QUEUES WITH BATCH DEPARTURES II

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1. Introduction; queue size after a departure. In this paper we continue the study of queues with batch departures undertaken by Foster and Nyunt in [4]. For a detailed description of the queueing model we refer to their paper. In brief, we assume units to arrive at the instants of a Poisson process with parameter λ and to be served in batches of fixed size k, the service time distribution $H(x)$ being arbitrary. It will be convenient for later use to take the mean of $H(x)$ as r/μ and to define $\rho = r\lambda/\mu$, the ratio of the mean service time to the mean inter-arrival time. (In [4] the mean of $H(x)$ was taken as $1/\mu$ and ρ was defined as λ/μ, so that basically the definition of ρ is unchanged in the present paper.) The traffic intensity is thus $\tau = \rho/k$ and we assume $\tau < 1$. We also need the Laplace Transform $\psi(s) = \int_0^\infty e^{-sx}\, dH(x)$. Let $\xi(t)$ denote the number of units in the system including the batch undergoing service (if any) at the instant t. We say that the system is in state E_j at the instant t if $\xi(t) = j$. Let

$$p_j^+(n) = P[\xi(\sigma_n + 0) = j]$$

where σ_n denotes the instant of departure of the nth batch. It has been shown in [4] that when $\rho < k$, the limiting probabilities

$$p_j^+ = \lim_{n\to\infty} p_j^+(n) \tag{1}$$

exist and their generating function is given by

$$P^+(z) = \sum_{j=0}^{\infty} p_j^+ z^j = \frac{\sum_{j=0}^{k-1} p_j^+(z^k - z^j)}{z^k/K(z) - 1} = \frac{(k-\rho)(z-1)\prod_{j=1}^{k-1}\left(\frac{z-\delta_j}{1-\delta_j}\right)}{z^k/K(z) - 1} \tag{2}$$

where $K(z) = \psi\{\lambda(1-z)\}$ and $1, \delta_1, \delta_2, \cdots \delta_{k-1}$ are the roots of the equation $z^k = K(z)$ on or within the unit circle.

From (2) it follows that

$$\sum_{j=0}^{k-1} p_j^+(z^k - z^j) = (k-\rho)(z-1)\prod_{j=0}^{k-1}(z-\delta_j)/(1-\delta_j)$$

from which we obtain by differentiation,

$$\sum_{j=0}^{k-1} p_j^+(k-j) = k - \rho. \tag{3}$$

We shall use this result (3) in the sequel.

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2. Queue size before an arrival. Now let us define the probability

$$p_j(n) = P[\xi(\tau_n - 0) = j]$$

where τ_n denotes the instant of arrival of the nth unit. We note first that the ordinary limit of $p_j(n)$ does not exist for $k > 1$. To avoid trivialities let us assume that the initial queue size is zero, i.e., $\xi(0) = 0$. Since units are served in batches of k it is clear that $p_j(n)$ is zero unless $n - 1 - j$ (number of units served up to the instant of the nth arrival) is a multiple of k, i.e.

$$(4) \qquad \begin{aligned} p_j(n) &= 0 \quad \text{if} \quad n \neq rk + j + 1 \\ &\neq 0 \quad \text{if} \quad n = rk + j + 1. \end{aligned}$$

Thus we consider the Césaro limit

$$p_j = \lim_{n\to\infty}(1/n)\sum_{m=1}^{n} p_j(m).$$

In view of the above relations (4), however, it is easy to show that this Césaro limit, if it exists, is also the Césaro limit of the sequence

$$(1/k)p_j(rk + j + 1) \qquad r = 0, 1, 2, \cdots.$$

We shall show that the ordinary limit of this sequence exists when $\rho < k$. It follows that the Césaro limit exists and

$$(5) \qquad p_j = (1/k) \lim_{r\to\infty} p_j(rk + j + 1).$$

Define the generating function $P(z) = \sum_{j=0}^{\infty} p_j z^j$.

3. Queue size at a random instant. Now let us introduce a third limiting probability distribution

$$(6) \qquad p_j^* = \lim_{t\to\infty} P[\xi(t) = j].$$

The existence of these limits (6) has previously been proved by Takács [9] and in this paper we shall assume their existence. Define $P^*(z) = \sum_{j=0}^{\infty} p_j^* z^j$.

4. Discussion of the main results. In this paper, we shall examine the relationships between the three distributions introduced above, $P(z)$, $P^+(z)$, $P^*(z)$, and specifically establish that

$$(7) \qquad P(z) = P^*(z) = (1/k)[(1 - z^k)/(1 - z)]P^+(z).$$

This result comprises Theorems 1 and 2 below. We use direct methods of proof on the limiting distribution and the proof of Theorem 2 is based on a method due to Khintchine [6] which he used for the case $k = 1$.

Theorem 2 has been previously proved by Takács [9] as a limiting result of the time-dependent case. It may be remarked that it can be derived from a result of Fabens [2]. A proof based on the supplementary variables method [1] can also be given [7].

We will assert these results heurisically before we give any rigorous proofs. Let us first consider the case $k = 1$. p_j, the probability that an arbitrary arrival finds j units in the system, is the proportion of arrivals that find j units ahead of it, that is, the proportion of transitions $E_j \to E_{j+1}$. Now if M_j denotes the number of transitions $E_j \to E_{j+1}$ over a given long period of time, then asymptotically $p_j = M_j / \sum_{j=0}^{\infty} M_j$.

Similarly, if N_j denotes the number of transitions $E_{j+1} \to E_j$ over a long period of time, then asymptotically, $p_j^+ = N_j / \sum_{j=0}^{\infty} N_j$. Now the number of transitions $E_j \to E_{j+1}$ and $E_{j+1} \to E_j$ differ at most by one and hence over a long period of time we have asymptotically, $M_j = N_j$ and $\sum_{j=0}^{\infty} M_j = \sum_{j=0}^{\infty} N_j$ from which it follows that $p_j = p_j^+$. The argument, clearly, makes no asumptions about the distributions and is therefore true for the general model $G/G/1$ where the input and the output are arbitrary. If, however, the input is Poisson, then on account of its Markov property, the arrival of a unit is independent of any other event in the system and so it follows that $p_j = p_j^*$. Thus for the model $E_1/G/1$ we have $P(z) = P^*(z) = P^+(z)$. An analogous argument may be applied to the case $k > 1$. If as before M_j and N_j denote the number of transitions $E_j \to E_{j+1}$ and $E_{j+k} \to E_j$ respectively, over a long period of time, we have, asymptotically,

$$M_j = N_0 + N_1 + \cdots + N_j \qquad j < k$$

$$= N_{j-k+1} + N_{j-k+2} \cdots + N_j \qquad j \geqq k.$$

For the two sides differ at most by one. Thus $\sum_{j=0}^{\infty} M_j = k \sum_{j=0}^{\infty} N_j$ and interpreting p_j and p_j^+ as before, we have

$$p_j = (1/k)(p_0^+ + p_1^+ + \cdots + p_j^+) \qquad j < k$$

$$= (1/k)(p_{j-k+1}^+ + p_{j-k+2}^+ \cdots + p_j^+) \qquad j \geqq k.$$

Here again we make no assumptions of the type of distributions and hence this relationship is true for the general model $G/G^k/1$ where the input and the output are arbitrary. If, however, the arrival process is Poisson, then for the same reason as in the case $k = 1$, we have $p_j = p_j^*$ and hence (7) is true for the model $E_1/G^k/1$.

5. Proofs of the main results. We now treat this problem more rigorously. Let

$$\xi_n = \xi(\tau_n - 0); \qquad \xi_n^+ = \xi(\sigma_n + 0).$$

THEOREM 1. *If $\rho < k$ the limiting probabilities defined by* (5) *exist and* $P(z) = (1/k)[(1 - z^k)/(1 - z)]P^+(z)$.

PROOF. If $j < k$, then

$$\xi_{nk+j+1} = j \Leftrightarrow \sigma_n < \tau_{nk+j+1} \Leftrightarrow \xi_n^+ \leqq j.$$

Therefore

$$p_j(nk + j + 1) = P[\sigma_n < \tau_{nk+j+1}] = \sum_{i=0}^{j} p_i^+(n). \tag{8}$$

If $j \geqq k$, say $j = sk + r$ $(s > 0, r < k)$, then

$$\xi_{nk+j+1} = ik + r \ (i = 0, 1, \cdots s) \Leftrightarrow \sigma_n < \tau_{nk+j+1} \Leftrightarrow \xi_n^+ \leqq j.$$

Therefore

$$\sum_{i=0}^{s} p_{ik+r}\,(nk + j + 1) = P[\sigma_n < \tau_{nk+j+1}] = \sum_{i=0}^{j} p_i^+(n). \tag{9}$$

Now in [4] it was shown that when $\rho < k$ the right hand sides of (8) and (9) tend to finite limits. Hence under the same conditions the probabilities on the left hand sides tend to finite limits. Thus the p_j defined by (5) exist, and we have from (8),

$$p_j = (1/k)(p_j^+ + p_{j-1}^+ + \cdots + p_0^+) \qquad j < k.$$

From (9) we have for $j = sk + r$,

$$\sum_{i=0}^{s} p_{j-ik} = (1/k)\sum_{i=0}^{j} p_i^+.$$

Replacing j by $j - k$ and subtracting, we obtain at once

$$p_j = (1/k)(p_j^+ + p_{j-1}^+ + \cdots + p_{j-k+1}^+) \qquad j \geqq k.$$

The theorem is now immediate on taking generating functions.

THEOREM 2. *If $\rho < k$ and the service time distribution is not a lattice distribution, then* $P^*(z) = (1/k)[(1 - z^k)/(1 - z)]P^+(z)$.

PROOF. The expected number of arrivals and services during an arbitrary period T are λT and $\lambda T/k$ respectively. But the expected length of each service is r/μ and so the expected time the server is busy is $(\lambda/k)T\cdot r/\mu = (\rho/k)T$. Hence the probability that at an arbitrary instant of time the server is busy is ρ/k and the probability that the server is idle is $1 - \rho/k$.

p_j^* is the probability that the number of units in the system at an arbitrary instant of time is j. Consider such an instant. This falls in an idle period $(j < k)$ with probability $1 - \rho/k$.

An idle period can begin with $i = 0, 1, 2, \cdots, k - 1$ units present. It ends when there are k units present and it is divided by the instants of arrival into $k - i$ intervals each of mean length $1/\lambda$.

The probability that an idle period selected at random begins with i units is jointly proportional to p_i^+ and to the expected length of such an idle period; which is $(k - i)/\lambda$.

The conditional probability, given that the idle period selected starts with i units, that at a random instant in the interval there are j $(= i, i + 1, \cdots, k - 1)$ units present is $1/(k - i)$.

We have therefore,

$$p_j^* = \sum_{i=0}^{j} C\cdot(k - i)p_i^+\cdot 1/(k - i) \qquad (j = 0, 1, 2, \cdots, k - 1)$$
$$= C\cdot\sum_{i=0}^{j} p_i^+,$$

where C is a normalising constant. Now we have $\sum_{j=0}^{k-1} p_j^* = 1 - \rho/k$. Therefore

$$C\sum_{j=0}^{k-1} \sum_{i=0}^{j} p_i^+ = 1 - \rho/k$$

i.e. $C\sum_{j=0}^{k-1}(k - j)p_j^+ = 1 - \rho/k$. Therefore by (3), $C = (1 - \rho/k)/(k - \rho) = 1/k$; hence

$$p_j^* = (1/k)\sum_{i=0}^{j} p_i^+. \tag{10}$$

Now consider $p_j^*(j \geqq k)$, then the arbitrary instant considered falls in a busy period. Denote by u the expired part of the service time that is going on at the instant considered. Then by an argument similar to that used by Khintchine [6] or as a consequence of a known result in renewal theory (see [8]), we get, provided $H(x)$ is not a lattice distribution,

$$P[x < u < x + dx] = (\rho/k)(\mu/\nu)\{1 - H(x)\}\, dx = (\lambda/k)\{1 - H(x)\}\, dx.$$

Thus the probability that j units arrive during the expired part of the service time is

$$m_j = \frac{\lambda}{k}\int_0^\infty e^{-\lambda x}\,\frac{(\lambda x)^j}{j!}\,\{1 - H(x)\}\, dx,$$

i.e.

$$M(z) = \sum_{j=0}^{\infty} m_j z^j = (1/k)[(1 - K(z))/(1 - z)]. \tag{11}$$

Now if ξ^* is the queue size at an arbitrary instant, by comparing ξ^* with the queue size ξ^+ after the immediately preceding service end point, we have

$$\xi^* = \max(\xi^+, k) + \nu^* \qquad (\xi^* \geqq k)$$

where ν^* is the number of arrivals during the expired part of the service time that is going on at the instant considered. Applying generating functions to the above equation, we get

$$\begin{aligned} P^*(z) - \sum_{j=0}^{k-1} p_j^* z^j &= \left\{P^+(z) - \sum_{j=0}^{k-1} p_j^+ z^j\right\} M(z) + \sum_{j=0}^{k-1} p_j^+ \cdot M(z)\cdot z^k \\ &= P^+(z)\cdot M(z) + \sum_{j=0}^{k-1} p_j^+(z^k - z^j)\cdot M(z). \end{aligned} \tag{12}$$

But by Equations (2) and (10)

$$\begin{aligned} &\sum_{j=0}^{k-1} p_j^+(z^k - z^j) = \left\{\frac{z^k}{K(z)} - 1\right\} P^+(z) \text{ and} \\ &\sum_{j=0}^{k-1} p_j^* z^j = \frac{1}{k}\,\frac{\sum_{j=0}^{k-1} p_j^+(z^k - z^j)}{z - 1} = \frac{1}{k}\left\{\frac{z^k}{K(z)} - 1\right\}\cdot\frac{P^+(z)}{z - 1}. \end{aligned} \tag{13}$$

Substituting from Equations (11) and (13) in Equation (12), we obtain the theorem.

EXAMPLE 1. If the service time distribution is Erlang E_r, it was shown in [4] that

$$P^+(z) = \prod_{j=1}^{r}((1 - \epsilon_j)/(1 - \epsilon_j z)) \tag{14}$$

where ϵ_j are the reciprocals of the roots outside the unit circle of the equation

$$[1 + (\rho/r)(1 - z)]^{-r} = z^k. \tag{15}$$

The ϵ_j $(j = 1, 2, \cdots, r)$ were shown to be distinct and hence

$$P^+(z) = \sum_{j=1}^{r} C_j/(1 - \epsilon_j z)$$

where $C_j = (1 - \epsilon_j)\prod_{i \neq j}((1 - \epsilon_i)/(1 - \epsilon_i/\epsilon_j))$, i.e.

$$p_m^+ = \sum_{j=1}^{r} C_j \epsilon_j^m. \tag{16}$$

Then by Theorems 1 and 2,

$$P(z) = P^*(z) = \frac{1}{k}\frac{1 - z^k}{1 - z}\prod_{j=1}^{r}\left(\frac{1 - \epsilon_j}{1 - \epsilon_j z}\right). \tag{17}$$

It follows that

$$\begin{aligned} p_m = p_m^* &= \frac{1}{k}\sum_{j=1}^{r} C_j \frac{1 - \epsilon_j^{m+1}}{1 - \epsilon_j} && m < k \\ &= \frac{1}{k}\sum_{j=1}^{r} C_j \epsilon_j^{m-k+1}\left(\frac{1 - \epsilon_j^k}{1 - \epsilon_j}\right) && m \geqq k. \end{aligned} \tag{18}$$

EXAMPLE 2. If the service time distribution is Exponential E_1, then we have from Example 2 of [4] and the above theorems,

$$P(z) = P^*(z) = \frac{1}{k}\frac{1 - z^k}{1 - z}\left(\frac{1 - \epsilon}{1 - \epsilon z}\right) \tag{19}$$

where ϵ is the reciprocal of the root outside the unit circle of the equation

$$\{1 + \rho(1 - z)\}^{-1} = z^k. \tag{20}$$

As in the previous example we can express these probabilities explicitly for later use,

$$\begin{aligned} p_m = p_m^* &= (1/k)(1 - \epsilon^{m+1}) && m < k \\ &= (1/k)\epsilon^{m-k+1}(1 - \epsilon^k) && m \geqq k. \end{aligned} \tag{21}$$

6. Relationship between $E_1/G^k/1$ and $E_k/G/1$. In this section, we investigate the relationships between the distributions for these two systems. We can think of the arrival process of the system $E_k/G/1$ as being composed of k phases each of which has an exponential distribution with parameter λ. Now if we think of a unit as being composed of k sub-units, one of which arrives at each phase then the situation is identical with the batch departure model $E_1/G^k/1$, a batch in the batch departure model corresponding to a unit in the Erlang arrival model. Thus if $\tau_1, \tau_2, \cdots$ are the instants of arrival in the batch departure model, the instants of arrival in the Erlang arrival model are $\tau_k, \tau_{2k}, \tau_{3k}, \cdots$ since a unit in the Erlang arrival model would have arrived when and only when all its k sub-units have arrived. Since the sub-units are served in batches, the instants of departure for both models are the same. It should be noted that the Erlang distribution considered here is composed of k independent exponential distributions, each with parameter λ, so that the Erlang distribution has mean k/λ, and that this differs from the Erlang distribution used by one of the authors in [3] where each exponential distribution had parameter λk so that the Erlang distribution had mean $1/\lambda$.

Let $\zeta(t)$ denote the number of units in the Erlang arrival model (including the units at the service counter if any) at the instant t. We shall study the three distributions related to the system $E_k/G/1$:

(i) $$q_j^+(n) = P[\zeta(\sigma_n + 0) = j]$$

(ii) $$q_j^*(t) = P[\zeta(t) = j]$$

(iii) $$q_j(n) = P[\zeta(\tau_{nk} - 0) = j].$$

If $\xi(t)$ denotes the number of units in the batch departure system at the instant t, then clearly

$$\zeta(t) = [\xi(t)/k] \tag{22}$$

where $[x]$ is the integral part of x.

(i) From Equation (22) we have

$$q_j^+(n) = p_{jk}^+(n) + p_{jk+1}^+(n) + \cdots + p_{jk+k-1}^+(n).$$

The probabilities on the right hand side are known to converge as $n \to \infty$ when $\rho < k$ and hence

$$q_j^+ = \lim_{n\to\infty} q_j^+(n) \qquad j = 0, 1, 2, \cdots \tag{23}$$

exist when $\rho < k$ and

$$q_j^+ = p_{jk}^+ + p_{jk+1}^+ + \cdots + p_{jk+k-1}^+ . \tag{24}$$

Define

$$Q^+(z) = \sum_{j=0}^{\infty} q_j^+ z^j.$$

In [4] the relationship between the generating functions $Q^+(z)$ and $P^+(z)$ was derived:

(25) $$Q^+(z) = \frac{1-z}{2\pi i}\int_C \frac{P^+(v)}{(1-v)(v^k-z)}\,dv$$

where C is the contour $|z| = 1 - \delta$. We shall not make use of this formula in the present paper.

(ii) From Equation (22) we have

$$q_j^*(t) = p_{jk}^*(t) + p_{jk+1}^*(t) + \cdots + p_{jk+k-1}^*(t).$$

It is known (cf [9]) that the probabilities on the right hand side converge as $t \to \infty$ when $\rho < k$, provided the service time distribution is not a lattic distribution. Under the same conditions, therefore,

(26) $$q_j^* = \lim_{t\to\infty} q_j^*(t) \qquad j = 0, 1, 2, \cdots$$

exist and

(27) $$q_j^* = p_{jk}^* + p_{jk+1}^* + \cdots + p_{jk+k-1}^*.$$

Define

$$Q^*(z) = \sum_{j=0}^{\infty} q_j^* z^j.$$

An analysis similar to that carried out in [4] will give the direct relationship between the generating functions

(28) $$Q^*(z) = \frac{1-z}{2\pi i}\int_C \frac{P^*(v)}{(1-v)(v^k-z)}\,dv.$$

Again we shall not use this formula in the present paper.

(iii) Now let us consider the limits

(29) $$q_j = \lim_{n\to\infty} q_j(n)$$

and the generating function $Q(z) = \sum_{j=0}^{\infty} q_j z^j$. By Equation (22)

$$q_j(n) = p_{jk}(nk) + p_{jk+1}(nk) + \cdots + p_{jk+k-1}(nk) = p_{jk+k-1}(nk)$$

since the other terms are zero by Equation (4). We have in Theorem 1 of the preceding section that when $\rho < k$ the limits, $(1/k)\lim_{n\to\infty} p_j(nk + j + 1) = p_j$ exist. Hence the limits q_j defined by (29) exist when $\rho < k$ and $q_j = kp_{jk+k-1}$. But by Theorem 1

$$p_{jk+k-1} = (1/k)(p_{jk+k-1}^+ + p_{jk+k-2}^+ + \cdots + p_{jk}^+).$$

Therefore $q_j = p_{jk+k-1}^+ + p_{jk+k-2}^+ + \cdots + p_{jk}^+$ so that by (24), $q_j = q_j^+$, i.e.

(30) $$Q(z) = Q^+(z).$$

This is consistent with the general result asserted heuristically in Section 1, that the queue size distribution just before arrivals is the same as that just after departures for the general model $G/G/1$.

EXAMPLE 3. If the service time distribution is Erlang E_r, then we have from Example 1 and Equations (24) and (30)

$$q_m = q_m^+ = \sum_{j=0}^{k-1} p_{mk+j}^+ = \sum_{j=1}^{r} C_j(\epsilon_j^{mk} + \epsilon_j^{mk+1} + \cdots + \epsilon_j^{mk+k-1}) \tag{31}$$
$$= \sum_{j=1}^{r} C_j\epsilon_j^{mk}((1 - \epsilon_j^k)/(1 - \epsilon_j)),$$

i.e.

$$Q(z) = Q^+(z) = \sum_{j=1}^{r} \frac{C_j}{1 - \epsilon_j}\left(\frac{1 - \epsilon_j^k}{1 - \epsilon_j^k z}\right). \tag{32}$$

Again from Example 1 and Equation (27)

$$q_m^* = \sum_{j=0}^{k-1} p_{mk+j}^* \qquad m \geqq 1$$
$$= \frac{1}{k}\sum_{j=1}^{r} C_j \frac{1 - \epsilon_j^k}{1 - \epsilon_j}(\epsilon_j^{mk-k+1} + \epsilon_j^{mk-k+2} + \cdots + \epsilon_j^{mk}) \tag{33a}$$
$$= \frac{1}{k}\sum_{j=1}^{r} C_j \epsilon_j^{mk-k+1}\left(\frac{1 - \epsilon_j^k}{1 - \epsilon_j}\right)^2$$

and

$$q_0^* = \sum_{j=0}^{k-1} p_j^*$$
$$= \frac{1}{k}\sum_{j=1}^{r} \frac{C_j}{1 - \epsilon_j}(1 - \epsilon_j + 1 - \epsilon_j^2 + \cdots + 1 - \epsilon_j^k) \tag{33b}$$
$$= 1 - \frac{1}{k}\sum_{j=1}^{r} C_j\epsilon_j \frac{1 - \epsilon_j^k}{(1 - \epsilon_j)^2},$$

since $\sum_{j=1}^{r} C_j/(1 - \epsilon_j) = 1$, i.e.

$$Q^*(z) = 1 - \frac{1}{k}\sum_{j=1}^{r} C_j \frac{(1 - \epsilon_j^k)}{(1 - \epsilon_j)^2}\left\{\epsilon_j - \sum_{m=1}^{\infty} \epsilon_j^{mk-k+1} z^m\right\} \tag{34}$$
$$= 1 - \frac{1 - z}{k}\sum_{j=1}^{r} \frac{C_j\epsilon_j}{(1 - \epsilon_j)^2} \cdot \frac{1 - \epsilon_j^k}{1 - \epsilon_j^k z}.$$

We shall obtain a similar set of formulae for the same model $E_k/E_r/1$ through batch arrivals in [5] and therein we shall show the equivalence of the two sets of results (cf [3] and [4]).

EXAMPLE 4. If the service time distribution is Exponential E_1, then, as in the previous example, we have from Example 2 and Equations (24) and (30)

$$q_j = q_j^+ = (1 - \epsilon)(\epsilon^{jk} + \epsilon^{jk+1} + \cdots + \epsilon^{jk+k-1}) = \epsilon^{jk}(1 - \epsilon^k), \tag{35}$$

i.e.

$$Q(z) = Q^{+}(z) = (1 - \epsilon^k)/(1 - \epsilon^k z). \tag{36}$$

Again from Example 2 and Equation (27)

$$\begin{aligned} q_j^* &= (1/k)(1 - \epsilon^k)(\epsilon^{jk-k+1} + \epsilon^{jk-k+2} + \cdots + \epsilon^{jk}) \qquad j \geqq 1 \\ &= (1/k)[(1 - \epsilon^k)^2/(1 - \epsilon)]\epsilon^{jk-k+1}, \end{aligned} \tag{37a}$$

and

$$\begin{aligned} q_0^* &= (1/k)(1 - \epsilon + 1 - \epsilon^2 + \cdots + 1 - \epsilon^k) \\ &= 1 - (\epsilon/k)[(1 - \epsilon^k)/(1 - \epsilon)]. \end{aligned} \tag{37b}$$

But from Equation (20), $\epsilon(1 - \epsilon^k)/(1 - \epsilon) = \rho$, i.e., $(\epsilon/k)[(1 - \epsilon^k)/(1 - \epsilon)] = \rho/k = \tau$. Substituting in (37a) and (37b), we obtain

$$q_j^* = \tau(1 - \epsilon^k)\epsilon^{(j-1)k} \qquad j \geqq 1,$$

with $q_0^* = 1 - \tau$. Therefore

$$\begin{aligned} Q^*(z) = q_0^* + \sum_{j=1}^{\infty} q_j^* z^j &= 1 - \tau + \tau(1 - \epsilon^k) \sum_{j=1}^{\infty} \epsilon^{(j-1)k} z^j \\ &= 1 - \tau + \tau(1 - \epsilon^k)z/(1 - \epsilon^k z). \end{aligned}$$

i.e.

$$Q^*(z) = 1 - \tau + \tau z Q(z). \tag{38}$$

This relationship between $Q^*(z)$ and $Q(z)$ will be seen to hold for the wider class of queueing models $G/E_1/1$ in [5].

REFERENCES

[1] Cox, D. R. (1955). The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables. *Proc. Cambridge Philos. Soc.* **51** 433–441.

[2] Fabens, A. J. (1961). The solution of queueing and inventory models by semi-Markov processes. *J. Roy. Statist. Soc. Ser. B* **23** 113–127.

[3] Foster, F. G. (1961). Queues with batch arrivals I. *Acta Math. Acad. Sci. Hungar.* **12** 1–10.

[4] Foster, F. G. and Nyunt, K. M. (1961). Queues with batch departures I. *Ann. Math. Statist.* **32** 1324–1332.

[5] Foster, F. G. and Perera, A. G. A. D. Queues with batch arrivals II. To be published.

[6] Khintchine, A. Ja. (1932). Mathematical theory of a stationary queue. *Mat. Sb.* **39** 73–84.

[7] Perera, A. G. A. D. (1962). Some theory of unit and batch size queues. Ph.D. thesis, Univ. of London.

[8] Smith, W. L. (1958). Renewal theory and its ramifications. *J. Roy. Statist. Soc. Ser. B* **20** 243–302.

[9] Takács, L. (1961). Transient behaviour of single-server queueing processes with Erlang input. *Trans. Amer. Math. Soc.* **100** 1–28.