w_2)\, dw_1\, dw_2$ which is easily evaluated to obtain the result.

The simplicity of proof in the above lemmas is not obtained for less restricted p's and a's so that the procedures here do not extend to the consideration of the distribution of Λ under alternative hypotheses. The density (2) can be termed that of generalized dependent beta variables.

LEMMA 3. *Let V be the ordinary beta variable with parameters D_1, D_2. Let $W = V^a(1-V)^b$, $a > 0$, $b > 0$. Then the density function of W is $h(w; D_1, D_2, a, b) =$*

$$(4) \quad \{[r^{D_1-a}(1-r)^{D_2-b}]/[a - ar - br] - [t^{D_1-1}(1-t)^{D_2-b}]/[a - at - bt]\}/B(D_1, D_2)$$

for

$$0 < w < [a/(a+b)]^a[b/(a+b)]^b,$$

$$w = r^a(1-r)^b = t^a(1-t)^b,\ r \leqq a/(a+b) \leqq t.$$

PROOF. $P(W \leqq w) = P(V \leqq r) + P(V \geqq t)$. Differentiation with respect to w yields the result.

The Mellin transform of W in Lemma 3 is

$$(5) \quad M(s; D_1, D_2, a, b) = B(a(s-1) + D_1, b(s-1) + D_2)/B(D_1, D_2).$$

This particular pair of inverse Mellin transforms, (4) and (5), does not seem to be in the literature. Other pairs of inverse transforms can be derived from these. For example, substitute in (4) using the relations

$$P(V \leqq r) = B_r(D_1, D_2)/B(D_1, D_2) = r^{D_1}F(D_1, 1 - D_2; D_1 + 1; r)/D_1 B(D_1, D_2)$$

and

$$(d/dr)F(D_1, 1 - D_2; D_1 + 1; r) = D_1(1 - D_2)F(D_1 + 1, 2 - D_2; D_1 + 2; r)/(D_1 + 1),$$

where B, B_r, F are the beta, incomplete beta and hypergeometric functions, respectively. (See [1], p. 58 and 87). Or, letting $w = \exp f$ and $s = c + it$, one finds $(2\pi)^{\frac{1}{2}}h(\exp f; D_1, D_2, a, b) \exp cf$ and $M(c + it; D_1, D_2, a, b)$ are inverse Fourier transforms (see [4], p. 60). (The author wishes to thank Professor A. Erdélyi for his comments on Lemma 3, communicated privately.)

THEOREM. *Under the conditions of Lemma 2, the density of W is $m(w) =$*

$$(6) \quad \int_{w/K}^{T} h(w/y; d_1 + d_2, d_3, \beta_1 + \beta_2, \beta_3) h(y; d_1, d_2, \beta_1, \beta_2)\, dy/y,$$

$$K = (\beta_1 + \beta_2)^{\beta_1+\beta_2}\beta_3^{\beta_3}/(\textstyle\sum \beta_i)^{\Sigma \beta_i}, \qquad T = \beta_1^{\beta_1}\beta_2^{\beta_2}/(\beta_1 + \beta_2)^{\beta_1+\beta_2},$$

$h(\cdot; D_1, D_2, a, b)$ given by (4), $0 < w < KT$.

PROOF. From (3), the Mellin transform of W is

$$B([\beta_1 + \beta_2](s - 1) + d_1 + d_2, \beta_3(s - 1) + d_3)B(\beta_1(s - 1) + d_1, \beta_2(s - 1) + d_2) \div B(d_1 + d_2, d_3)B(d_1, d_2) = M(s; d_1 + d_2, d_3, \beta_1 + \beta_2, \beta_3)M(s; d_1, d_2, \beta_1, \beta_2).$$

Hence the conclusion follows by the well-known convolution theorem for Mellin transforms. (See [4], p. 8.)

Application of a simple transformation yields the following

COROLLARY. *Under the conditions of Lemma* 2, *the density of* $\Lambda = \prod (A_i w_i)^{\beta_i} = AW$ *with* $A_i > 0$, $\prod (A_i^{\beta_i} = A$, *is* $m(\lambda/A)/A$, $0 < \lambda < KAT$.

3. The case $k = 2$. When analogous Lemmas 1 and 2 are worked out, it is seen that W_1 is an ordinary beta variable so that Lemma 3 applies directly and the density of $W = W_1^{\beta_1}(1 - W_1)^{\beta_2}$ is $h(w; d_1, d_2, \beta_1, \beta_2)$.

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