A COMBINATORIAL METHOD FOR PRODUCTS OF TWO POLYKAYS WITH SOME GENERAL FORMULAE

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1. Introduction and summary. Wishart has demonstrated and Kendall has justified the application of a combinatorial method to products of k-statistics. Wishart's combinatorial method is, essentially, that introduced by Fisher though modifications were necessary since Fisher's method is applied directly to the writing of cumulant results while Wishart's method applies to products of k-statistics. It is the purpose of this paper to show that the combinatorial method may be further modified and extended so as to produce products of polykays and to present general formulae resulting from the new combinatorial method.

2. Notation and background material. The notation, based on MacMahon [9], is similar to that of [4] except that P represents any partition of order π and not just one with no unit parts. Similarly Q is a partition of q having order χ. We use $[P]$ as the augmented monomial symmetric function [3], [8] and m_P' as the average $[P]$. It is known that $Em_P' = \mu_P'$ where $\mu_P' = \mu_{p_1}' \cdots \mu_{p_\pi}'$. We let $C(P)$ denote the combinatorial coefficient which is the number of ways the partition can be formed from p distinct units. In this notation the general formula for k_p, see [2], becomes

$$(2.1)\qquad \begin{aligned} k_p &= \sum (-1)^{\pi-1}(\pi-1)!\,C(P)m_P' \\ &= \sum (-1)^{\pi-1}(\pi-1)!\,C(P)[P]/n^{(\pi)}. \end{aligned}$$

This notation can be modified and extended to polykays [11], [12] with the use of subscripts. Thus $p_1 \cdots p_\pi$ is a specified partition of order π with $p_1 \geqq p_2 \geqq \cdots \geqq p_\pi$. A partition of p_i, of order π_i, is indicated by P_i with combinatorial coefficient

$$(2.2)\qquad C(P_i) = p_i!/(p_{i1}!)^{\pi_{i1}} \cdots (p_{is}!)^{\pi_{is}}\pi_{i1}! \cdots \pi_{is}!\,.$$

Then a partition of the specified $P = p_1 \cdots p_\pi$ which is obtained by partitioning one or more of the p_i is indicated by P_I and the value of the generalized k-statistic [13] p. 2 is

$$(2.3)\qquad k_{p_1 \cdots p_\pi} = k_P = \sum (-1)^{\Sigma(\pi_i-1)} \prod (\pi_i - 1)! \prod C(P_i)[P_I]/n^{(\Sigma\pi_i)}$$

and for the product we have

Received 30 October 1962; revised 26 February 1964.

[1] Research supported by National Science Foundation Grant NSF-GP-12.

[2] Research supported in part by Office of Naval Research under contract Nonr 1224(41) at University of Michigan.

$$k_P k_Q = \sum \prod \{(-1)^{\pi_i - 1}(-1)^{\chi_j - 1}(\pi_i - 1)!\,(\chi_j - 1)!\,C(P_i)C(Q_j)\} \cdot [P_I Q_J]/n^{(\rho)} \tag{2.4}$$

where ρ is the order of $P_I Q_J$.

3. Modification of Wishart's method. The problem is to expand the right hand side of (2.4) in terms of polykays. This is more general than the problem of Wishart which was, for products of two factors, to expand

$$k_p k_q = \sum (-1)^{\pi - 1}(-1)^{\chi - 1}(\pi - 1)!\,(\chi - 1)!\,C(P)C(Q)[PQ]/n^{(\rho)} \tag{3.1}$$

in polykays. The general procedure, following Wishart [13] is to take expected values to obtain μ's, change these to parent cumulants, and then obtain the formulae as linear functions of the polykays by estimation. This process, as Kendall pointed out [7], is very similar to an application of the Irwin-Kendall principle [6] though, in obtaining these products, no finite population is involved.

The main task is the transformation from the μ's to the cumulants and modifications can be made in Wishart's method to do this for the more general problem here. The essential modification is based on the fact that only the bipartitions (arrays) which represent $P_I Q_J$, and not all those of PQ, need be used in the process. We define bipartitions of $P_I Q_J$ to be admissible bipartitions of PQ.

We illustrate with the example of Wishart [13], p. 4 giving the combinatorial development of k_2^2. There are 7 bipartitions listed by Wishart. In obtaining the bipartitions appropriate to $k_2 k_{11}$, the bipartitions 22 and $\begin{matrix}20\\02\end{matrix}$ are not admissible since 2 in the second column is not a partition of 11. The admissible bipartitions $P_I Q_J$ are then

$$\begin{matrix} (a) & (b) & (c) & (d) & (e) \\ 2\ 1 & 2\ 0 & 1\ 1 & 1\ 1 & 1\ 0 \\ 0\ 1 & 0\ 1 & 1\ 1 & 1\ 0 & 1\ 0 \\ & 0\ 1 & & 0\ 1 & 0\ 1 \\ & & & & 0\ 1 \end{matrix} \tag{3.2}$$

We form the combined bipartitions of a given bipartition by combining rows. Those which are admissible are called the c-bipartitions of the bipartition. Thus (a) and (c) of (3.2) have no c-bipartitions, but (b), (d), (e) do have c-bipartitions since rows can be combined as long as the resulting second element is not greater than 1.

In the Wishart technique the n-coefficient, the non-combinatorial factor of the coefficient associated with a bipartition, is obtained from (3.1). In our modification the corresponding n-coefficient, obtained from (2.4) is

$$n\text{-coefficient} = \prod \{(-1)^{\pi_i - 1}(-1)^{\chi_j - 1}(\pi_i - 1)!(\chi_j - 1)!\}n^{(\rho)}/n^{(\Sigma \pi_i)}n^{(\Sigma \chi_j)}. \tag{3.3}$$

The sum of the n-coefficients, for the bipartition and all its c-bipartitions, is called the bipartition function. Thus the bipartition function of (a) in (3.2) is $(1/n)(1/n^{(2)}) \cdot n^{(2)} = 1/n$ since there is no c-bipartition, and the bipartition function for (b) is $(1/n)(1/n^{(2)}) \cdot n^{(3)} + 2(1/n)(1/n^{(2)}) \cdot n^{(2)} = 1$. Similar treatment of (c), (d), (e) shows bipartition functions of $-1/n^{(2)}$, 0, 0 respectively. Since the combinatorial coefficient, the number of ways the bipartition can be formed in (a), (b), (c), is 2, 1, 2 respectively, and since the polykays have subscripts indicated by the sums of the elements in the rows, we have

$$k_2 k_{11} = 2k_{31}/n - 2k_{22}/n^{(2)} + k_{211} \, .$$

4. Rules for pattern functions. The application of (3.3) to bipartitions and c-bipartitions makes possible the determination of certain rules for general bipartition functions. We first define a pattern to be a generalized bipartition in which the positions of the specific partitions of the P_I and Q_J are fixed though rows or columns may be interchanged. Thus

$$\begin{matrix} p_{11} & q_1 \\ p_{12} & q_{21} \\ p_2 & q_{22} \end{matrix}$$

is a pattern and other equivalent patterns are obtained by interchanging rows or columns. Any pattern resulting from admissible combination of the rows of a pattern is a c-pattern of the pattern. Thus the pattern above has no c-pattern since only parts of the same partition of p_i or q_j can be combined. This concept of pattern is more general than that of Fisher and Wishart and becomes a pattern in the Fisher-Wishart sense only when the first column consists of a single partition of p (with perhaps 0 terms) and the second column consists of a single partition of q.

For the general bipartition called a pattern the results of the section above give, with pattern function replacing bipartition function:

Rule 1. *General rule.* The pattern function may be obtained by applying (3.3) to the pattern, to each of its c-patterns, and by adding the results.

For the Fisher-Wishart results, (3.3) becomes

$$(4.1) \quad n\text{-coefficient} = (-1)^{\pi-1}(-1)^{\chi-1}(\pi - 1)!\,(\chi - 1)!\,n^{(\rho)}/n^{(\pi)}n^{(\chi)}$$

and, since all rows may be added, the partition parts of p and q may be replaced by the unspecified x.

We see at once that the bipartition functions for all bipartitions having the same pattern are the same no matter what the values of p_i and q_j provided only that the partition parts exist. Thus

Rule 2. *Pattern rule.* The bipartition functions of all bipartitions having the same pattern are identical and equal to the pattern function.

In the Fisher analysis this becomes the Fisher rule of patterns.

We next have a very useful rule.

Rule 3. *First rule of* 0 *elements.* The pattern function is 0 for any pattern (bipartition) which has at least one row with a zero element, the other element not being either a p_i or a q_j. Thus the pattern function of

$$\begin{matrix} p_{11} & q_1 \\ p_{12} & 0 \end{matrix} \text{ is 0 but that of } \begin{matrix} p_1 & q_1 \\ p_2 & 0 \end{matrix}$$

is not. The proof consists in adding the n-coefficients for the pattern and all its c-patterns. Tukey's rule [12], p. 45 as applied to bipartitions is a special case of this rule since any unit subscript over and above the subscripts of the original set must result from associating a unit part of a partition with a 0 in the same row. But the first rule of 0 elements, as applied to patterns, is more general in that it eliminates patterns which Tukey's rule does not. Even for Wishart analyses, the rule appears to be more adequate than the Wishart recommendation [13], p. 4 which applies to unit parts.

We define an extended pattern to be one which consists of an initial pattern plus additional rows in which the elements are p_i, but not parts of p_i, matched with 0 or q_j matched with 0. Then we have

RULE 4. *Second rule of* 0 *elements.* The pattern function of the extended pattern equals that of the initial pattern.

This rule is very useful in appropriate cases in simplifying the calculation of the factor pattern since, in determining the coefficient, one may cross out any row in the pattern in which a p_i term or a q_j term is in a row with a 0. It should be applied after the first rule of 0 elements. Suppose there are R rows of the initial pattern with both p and q elements, S with q elements only, and T with p elements only. If the contribution of the signs and factorials is indicated by C, the value of the n-coefficient is $Cn^{(R+S+T)}/n^{(R+S)}n^{(R+T)}$. Consider an extended pattern which results from adding the row p_i 0. Then the contribution to the partition function from the c-patterns resulting from combining the new row with the initial rows in all possible ways, but not the initial rows with themselves, is

$$Cn^{(R+S+T+1)}/n^{(R+T+1)}n^{(R+S)} + CSn^{(R+S+T)}/n^{(R+T+1)}n^{(R+S)}$$

which reduces to the value of the n-coefficient above. Since this result holds for each c-pattern of the initial pattern, it follows that the two pattern functions are the same. A similar argument holds when 0 q_j is added.

It follows as a corollary of Rule 4 that any patterns composed only of rows with p_i or q_j paired with 0 have pattern functions unity. Thus all rows except one may be crossed out by Rule 4 and the coefficient is then $(1/n)(n) = 1$. In general the application of the two rules of 0 elements leads us to partitions with no 0 terms which we now consider.

We collect each row which can not be combined with any other row at the lower part of the pattern and indicate this collection by A, having a rows. The pattern above A is called the reduced pattern. Then the signed factorial contribution to every n-coefficient of each c-pattern, and hence to the pattern function, is the same. This is denoted by C_A and is the product of the signed

TABLE 1

Pattern functions for patterns with given specifications

Formula Number	Specification	Pattern Function
(4.1)	All rows are p_i 0 or 0 q_j	1
(4.2)	Pattern is all A	$C_A/n^{(\rho)}$
(4.3)	$r_1 = s_1 = 2$	$n/n^{(2)}$
(4.4)	$r_1 = s_1 = 3$	$n^2/n^{(3)}$
(4.5)	$r_1 = s_1 = 4$	$n(n + 1)/n^{(4)}$
(4.6)	$r_1 = s_1 = 5$	$n^3(n + 5)/n^{(5)}$
(4.7)	$r_1 = s_1 = \rho$	$\sum (u - 1)!\Delta^u(0^\rho)/n^{(u)}u$
(4.8)	$r_1 = 2, s_1 = 2, R_1 = 3, S_2 = 1$ $(r_1 = 2, s_1 = 2, R_2 = 1, S_1 = 3)$	$-n/n^{(3)}$
(4.9)	$r_1 = 2, s_1 = 2, R_2 = 1, S_2 = 1$	$(n - 1)/n^{(3)}$
(4.10)	$r_1 = 2, s_1 = 2$ with other $R_i = 1$, $S_j = 1$	$(n - a)/n^{(a+2)}$
(4.11)	$r_1 = 3, s_1 = 3, R_2 = 1, S_2 = 1$	$(n - 1)^2/n^{(4)}$
(4.12)	$r_1 = 3, s_1 = 3$ with other $R_i = 1$, $S_j = 1$	$(n - a)^2/n^{(a+3)}$
(4.13)	$r_1 = s_1 = 2, r_2 = s_2 = 2$	$(n^2 - 3n + 1)/n^{(4)}$
(4.14)	$r_1 = s_1 = 3, r_2 = s_2 = 2$	$n(n^2 - 5n + 5)/n^{(5)}$
(4.15)	$r_1 = s_1 = 4, r_2 = s_2 = 2$	$(n - 1)(n^3 - 5n^2 + n + 4)/n^{(6)}$
(4.16)	$r_1 = s_1 = 3, r_2 = s_2 = 3$	$(n^4 - 8n^3 + 26n^2 - 91n + 196)/n^{(6)}$
(4.17)	$r_1 = 4, s_1 = 2, s_2 = 2$ $(r_1 = 2, r_2 = 2, s_1 = 4)$	$-n^{(2)}/n^{(4)}$
(4.18)	$r_1 = 5, s_1 = 3, s_2 = 2$ $(r_1 = 3, r_2 = 2, s_1 = 5)$	$-n^2(n - 1)/n^{(5)}$
(4.19)	$r_1 = 6, s_1 = 3, s_2 = 3$ $(r_1 = 3, r_2 = 3, s_1 = 6)$	$-n^{(2)}(n^2 - n + 4)/n^{(5)}$
(4.20)	$r_1 = 6, s_1 = 4, s_2 = 2$ $(r_1 = 4, r_2 = 2, s_1 = 6)$	$-n^{(2)}(n - 1)(n + 4)/n^{(6)}$
(4.21)	$r_1 = 2, r_2 = 2, r_3 = 2$ $(s_1 = 2, s_2 = 2, s_3 = 2)$	$(n^3 - 9n^2 + 23n - 14)/n^{(6)}$
(4.22)	$r_1 = 6, s_1 = 2, s_2 = 2, s_3 = 2$ $(r_1 = 2, r_2 = 2, r_3 = 2, s_1 = 6)$	$2n^{(3)}/n^{(6)}$
(4.23)	$r_1 = 4, r_2 = 2, s_1 = 2, s_2 = 2, s_3 = 2$ $(r_1 = 2, r_2 = 2, r_3 = 2, s_1 = 4, s_2 = 2)$	$-(n - 2)(n^2 - 5n + 2)/n^{(6)}$

factorials in A excluding all parts of partitions which have parts in the rows above A.

In developing a notation for more complex situations we let r_i be the number of parts of a partition of p_i appearing in the reduced pattern and R_i the number of parts appearing in the pattern. Similarly s_j and S_j are the numbers of parts of q_j. Once the values of C_A and a are determined, we need pay no more attention to the elements of A in determining the pattern function. Specification of the r's and s's enables us to identify groups of reduced patterns which are covered by a general formula for the pattern function. Thus $r_1 = R_1 = \rho$, $s_1 = S_1 = \rho$ identify the ρ-row, two-column patterns of Fisher [5], pp. 223-226. Formulae for pattern functions are presented in Table 1 for patterns specified by values of r_i, s_j, R_i, and S_j. Unless otherwise specified it is understood that $R_i = r_i$ and $S_j = s_j$ for $i, j > 1$ and that all the unspecified r_i and s_j are 0. The formulae (4.3)—(4.7) featuring only r_1 and s_1 were given essentially by Fisher.

As an illustration, some results of this section are applied to the bipartitions (3.2). The bipartition functions of (d) and (e) are 0 by the first rule of 0 elements; that of (a) is $1/n$ since the second row may be crossed out by the second rule of 0 elements; that of (b) is 1 by the corollary of the second rule of 0 elements or by (4.1); and that of (c) is $-1/n^{(2)}$ by (4.2) with $C_A = -1$ and $\rho = 2$. The results of this section are applied to obtain general formulae in the next section.

5. Some general formulae. The section above gives the pattern function for many patterns. The subscripts of the associated polykays are obtained by adding the row entries in the bipartition. The numerical coefficient is obtained combinatorially. Thus the numerical coefficient in (3.2) is 2 for (a), 1 for (b), and 2 for (c).

The determination of the numerical coefficient for specific bipartitions can be a real task [8] [14]. It is commonly less difficult for general formulae since the results are not stated so specifically, though the specific results should be obtained readily. In this section combinatorial concepts are used in stating the results so that the determination of the combinatorial coefficient is simple. The combinatorial method is illustrated by application to $k_{p_1p_2}k_2$. Here the pattern $\begin{matrix} p_1 & 2 \\ p_2 & 0 \end{matrix}$ represents the bipartition $\begin{matrix} p_1 & 2 \\ p_2 & 0 \end{matrix}$ as well as the bipartition $\begin{matrix} p_1 & 0 \\ p_2 & 2 \end{matrix}$ and the pattern $\begin{matrix} p_1 & 1 \\ p_2 & 1 \end{matrix}$ represents 2 bipartitions, for distinct units, since the units can be associated with the p's in 2! ways.

Pattern	$\begin{matrix} p_1 & 0 \\ p_2 & 0 \\ 0 & 2 \end{matrix}$	$\begin{matrix} p_1 & 2 \\ p_2 & 0 \end{matrix}$	$\begin{matrix} p_1 & 1 \\ p_2 & 1 \end{matrix}$	$\begin{matrix} p_{11} & 1 \\ p_{12} & 1 \\ p_2 & 0 \end{matrix}$	$\begin{matrix} p_1 & 0 \\ p_{21} & 1 \\ p_{22} & 1 \end{matrix}$
Pattern function	1	$1/n$	$-1/n^2$	$-1/(n-1)$	$-1/(n-1)$
Combinatorial coefficient	1	1	1	$C(P_1 \mid 2)$	$C(P_2 \mid 2)$

The pattern function is determined from the results above. The combinatorial coefficient for the pattern is determined by application of (2.2) to all partitions of p_i and q_j. We denote 2-part partitions of p_i by $P_i \mid 2$ with $C(P_i \mid 2)$ the associated combinatorial coefficient. We indicate the collection of polykays having the same pattern function and combinatorial coefficient by a symbolic addition of the P and Q which is here represented by $\oplus$. Thus $k_{p\oplus 2}$, as applied to this problem, is $k_{p_1+2,p_2} + k_{p_1,p_2+2}$ and $k_{p\oplus 11} = 2k_{p_1+1,p_2+1}$. Furthermore we denote P with $p_{i1}p_{i2} \oplus 11$ replacing p_i by $P{:}P_i \mid 2 \oplus 11$. The patterns above then give the general formula

$$(5.1)\quad k_{p_1p_2}k_2 = k_Pk_2 = k_{P2} + k_{P\oplus 2}/n - k_{P\oplus 11}/n^{(2)} + \sum C(P_i \mid 2)k_{P:P_i\mid 2\oplus 11}/(n-1).$$

An illustration helps to clarify the symbolism and to show how easily the general formula is applied to specific cases. Thus with $p_1 = 4$ and $p_2 = 2$, we have

$$(5.2)\quad k_{42}k_2 = k_{422} + (k_{62} + k_{44})/n - 2k_{53}/n^{(2)} + 2(4k_{422} + 3k_{332})/(n-1) + 2k_{422}/(n-1)$$

as given in [10] p. 141.

The formulae of this section feature subscripts. To simplify the typing and typesetting we use $k_P = k(P)$. Then (5.1) can be generalized to give, for a specific $P = p_1 \cdots p_\pi$, with the logical extension of notation

$$(5.3)\quad k(P)k(2) = k(P2) + k(P \oplus 2)/n - k(P \oplus 11)/n^{(2)} + \sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 11)/(n-1).$$

A combinatorial treatment of k_Pk_{11} gives

$$(5.4)\quad k(P)k(11) = k(P11) + 2k(P \oplus 1, 1)/n + k(P \oplus 11)/n^{(2)} - \sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 11)/n^{(2)}.$$

As special cases of (5.3) and (5.4) when P is p we have, since 11 can not be "added" to p,

$$(5.5)\quad k(p)k(2) = k(p2) + k(p+2)/n + 2\sum C(P \mid 2)k(p_1 + 1, p_2 + 1)/(n-1)$$

$$(5.6)\quad k(p)k(11) = k(p11) + 2k(p+1, 1)/n - 2\sum C(P \mid 2)k(p_1 + 1, p_2 + 1)/n^{(2)}.$$

Formulae equivalent to (5.6) were given by Barton, David and Fix [1] for $p = r$ odd and $p = r$ even using binomial coefficients. The need for two formulae is avoided by using the combinatorial coefficient $C(P \mid 2)$ rather than the binomial coefficient.

Values of $k(P)k(Q)$ for Q of weight 2 are available in Table 2. The entry in the divisor column divides each entry to its left. Thus (5.3) and (5.4) immediately result.

Values of $k(P)k(Q)$ for Q of weight 3 are available in Table 3.

The notation of Table 3 is a logical extension of that above. Thus $P + 1{:}P_i \mid 2 \oplus 11$ indicates all the terms in which 1 is added to an element of the specified P and another element p_i is replaced by $P_i \mid 2 \oplus 11$. The divisor is used as in Table 2. Thus the first formula is

$$\begin{aligned} k(P)k(3) = {} & k(P3) + k(P \oplus 3)/n - 3k(P \oplus 21)/n^{(2)} \\ & + 2k(P \oplus 111)/n^{(3)} + 3n\sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 21)/n^{(2)} \\ & + n^2\sum C(P_i \mid 3)k(P{:}P_i \mid 3 \oplus 111)/n^{(3)} \\ & - 3n\sum C(P_i \mid 2)k(P \oplus 1{:}P_i \mid 2 \oplus 11)/n^{(3)}. \end{aligned} \tag{5.7}$$

TABLE 2

Values of $k(P)k(Q)$ when Q is of weight 2

Term	Product			
	$k(P)k(2)$	$k(P)k(11)$	Div.	$n^2k(P)k_1^2$
$k(P2)$	1		1	n
$k(P11)$		1	1	n^2
$k(P \oplus 2)$	1		n	1
$k(P \oplus 1, 1)$		2	n	$2n$
$k(P \oplus 11)$	−1	1	$n^{(2)}$	1
$\sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 11)$	n	−1	$n^{(2)}$	0
$n^2k(P)k_1^2$	n	n^2		Check

TABLE 3

Values of $k(P)k(Q)$ when Q is of weight 3

Term	Product				
	$k(P)k(3)$	$k(P)k(21)$	$k(P)k(111)$	Div.	$n^3k(P)k_1^3$
$k(P3)$	1			1	n
$k(P21)$		1		1	$3n^2$
$k(P111)$			1	1	n^3
$k(P \oplus 3)$	1			n	1
$k(P \oplus 2, 1)$		1		n	$3n$
$k(P \oplus 1, 2)$		1		n	$3n$
$k(P \oplus 1, 11)$			3	n	$3n^2$
$k(P \oplus 21)$	−3	1		$n^{(2)}$	3
$k(P \oplus 11, 1)$		−1	3	$n^{(2)}$	$3n$
$k(P \oplus 111)$	2	−1	1	$n^{(3)}$	1
$\sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 21)$	$3n$	−1		$n^{(2)}$	0
$\sum C(P_i \mid 2)k(P1{:}P_i \mid 2 \oplus 11)$		n	−3	$n^{(2)}$	0
$\sum C(P_i \mid 3)k(P{:}P_i \mid 3 \oplus 111)$	n^2	$-n$	2	$n^{(3)}$	0
$\sum C(P_i \mid 2)k(P \oplus 1{:}P_i \mid 2 \oplus 11)$	$-3n$	$n + 1$	−3	$n^{(3)}$	0
$n^3k(P)k_1^3$	n	$3n^2$	n^3		Check

TABLE 4

Values of $k(P)k(Q)$ where Q is of weight 4

Term	Product						
	$k(P)k(4)$	$k(P)k(31)$	$k(P)k(22)$	$k(P)k(211)$	$k(P)k(1^4)$	Div.	$n^4k(P)k_1^4$
$k(P4)$	1					1	n
$k(P31)$		1				1	$4n^2$
$k(P22)$			1			1	$3n^2$
$k(P211)$				1		1	$6n^3$
$k(P1111)$					1	1	n^4
$k(P \oplus 4)$	1					n	1
$k(P \oplus 3, 1)$		1				n	$4n$
$k(P \oplus 2, 2)$			2			n	$6n$
$k(P \oplus 1, 3)$		1				n	$4n$
$k(P \oplus 2, 11)$				1		n	$6n^2$
$k(P \oplus 1, 21)$				2		n	$12n^2$
$k(P \oplus 1, 111)$					4	n	$4n^3$
$k(P \oplus 31)$	−4	1				$n^{(2)}$	4
$k(P \oplus 22)$	−3		1			$n^{(2)}$	3
$k(P \oplus 21, 1)$		−3		2		$n^{(2)}$	$12n$
$k(P \oplus 11, 2)$			−2	1		$n^{(2)}$	$6n$
$k(P \oplus 11, 11)$				−1	6	$n^{(2)}$	$6n^2$
$k(P \oplus 211)$	12	−3	−2	1		$n^{(3)}$	6
$k(P \oplus 111, 1)$		2		−2	4	$n^{(3)}$	$4n$
$k(P \oplus 1111)$	−6	2	1	−1	1	$n^{(4)}$	1
$\sum C(P_i \mid 2)k(P:P_i \mid 2 \oplus 31)$	$4n$	−1				$n^{(2)}$	0
$\sum C(P_i \mid 2)k(P:P_i \mid 2 \oplus 22)$	$3n$		−1			$n^{(2)}$	0
$\sum C(P_i \mid 2)k(P1:P_i \mid 2 \oplus 21)$		$3n$		−2		$n^{(2)}$	0
$\sum C(P_i \mid 2)k(P2:P_i \mid 2 \oplus 11)$			$2n$	−1		$n^{(2)}$	0
$\sum C(P_i \mid 2)k(P11:P_i \mid 2 \oplus 11)$				n	−6	$n^{(2)}$	0

$\sum C(P_i \mid 3)k(P{:}P_i \mid 3 \oplus 211)$	$6n^2$	$-3n$	$-2n$	2		$n^{(3)}$	0
$\sum C(P_i \mid 2)k(P \oplus 1{:}P_i \mid 2 \oplus 21)$	$-12n$	$3n$	4	-2		$n^{(3)}$	0
$\sum C(P_i \mid 2)k(P \oplus 2{:}P_i \mid 2 \oplus 11)$	$-6n$	3	$2(n-1)$	-1		$n^{(3)}$	0
$\sum C(P_i \mid 3)k(P1{:}P_i \mid 3 \oplus 111)$		n^2		$-2n$	8	$n^{(3)}$	0
$\sum C(P_i \mid 2)k(P \oplus 1, 1{:}P_i \mid 2 \oplus 11)$		$-3n$		$2(n+1)$	-12	$n^{(3)}$	0
$\sum C(P_i \mid 4)k(P{:}P_i \mid 4 \oplus 1111)$	$n^2(n+1)$	$-n(n+1)$	$-n^{(2)}$	$2n$	-6	$n^{(4)}$	0
$\sum C(P_i \mid 3)k(P \oplus 1{:}P_i \mid 3 \oplus 111)$	$-4n(n+1)$	n^2+n+4	$4(n-1)$	$-2(n+1)$	8	$n^{(4)}$	0
$\sum C(P_i \mid 2)k(P \oplus 11{:}P_i \mid 2 \oplus 11)$	$12n$	$-3(n+1)$	$-2n$	$n+3$	-6	$n^{(4)}$	0
$\sum C(P_i \mid 2)C(P_j \mid 2)$ $k(P{:}P_i \mid 2 \oplus 11{:}P_j \mid 2 \oplus 11)$	$-n(n-1)$	$n-1$	$(n^2-3n+3)/3$	$-n/3$	1	$n^{(4)}$	0
$n^4k(P)k_1^4$	n	$4n^2$	$3n^2$	$6n^3$	n^4		Check

Values of $k(P)k(Q)$ for Q of weight 4 are given in Table 4.

The formula (5.3) and (5.7), and others of the tables, are very general and have special cases which are also quite general. For example with $p_i = r$ and $\pi = s$, (5.3) and (5.7) become

$$(5.8) \quad \begin{aligned} k(r^s)k(2) &= k(r^s2) + sk(r+2, s^{r-1})/n - s^{(2)}k(\{r+1\}^2r^{s-2})/n^{(2)} \\ &\quad + 2sn\sum C(R \mid 2)k(r^{s-1}, r_1+1, r_2+1)/n^{(2)}. \end{aligned}$$

$$(5.9) \quad \begin{aligned} k(r^s)k(3) &= k(r^s3) + sk(r+3, r^{s-1})/n - 3s^{(2)}k(r+2, r+1, r^{s-2})/n^{(2)} \\ &\quad + 2s^{(2)}k(\{r+1\}^3r^{s-3})/n^{(3)} \\ &\quad + 3ns\sum C(R \mid 2)\{k(r^{s-1}, r_1+2, r_2+1) \\ &\quad + k(r^{s-1}, r_1+1, r_2+2)\}/n^{(2)} \\ &\quad - 6ns^{(2)}\sum C(R \mid 2)k(r+1, r^{s-2}, r_1+1, r_2+1)/n^{(3)} \\ &\quad + 6n^2s\sum C(R \mid 3)k(r^{s-1}, s_1+1, s_2+1, s_3+1)/n^{(3)}. \end{aligned}$$

Formulae (5.8) and (5.9) are general enough to include many products needed in making moment estimates of moment functions [10] and also certain formulae with $r = 2, 3$ given by Barton, David and Fix [1].

We complete this section with the formulae for expansion of $k(p)k(q)$ and $k(1^p)k(1^q)$ where, without loss of generality, $p \geqq q$. In the first case we have the Fisher technique. Then, in the notation of this paper,

$$(5.10) \qquad k(p)k(q) = k(pq) + \sum \rho(n)C(P \mid \rho)C(Q \mid \rho)k(P \mid \rho \oplus Q \mid \rho)$$

where $\rho(n) = \sum (u-1)!\Delta^u(0^\rho)/n^{(u)}u$ as given by Fisher and where $k(P \mid \rho \oplus Q \mid \rho)$ represents the sum of $\rho!$ k values obtained by adding the $\rho!$ permutations of $Q \mid \rho$ to the values of $P \mid \rho$ for fixed Q and P. The coefficients in the expansion of $k(p)k(q) - k(pq)$ are those of Fisher's $\kappa(pq)$ is terms of cumulants and are tabulated [5], pp. 210–213 for many values of p and q.

In the second case the combinatorial procedure is very simple. With ρ the number of rows containing 2 units, the formula is

$$(5.11) \quad \begin{aligned} k(1^p)k(1^q) &= \sum \binom{p}{\rho}\binom{q}{\rho} \rho!\,k(2^\rho1^{p+q-2\rho})/n^{(\rho)} \\ &= \sum p^{(\rho)}q^{(\rho)}k(2^\rho1^{p+q-2\rho})/n^{(\rho)}\rho!. \end{aligned}$$

6. Checking. These formulae have been checked by various methods. A useful formula for checking $k(P)k(Q)$ simultaneously for all Q having a given weight uses [4]

$$(6.1) \qquad k(P)k_1^r = \sum \binom{r}{u} C(U) \sum C(T)k(P \oplus U, T)/n^{r-\tau}$$

where $0 \leqq u \leqq r$, $t = r - u$, τ is the order of T, v is the order of U and $k(P \oplus U, T)$ is the sum of $\pi^{(v)}$ k-functions having subscripts which result from adding the υ values of U to υ values of the P in every possible way and suffixing the other $\pi - \upsilon$ values of P and the τ values of T. Thus when $r = 2$

$$k(P)k_1^2 = k(P11) + k(P2)/n + 2k(P \oplus 1, 1)/n + k(P \oplus 2)/n^2 + k(P \oplus 11)/n^2. \tag{6.2}$$

Expanding k_1^2 we get $k(P)\{k(2)/n + k(11)\}$ and application of (5.3) and (5.4) gives (6.2). This technique is shown in Table 2 where the values of $n^2k(P)k_1^2$ are expanded by rows and by columns. The entries in the last column are obtained by multiplying the element in the row by those in the last row, dividing by the divisor, and adding. Corresponding checks are provided for Table 3 and Table 4.

Another method, useful when the specified Q has unit parts, is to apply the result to deviates [4], express in terms of polykays with no unit parts, and thus arrive at a formula which may be determined independently. Thus (5.4) becomes, in the notation of [4], with P not having unit parts

$$k(P)d(11) = d(P11) + k(P \oplus 11)/n^{(2)} + 2d(P \oplus 1, 1)/n - \sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 11)/n^{(2)} \tag{6.2}$$

which, with $d(11) = -k(2)/n$, $d(P11) = k(P \oplus 2)/n^2 - k(P2)/n$, $d(P \oplus 1, 1) = -k(P \oplus 2)/n$, gives (5.3). The method can also be applied to specific cases when p and q have unit parts. Thus the simple combinatorial result $k_{11}^2 = 2k_{22}/n^{(2)} + 4k_{211}/n + k_{1111}$ leads easily to the formula for k_2^2.

REFERENCES

[1] Barton, D. E., David, F. N. and Fix, E. (1960). The polykays of the natural numbers. *Biometrika* **48** 53–59.

[2] Cornish, E. O. and Fisher, R. A. (1937). Moments and cumulants in the specification of distributions. *Rev. Inst. Internat. Statist.* **4** 1–14.

[3] David, F. N. and Kendall, M. G. (1949). Tables of symmetric functions, Part I. *Biometrika* **36** 431–449.

[4] Dwyer, P. S. (1964). Properties of polykays of deviates. *Ann. Math. Statist.* **35** 1167–1173.

[5] Fisher, R. A. (1928). Moments and product moments of sampling distributions. *Proc. London Math. Soc.* (*2*). **30** 199–238.

[6] Irwin, J. O. and Kendall, M. G. (1944). Sampling moments of moments for a finite population. *Ann. Eugenics* **12** 138–142.

[7] Kendall, M. G. (1952). Moment-statistics in samples from a finite population. *Biometrika* **39** 14–16.

[8] Kendall, M. G. and Stuart, A. (1958). *The Advanced Theory of Statistics*, **1**. Griffin, London.

[9] MacMahon, P. A. (1915). *Combinatory Analysis*, **1**. Cambridge Univ. Press.

[10] Schaeffer, Esther and Dwyer, P. S. (1963). Computation with multiple k-statistics. *J. Amer. Statist. Assoc.* **58** 120–151.

[11] Tukey, J. W. (1950). Some sampling simplified. *J. Amer. Statist. Assoc.* **45** 501–519.

[12] Tukey, J. W. (1956). Keeping moment-like sampling computations simple. *Ann. Math. Statist.* **27** 37–54.

[13] Wishart, J. (1952). Moment coefficients of the k-statistics in samples from a finite population. *Biometrika* **39** 1–13.

[14] Wishart, J. (1952). The combinatorial development of the cumulants of the k-statistics. *Trabajos Estadist.* **3** 13–26.