SMALL-SAMPLE DISTRIBUTIONS OF THE TWO-SAMPLE CRAMÉR-VON MISES' W^2 AND WATSON'S U^2

By E. J. Burr[1]

University of New England, Armidale

1. Introduction and summary. The hypothesis that the two independent random samples $u_1, \cdots, u_m$ and $v_1, \cdots, v_n$ come from the same (unknown) continuous distribution may be tested by the two-sample Cramér-von Mises criterion

$$W^2 = mn(m+n)^{-2}\sum d_i^2,$$

or by

$$U^2 = mn(m+n)^{-2}\sum(d_i - \bar{d})^2,$$

as proposed by Watson [4]. Here d_i is the difference between the sample distribution functions at the ith point in the pooled sample; more precisely, if there are m_i members of the first sample and n_i members of the second sample contained in the first i members of the pooled set of $m + n$ members arranged in order of magnitude, then

$$d_i = n_i/n - m_i/m, \qquad i = 1, 2, \cdots, m+n;$$

and $\bar{d} = \sum d_i/(m+n)$. Watson's U^2 is particularly appropriate when the sample members lie on a circle with no preferred initial point.

The limiting distributions of W^2 and U^2 for large m, n are known [1], [4]. Comparisons between the upper tails of the limiting distributions and small-sample distributions have been made by Anderson [2] for W^2 with $m, n \leqq 7$ and $m = n = 8$; by the writer [3] for W^2 with $m = n \leqq 12$; and by Watson [5] for U^2 and W^2 with $m = n = 10$. In this paper similar comparisons are made, for both U^2 and W^2, for all sample pairs with $m, n \geqq 4$ and $m + n \leqq 17$.

For each such m, n, the true probability P of attaining or exceeding U^2 is tabulated for every attainable U^2 with $P \leqq 0.1$. The correction factor $R = P/P_\infty$ is also given, where P_∞ is the approximation to P obtained by using the limiting distribution. For W^2, the values of P and R are tabulated for the smallest W^2 significant at each of the levels

$$(10, 8, 6, 5, 4, 3, 2.5, 2, 1.5) \times 10^{-a}, \qquad a = 2, 3, 4,$$

except for the largest and second largest values of W^2, for which special formulae are supplied. It is found that for all U^2 and W^2 other than the largest and second largest values of each, when P lies between 0.1 and 0.01, R lies between 0.65 and

Received 3 September 1963; revised 28 January 1964.

[1] Research partially supported by the Office of Naval Research, Contract Nonr-248(63) to the Johns Hopkins University.

1.35 (except when $m = n = 5$), and when P lies between 0.01 and 0.001, R lies between 0.47 and 1.02. Less precisely, we may sum up by saying that the limiting distribution yields upper-tail areas correct within ± 35 per cent between the 10 per cent and 1 per cent levels, but it overestimates the true P by a factor between 1 and 2 between the 1 per cent and 0.1 per cent levels.

2. Method of calculation. In the pooled ordered sample let r_j members of the second sample precede the jth member of the first sample, so that $0 \leqq r_1 \leqq \cdots \leqq r_m \leqq n$. Then the values of d_i are

$$1/n,\ 2/n,\ \cdots,\ r_1/n,\ r_1/n - 1/m,\ (r_1 + 1)/n - 1/m,\ \cdots,\ r_2/n - 1/m,$$
$$r_2/n - 2/m,\ \cdots,\ n/n - m/m.$$

To express the variable part of W^2 as a sum of squares of integers in the simplest possible way, write $s_j = 2mr_j - (2j - 1)n, j = 1, \cdots, m$. Then it is found that

$$W^2 = \sum s_j^2/4m^2n(m + n) + n(n + 2m)/12mn(m + n),$$
$$U^2 = (\sum s_j^2 - m\bar{s}^2)/4m^2n(m + n) + n(n + 2m)/12mn(m + n),$$

where $\bar{s} = \sum s_j/m$.

There are $\binom{m+n}{n}$ permutations of the members of the first sample with those of the second. Under the hypothesis, all these permutations are equally probable. To each permutation there corresponds a set of integers $r_1, \cdots, r_m$, and hence an integer $\sum s_j^2$ and an integer $\sum s_j^2 - m\bar{s}^2$. Hence, to find the distributions of W^2 and U^2 for given m, n, it is necessary to count the permutations giving rise to each attainable value of $\sum s_j^2$ and of $\sum s_j^2 - m\bar{s}^2$.

The calculations were carried out on the Ferranti SIRIUS computer at the Melbourne Computer Centre.

3. Results. Table 1 gives the values of P and R for all attainable U^2 for which $P \leqq 0.1$. Values of P are given in floating point decimal form; for example, the first entry under P represents 7.14×10^{-2}.

The attainable values of W^2 with $P \leqq 0.1$ are about ten times as numerous as those of U^2, so that complete tabulation is hardly practicable. Table 2 gives the values of P and R for W^2 selected as described in Section 1. The largest and second largest values of W^2 have been omitted to conserve space. The values of P for these extreme W^2 are simple multiples (see next paragraph) of the probability $p_0 = \binom{m+n}{n}^{-1}$. For each m, n, the value of p_0 may be found rapidly from the fact that the smallest P shown in Table 1 equals $(m + n)p_0$, or that the smallest P shown in Table 2 equals $8p_0$ or $6p_0$ according as $m = n$ or $m \neq n$.

The largest attainable W^2 is $(2mn + 1)/6(m + n)$, and its significance level is $2p_0$. The largest but one is $(2mn - 11)/6(m + n) + 1/mn$, and its significance level is $4p_0$.

The largest attainable U^2 is $(mn + 2)/12(m + n)$, and its significance level is $(m + n)p_0$. The largest but one is $(mn + 2)/12(m + n) - (m - 1)(n - 1)/mn(m + n)$, and its significance level is $3(m + n)p_0$.

The range of sample sizes considered here is too small to reveal any simple empirical formula for R such as that found for $R(W^2)$ with $m = n$ [3], but the following summary may be useful.

If we exclude the largest and second largest values of U^2 for each m, n, then for all sample sizes included in Table 1 and all U^2 with significance levels between 0.1 and 0.01, we have $0.65 \leqq R(U^2) \leqq 1.34$. The corresponding range for $R(W^2)$ is $0.76 \leqq R(W^2) \leqq 1.35$, except for one value with $m = n = 5$ for which $R = 1.63$. For significance levels between 0.01 and 0.001, still excluding the largest and second largest values, both $R(U^2)$ and $R(W^2)$ satisfy $0.47 \leqq R \leqq 1.02$.

Closer bounds for $R(U^2)$ are given by the following.

$$m + n = 12: \quad R(U^2) = 1.24 \pm 0.13;$$

$$m + n = 13: \quad R(U^2) = 1.06 \pm 0.19;$$

$$m + n = 14: \quad R(U^2) = 1.02 \pm 0.22;$$

$$m + n = 15: \quad R(U^2) = 1.00 - 4.0(U^2 - 0.182) \pm 0.12;$$

$$m + n = 16: \quad R(U^2) = 1.00 - 3.7(U^2 - 0.170) \pm 0.14;$$

$$m + n = 17: \quad R(U^2) = 1.00 - 3.3(U^2 - 0.160) \pm 0.12.$$

These bounds hold for all $U^2 > 0.153$ (these include all U^2 with $P \leqq 0.1$) except the largest and second largest values. (For $m + n \leqq 11$, there are no such U^2.) The last three formulae yield some clues to the possible behaviour of $R(U^2)$ for larger $m + n$, but extrapolation is risky.

The dependence of $R(W^2)$ on m and n is more complicated, but the empirical formula found for equal samples continues to give good approximations when m and n are nearly equal. Thus for $(m, n) = (8, 8)$ or $(7, 9)$ we have the bounds $R(W^2) = 1 + (1/8)f(W^2) \pm 0.12$, and for $(m, n) = (8, 9)$ or $(7, 10)$ we have $R(W^2) = 1 + (2/17)f(W^2) \pm 0.12$, where $f(\cdot)$ is the correction function described in [3]. But for each $m + n$, with increasing $|m - n|$ the upper and lower bounds of $R(W^2)$ both decrease.

4. Examples.

Example 1. Let $m = 7$, $n = 9$, $W^2 = 1.0273$. From Table 2 we see that this W^2 lies between 0.9936, for which $P = 0.00140$ and 1.0491, the smallest W^2 with $P \leqq 0.001$. Hence for $W^2 = 1.0273$ we have $0.00100 < P < 0.00140$. Alternatively, we may interpolate $f(1.0273) = -3.32$ from Table 4 of [3] and so obtain the bounds

$$R = 1 + (1/8)(-3.32) \pm 0.12 = 0.585 \pm 0.12.$$

By interpolation in Table 3 of [3], $P_\infty = 0.002124$, so that the corresponding bounds for the true P are

$$P = (0.002124)(0.585 \pm 0.12) = (1.24 \pm 0.25) \times 10^{-3}.$$

TABLE 1. Probability P of attaining or exceeding U^2.

m, n	U^2	P	R
4, 5	.2037	7.14 -2	1.99
4, 6	.2167	4.76 -2	1.71
4, 7	.1688	1.00 -1	1.40
	.2273	3.33 -2	1.48
4, 8	.1632	9.70 -2	1.22
	.1814	7.27 -2	1.31
	.2361	2.42 -2	1.28
4, 9	.1752	7.27 -2	1.16
	.1923	5.45 -2	1.21
	.2436	1.82 -2	1.11
4,10	.1571	9.79 -2	1.09
	.1643	8.39 -2	1.07
	.1857	5.59 -2	1.09
	.2018	4.20 -2	1.13
	.2500	1.40 -2	0.97
4,11	.1556	9.89 -2	1.07
	.1677	7.69 -2	1.05
	.1737	6.59 -2	1.02
	.1950	4.40 -2	1.03
	.2101	3.30 -2	1.04
	.2556	1.10 -2	0.85
4,12	.1562	8.79 -2	0.96
	.1654	7.91 -2	1.03
	.1771	6.15 -2	1.01
	.1823	5.27 -2	0.96
	.2031	3.52 -2	0.97
	.2174	2.64 -2	0.96
	.2604	8.79 -3	0.75
4,13	.1538	1.00 -1	1.04
	.1629	8.57 -2	1.07
	.1652	7.14 -2	0.93
	.1742	6.43 -2	1.00
	.1855	5.00 -2	0.97
	.1900	4.29 -2	0.91
	.2104	2.86 -2	0.91
	.2240	2.14 -2	0.89
	.2647	7.14 -3	0.66
5, 5	.2250	3.97 -2	1.68
5, 6	.1818	7.14 -2	1.29
	.2424	2.38 -2	1.43
5, 7	.1712	7.58 -2	1.11
	.1998	4.55 -2	1.17
	.2569	1.52 -2	1.21
5, 8	.1654	8.08 -2	1.06
	.1769	7.07 -2	1.16
	.1885	5.05 -2	1.04
	.2154	3.03 -2	1.06
	.2692	1.01 -2	1.03
5, 9	.1591	9.79 -2	1.13
	.1623	8.39 -2	1.03
m, n	U^2	P	R
5, 9	.1655	6.99 -2	0.92
	.1814	5.59 -2	1.00
	.1909	4.90 -2	1.06
	.2036	3.50 -2	0.97
	.2290	2.10 -2	0.96
	.2798	6.99 -3	0.87
5,10	.1609	9.99 -2	1.20
	.1742	6.99 -2	1.09
	.1769	5.99 -2	0.98
	.1956	4.00 -2	0.95
	.2036	3.50 -2	0.97
	.2169	2.50 -2	0.90
	.2409	1.50 -2	0.87
	.2889	5.00 -3	0.75
5,11	.1560	9.89 -2	1.07
	.1605	9.16 -2	1.09
	.1696	7.69 -2	1.09
	.1742	6.59 -2	1.03
	.1878	5.13 -2	1.04
	.1901	3.66 -2	0.78
	.2082	2.93 -2	0.89
	.2151	2.56 -2	0.89
	.2287	1.83 -2	0.84
	.2514	1.10 -2	0.79
	.2969	3.66 -3	0.64
5,12	.1549	9.62 -2	1.02
	.1608	8.79 -2	1.05
	.1628	8.24 -2	1.02
	.1647	7.97 -2	1.03
	.1667	7.42 -2	1.00
	.1686	6.87 -2	0.96
	.1726	6.32 -2	0.95
	.1784	5.77 -2	0.98
	.1824	5.22 -2	0.95
	.1863	4.95 -2	0.98
	.1980	3.85 -2	0.96
	.2000	3.30 -2	0.85
	.2020	2.75 -2	0.74
	.2196	2.20 -2	0.84
	.2255	1.92 -2	0.82
	.2392	1.37 -2	0.77
	.2608	8.24 -3	0.71
	.3039	2.75 -3	0.55
6, 6	.1713	9.09 -2	1.34
	.2060	3.90 -2	1.14
	.2639	1.30 -2	1.19
6, 7	.1538	9.85 -2	1.03
	.1722	8.33 -2	1.25
	.1758	6.82 -2	1.10
	.1832	6.06 -2	1.13
	.1905	5.30 -2	1.14
	.1941	3.79 -2	0.87
	.2271	2.27 -2	1.01
	.2820	7.58 -3	0.99
6, 8	.1607	9.32 -2	1.11
	.1622	8.39 -2	1.03
	.1682	7.46 -2	1.03
m, n	U^2	P	R
6, 8	.1741	6.53 -2	1.01
	.1786	5.59 -2	0.95
	.1860	5.13 -2	1.01
	.1964	4.20 -2	1.01
	.2039	3.73 -2	1.04
	.2083	3.26 -2	1.00
	.2143	2.33 -2	0.80
	.2455	1.40 -2	0.89
	.2976	4.66 -3	0.83
6, 9	.1556	9.89 -2	1.07
	.1580	8.99 -2	1.02
	.1630	7.79 -2	0.97
	.1654	7.49 -2	0.98
	.1728	6.89 -2	1.04
	.1802	6.29 -2	1.10
	.1876	5.09 -2	1.03
	.1926	3.90 -2	0.87
	.2000	3.30 -2	0.85
	.2148	2.70 -2	0.94
	.2222	2.40 -2	0.96
	.2247	2.10 -2	0.89
	.2321	1.50 -2	0.73
	.2617	8.99 -3	0.79
	.3111	3.00 -3	0.70
6,10	.1562	9.59 -2	1.05
	.1594	9.19 -2	1.07
	.1635	8.19 -2	1.03
	.1646	7.79 -2	1.00
	.1677	7.59 -2	1.04
	.1688	7.19 -2	1.01
	.1729	6.79 -2	1.03
	.1760	5.79 -2	0.94
	.1812	5.39 -2	0.97
	.1896	4.80 -2	1.01
	.1927	4.40 -2	0.99
	.1969	4.20 -2	1.02
	.1979	3.80 -2	0.94
	.2052	3.00 -2	0.86
	.2094	2.60 -2	0.81
	.2135	2.20 -2	0.74
	.2312	1.80 -2	0.86
	.2385	1.60 -2	0.89
	.2396	1.40 -2	0.79
	.2479	9.99 -3	0.67
	.2760	5.99 -3	0.70
	.3229	2.00 -3	0.59
6,11	.1569	9.48 -2	1.05
	.1586	8.93 -2	1.02
	.1604	8.65 -2	1.03
	.1622	8.38 -2	1.03
	.1640	7.83 -2	1.00
	.1658	7.28 -2	0.96
	.1676	6.73 -2	0.92
	.1711	6.46 -2	0.95
	.1747	6.18 -2	0.97
	.1765	5.91 -2	0.96
	.1800	5.49 -2	0.96
	.1836	5.36 -2	1.00
	.1854	5.08 -2	0.99
	.1872	4.81 -2	0.97

TABLE 1 - Continued

m, n	U^2	P	R
6,11	.1890	4.53 -2	0.94
	.1925	4.12 -2	0.92
	.1961	3.85 -2	0.92
	.1979	3.57 -2	0.89
	.1996	3.43 -2	0.88
	.2050	3.16 -2	0.90
	.2086	2.88 -2	0.88
	.2121	2.61 -2	0.86
	.2139	2.34 -2	0.80
	.2210	2.06 -2	0.81
	.2246	1.79 -2	0.75
	.2264	1.51 -2	0.66
	.2460	1.24 -2	0.79
	.2531	1.10 -2	0.81
	.2620	6.87 -3	0.61
	.2888	4.12 -3	0.62
	.3333	1.37 -3	0.49
7, 7	.1578	9.79 -2	1.10
	.1695	7.34 -2	1.04
	.1782	5.71 -2	0.96
	.1986	4.90 -2	1.23
	.2015	3.26 -2	0.87
	.2161	2.86 -2	1.02
	.2511	1.22 -2	0.87
	.3036	4.08 -3	0.82
7, 8	.1556	9.56 -2	1.03
	.1579	9.09 -2	1.03
	.1603	8.16 -2	0.97
	.1627	7.93 -2	0.98
	.1651	7.46 -2	0.97
	.1675	7.23 -2	0.99
	.1722	6.76 -2	1.01
	.1770	6.29 -2	1.04
	.1794	5.83 -2	1.00
	.1818	4.90 -2	0.88
	.1913	4.43 -2	0.97
	.1936	3.96 -2	0.91
	.2008	3.50 -2	0.92
	.2079	3.03 -2	0.92
	.2151	2.80 -2	0.98
	.2222	2.33 -2	0.94
	.2246	1.86 -2	0.79
	.2365	1.63 -2	0.87
	.2389	1.17 -2	0.65
	.2722	6.99 -3	0.75
	.3222	2.33 -3	0.67
7, 9	.1560	9.93 -2	1.08
	.1600	9.09 -2	1.07
	.1639	7.97 -2	1.01
	.1659	7.41 -2	0.98
	.1719	7.27 -2	1.08
	.1758	6.43 -2	1.03
	.1798	5.87 -2	1.02
	.1818	4.90 -2	0.89
	.1838	4.62 -2	0.87
	.1878	4.34 -2	0.88
	.1897	4.20 -2	0.89
	.1957	3.92 -2	0.93
	.1996	3.64 -2	0.94
7, 9	.2016	3.36 -2	0.90
	.2036	3.08 -2	0.86
	.2116	2.80 -2	0.91
	.2155	2.52 -2	0.89
	.2195	2.24 -2	0.85
	.2215	1.96 -2	0.78
	.2314	1.68 -2	0.81
	.2433	1.40 -2	0.85
	.2453	1.12 -2	0.71
	.2552	9.79 -3	0.75
	.2592	6.99 -3	0.58
	.2909	4.20 -3	0.65
	.3385	1.40 -3	0.56
7,10	.1546	9.96 -2	1.05
	.1563	9.70 -2	1.06
	.1580	8.92 -2	1.01
	.1597	8.74 -2	1.02
	.1613	8.39 -2	1.01
	.1630	8.04 -2	1.00
	.1647	7.69 -2	0.99
	.1681	7.34 -2	1.01
	.1698	7.17 -2	1.02
	.1714	6.64 -2	0.98
	.1748	6.29 -2	0.99
	.1765	5.94 -2	0.97
	.1798	5.42 -2	0.94
	.1815	5.24 -2	0.94
	.1849	5.07 -2	0.97
	.1866	4.90 -2	0.97
	.1899	4.81 -2	1.02
	.1916	4.46 -2	0.98
	.1933	4.11 -2	0.93
	.1950	3.93 -2	0.92
	.1966	3.76 -2	0.91
	.1983	3.41 -2	0.85
	.2000	3.32 -2	0.86
	.2034	2.97 -2	0.82
	.2067	2.80 -2	0.83
	.2084	2.62 -2	0.80
	.2134	2.54 -2	0.86
	.2185	2.36 -2	0.88
	.2218	2.19 -2	0.87
	.2235	2.01 -2	0.83
	.2269	1.84 -2	0.81
	.2302	1.75 -2	0.82
	.2319	1.57 -2	0.77
	.2353	1.40 -2	0.73
	.2403	1.22 -2	0.70
	.2471	1.05 -2	0.69
	.2622	8.74 -3	0.77
	.2639	6.99 -3	0.64
	.2723	6.12 -3	0.66
	.2773	4.37 -3	0.52
	.3076	2.62 -3	0.57
	.3529	8.74 -4	0.46
8, 8	.1562	9.95 -2	1.09
	.1631	8.95 -2	1.12
	.1680	8.21 -2	1.13
	.1709	6.71 -2	0.98
	.1787	5.72 -2	0.97
8, 8	.1836	4.97 -2	0.93
	.1875	4.72 -2	0.96
	.1943	4.23 -2	0.98
	.1992	3.73 -2	0.95
	.2031	3.23 -2	0.89
	.2178	2.49 -2	0.92
	.2256	1.99 -2	0.85
	.2344	1.74 -2	0.89
	.2412	1.49 -2	0.87
	.2500	9.95 -3	0.69
	.2617	8.70 -3	0.76
	.2959	3.73 -3	0.64
	.3438	1.24 -3	0.55
8, 9	.1552	9.93 -2	1.06
	.1569	9.65 -2	1.07
	.1585	9.37 -2	1.07
	.1601	8.95 -2	1.06
	.1618	8.32 -2	1.01
	.1634	7.90 -2	0.99
	.1650	7.62 -2	0.99
	.1667	7.20 -2	0.97
	.1699	7.06 -2	1.01
	.1716	6.78 -2	1.00
	.1732	6.71 -2	1.03
	.1748	6.57 -2	1.04
	.1765	6.29 -2	1.02
	.1781	6.15 -2	1.04
	.1797	6.01 -2	1.04
	.1814	5.66 -2	1.02
	.1830	5.38 -2	1.00
	.1846	5.24 -2	1.00
	.1863	4.96 -2	0.98
	.1879	4.55 -2	0.93
	.1895	4.41 -2	0.93
	.1912	4.27 -2	0.93
	.1928	3.85 -2	0.86
	.1944	3.71 -2	0.86
	.1961	3.57 -2	0.86
	.1994	3.43 -2	0.88
	.2010	3.29 -2	0.87
	.2026	3.15 -2	0.86
	.2075	3.01 -2	0.90
	.2092	2.87 -2	0.89
	.2108	2.66 -2	0.85
	.2124	2.38 -2	0.79
	.2190	2.31 -2	0.87
	.2239	2.03 -2	0.84
	.2255	1.75 -2	0.75
	.2271	1.61 -2	0.71
	.2402	1.47 -2	0.84
	.2418	1.33 -2	0.79
	.2484	1.19 -2	0.80
	.2582	9.09 -3	0.74
	.2598	8.39 -3	0.71
	.2647	6.99 -3	0.65
	.2729	5.59 -3	0.61
	.2827	4.90 -3	0.65
	.2843	3.50 -3	0.48
	.3170	2.10 -3	0.55
	.3628	6.99 -4	0.45

TABLE 2. Probability P of attaining or exceeding W^2 (selected values).

m, n / W^2	100P	R
4, 5		
0.4037	9.52	1.35
0.4093	7.94	1.16
0.4704	4.76	1.00
4, 6		
0.3833	8.57	1.07
0.4250	7.62	1.23
0.4833	5.71	1.30
0.4917	4.76	1.14
0.5333	3.81	1.16
0.5500	2.86	0.96
4, 7		
0.3766	9.70	1.16
0.4026	7.88	1.11
0.4416	5.45	0.97
0.4968	4.85	1.19
0.5520	3.64	1.23
0.5974	2.42	1.07
0.6169	1.82	0.89
4, 8		
0.4028	8.89	1.25
0.4132	6.87	1.03
0.4653	5.25	1.08
0.4861	4.85	1.12
0.5069	3.64	0.95
0.6111	2.42	1.15
0.6528	1.62	0.97
0.6736	1.21	0.82
4, 9		
0.3697	9.79	1.12
0.4017	7.83	1.10
0.4573	5.59	1.09
0.4722	4.76	1.01
0.4936	3.92	0.95
0.5534	2.80	0.96
0.6090	2.24	1.05
0.6133	1.96	0.94
0.6731	1.40	0.94
0.7222	.839	0.75
4,10		
0.3643	9.99	1.11
0.3929	7.99	1.06
0.4393	5.99	1.05
0.4572	5.00	0.97
0.5072	4.00	1.05
0.5357	3.00	0.93
0.5786	2.40	0.95
0.6036	2.00	0.91
0.6607	1.40	0.88
0.7179	.999	0.87
0.7429	.799	0.80
0.7643	.599	0.67
4,11		
0.3647	9.82	1.09
0.4010	7.91	1.10
0.4359	5.86	1.01
0.4662	4.98	1.03
0.4980	3.96	0.98
0.5525	2.93	1.00
0.5722	2.34	0.89
0.6162	1.90	0.93
0.6480	1.47	0.86
0.7465	.879	0.89
0.7571	.733	0.79
0.7798	.586	0.72
0.8010	.440	0.60
4,12		
0.3750	9.23	1.10
0.3958	7.80	1.05
0.4479	5.82	1.08
0.4687	4.95	1.03
0.5000	3.96	0.99
0.5521	2.64	0.90
0.5937	2.42	1.04
0.6042	1.98	0.90
0.6562	1.43	0.88
0.6979	.989	0.77
0.7396	.769	0.75
0.7917	.549	0.72
0.8125	.440	0.64
0.8333	.330	0.54
4,13		
0.3620	9.92	1.09
0.3880	7.98	1.03
0.4401	5.97	1.05
0.4661	4.96	1.02
0.4989	3.95	0.98
0.5419	2.94	0.94
0.5713	2.44	0.92
0.6210	1.93	0.97
0.6527	1.34	0.81
0.7127	.924	0.78
0.7330	.756	0.71
0.7726	.588	0.69
0.8224	.420	0.65
0.8416	.336	0.58
0.8620	.252	0.48
5, 5		
0.4500	8.73	1.63
0.4900	4.76	1.13
0.5700	3.17	1.19
5, 6		
0.3727	9.96	1.17
0.4000	7.79	1.08
0.4697	5.63	1.18
0.4879	4.76	1.11
0.5455	3.90	1.27
5, 6		
0.5697	2.60	0.98
0.5879	2.16	0.90
0.6576	1.73	1.07
0.6636	1.30	0.83
5, 7		
0.3718	9.85	1.15
0.4099	7.58	1.11
0.4385	5.81	1.01
0.4766	4.80	1.05
0.5337	3.54	1.08
0.5575	2.78	0.97
0.6290	2.27	1.20
0.6337	1.77	0.96
0.6718	1.26	0.85
0.7432	.758	0.76
5, 8		
0.3692	9.63	1.10
0.3942	7.93	1.06
0.4538	5.91	1.13
0.4712	4.97	1.05
0.5135	3.88	1.05
0.5462	2.95	0.97
0.5865	2.49	1.03
0.6231	1.86	0.95
0.7019	1.40	1.11
0.7096	.932	0.77
0.7442	.777	0.78
0.8115	.466	0.68
5, 9		
0.3690	9.69	1.11
0.3976	7.99	1.09
0.4389	5.99	1.05
0.4770	4.90	1.07
0.5182	3.90	1.09
0.5468	3.00	0.99
0.5786	2.50	0.99
0.6008	2.00	0.90
0.6611	1.50	0.95
0.7024	.999	0.79
0.7690	.699	0.81
0.7722	.599	0.70
0.8071	.500	0.71
0.8579	.400	0.75
0.8706	.300	0.60
5,10		
0.3689	9.86	1.13
0.4089	7.66	1.12
0.4422	5.79	1.03
0.4689	4.93	1.03
0.5089	3.73	0.99
0.5622	2.93	1.05
0.6022	2.20	0.99
0.6089	1.86	0.88
0.6489	1.47	0.86
5,10		
0.7222	.999	0.89
0.7489	.799	0.82
0.8222	.599	0.93
0.8289	.400	0.64
0.9089	.266	0.66
0.9222	.200	0.53
5,11		
0.3585	9.94	1.07
0.3926	7.97	1.05
0.4426	6.00	1.07
0.4699	4.95	1.04
0.5017	3.94	1.00
0.5494	2.98	1.00
0.5767	2.47	0.97
0.6153	1.97	0.96
0.6540	1.47	0.89
0.7131	.962	0.81
0.7358	.778	0.75
0.7881	.595	0.76
0.8153	.458	0.68
0.8722	.366	0.74
0.8790	.275	0.58
0.9108	.229	0.57
0.9540	.183	0.58
0.9676	.137	0.47
5,12		
0.3608	9.92	1.08
0.3931	7.85	1.04
0.4402	5.98	1.05
0.4696	4.98	1.05
0.5029	3.94	1.01
0.5441	2.97	0.96
0.5745	2.49	0.96
0.6147	1.94	0.94
0.6480	1.49	0.87
0.7137	.937	0.79
0.7510	.776	0.81
0.7794	.582	0.71
0.8255	.485	0.76
0.8412	.388	0.66
0.9128	.291	0.74
0.9235	.194	0.52
0.9941	.129	0.51
1.0078	.0970	0.41
6, 6		
0.3750	9.31	1.11
0.4306	6.71	1.12
0.4861	5.41	1.25
0.5139	3.90	1.06
0.5972	2.81	1.24
0.6250	1.95	1.00
0.6806	1.30	0.91
0.7639	.866	0.97

TABLE 2 - Continued.

m, n W^2	100P	R
6, 7		
0.3755	9.67	1.15
0.4029	7.93	1.12
0.4451	5.94	1.08
0.4652	4.90	1.00
0.5055	3.96	1.03
0.5550	2.91	1.01
0.5879	2.33	0.97
0.6392	1.98	1.11
0.6685	1.40	0.92
0.7234	.932	0.83
0.7656	.699	0.79
0.7766	.583	0.70
0.8516	.466	0.85
0.8553	.350	0.65
6, 8		
0.3661	9.52	1.07
0.4018	7.86	1.10
0.4434	5.93	1.07
0.4732	4.86	1.04
0.5089	4.00	1.06
0.5417	3.00	0.96
0.5923	2.40	1.02
0.6190	1.93	0.96
0.6786	1.27	0.88
0.7232	.999	0.89
0.7560	.799	0.86
0.7827	.599	0.74
0.8304	.466	0.75
0.8423	.400	0.69
0.9286	.266	0.74
0.9345	.200	0.57
6, 9		
0.3704	9.71	1.12
0.4000	7.95	1.10
0.4370	5.99	1.04
0.4741	4.88	1.05
0.5074	3.84	1.01
0.5518	2.92	0.99
0.5889	2.44	1.02
0.6185	1.92	0.95
0.6741	1.40	0.95
0.7111	.999	0.83
0.7555	.759	0.81
0.8185	.559	0.85
0.8333	.480	0.79
0.8556	.400	0.74
0.9037	.280	0.67
0.9111	.240	0.60
0.9333	.200	0.57
1.0037	.120	0.50
6,10		
0.3646	9.84	1.10
0.3979	7.89	1.08
0.4479	5.82	1.07
0.4687	4.95	1.03

m, n W^2	100P	R
6,10		
0.5104	3.97	1.06
0.5479	2.97	0.99
0.5771	2.47	0.97
0.6146	2.00	0.97
0.6604	1.50	0.94
0.7271	.999	0.91
0.7521	.749	0.79
0.7812	.599	0.74
0.8187	.500	0.76
0.8437	.400	0.69
0.9021	.300	0.72
0.9229	.250	0.67
0.9521	.200	0.63
0.9729	.150	0.53
1.0562	.0999	0.55
1.0646	.0749	0.43
6,11		
0.3609	9.97	1.09
0.3966	8.00	1.08
0.4385	5.96	1.04
0.4661	4.99	1.03
0.5044	3.99	1.03
0.5499	2.96	0.99
0.5766	2.49	0.97
0.6114	1.99	0.95
0.6560	1.49	0.91
0.7264	.986	0.90
0.7522	.792	0.83
0.7950	.598	0.80
0.8146	.485	0.72
0.8476	.388	0.69
0.8957	.275	0.63
0.9198	.242	0.64
0.9581	.194	0.63
0.9875	.145	0.55
1.0285	.0970	0.46
1.1096	.0646	0.47
1.1185	.0485	0.37
7, 7		
0.3826	9.32	1.16
0.4031	7.93	1.12
0.4643	5.59	1.14
0.4847	4.90	1.12
0.5255	3.55	1.04
0.5663	2.97	1.10
0.5867	2.33	0.97
0.6480	1.69	0.99
0.6684	1.46	0.96
0.7704	.816	0.95
0.8112	.524	0.76
0.8520	.408	0.74
0.8724	.350	0.71
0.9541	.233	0.74

m, n W^2	100P	R
7, 8		
0.3615	9.95	1.09
0.3960	7.99	1.08
0.4413	5.91	1.05
0.4794	4.94	1.10
0.5127	3.98	1.08
0.5472	2.98	0.99
0.5794	2.49	0.99
0.6091	1.99	0.94
0.6615	1.49	0.94
0.7270	.963	0.88
0.7698	.777	0.90
0.7984	.591	0.80
0.8448	.497	0.87
0.8651	.373	0.73
0.8936	.280	0.64
0.9079	.249	0.61
0.9591	.186	0.61
1.0413	.124	0.63
1.0436	.0932	0.48
7, 9		
0.3646	9.93	1.11
0.3963	7.94	1.07
0.4420	5.91	1.05
0.4697	5.00	1.05
0.5055	3.95	1.02
0.5570	2.95	1.03
0.5769	2.48	0.97
0.6186	1.98	0.98
0.6622	1.49	0.94
0.7197	.996	0.87
0.7574	.787	0.85
0.8051	.594	0.84
0.8388	.490	0.83
0.8686	.385	0.77
0.8983	.297	0.70
0.9360	.245	0.70
0.9539	.192	0.61
0.9936	.140	0.55
1.0491	.0874	0.46
1.1186	.0699	0.54
1.1225	.0524	0.41
7,10		
0.3597	9.98	1.08
0.3941	7.99	1.07
0.4403	5.99	1.06
0.4723	5.00	1.07
0.5042	3.97	1.02
0.5445	2.99	0.97
0.5815	2.50	1.00
0.6118	1.98	0.95
0.6605	1.47	0.93
0.7218	.987	0.88
0.7555	.792	0.85
0.7941	.596	0.79
0.8235	.494	0.77
0.8672	.391	0.77

m, n W^2	100P	R
7,10		
0.8983	.298	0.70
0.9227	.247	0.66
0.9529	.195	0.62
1.0143	.144	0.63
1.0395	.0926	0.47
1.1008	.0720	0.50
1.1227	.0514	0.40
1.1874	.0411	0.46
1.1924	.0309	0.35
8, 8		
0.3750	9.63	1.14
0.4062	7.63	1.10
0.4531	5.69	1.08
0.4844	4.82	1.10
0.5156	3.88	1.07
0.5625	2.98	1.08
0.6094	2.08	0.98
0.6250	1.94	1.00
0.6719	1.46	0.98
0.7344	.979	0.93
0.7656	.730	0.83
0.8281	.544	0.87
0.8594	.420	0.79
0.8906	.388	0.87
0.9219	.249	0.66
0.9688	.186	0.64
1.0000	.140	0.57
1.0625	.0932	0.53
1.1406	.0622	0.54
8, 9		
0.3611	9.98	1.09
0.3954	7.96	1.07
0.4404	5.97	1.05
0.4722	4.99	1.06
0.5033	3.99	1.02
0.5490	2.99	1.00
0.5833	2.48	1.01
0.6176	2.00	0.99
0.6601	1.50	0.94
0.7230	.995	0.89
0.7467	.798	0.81
0.7949	.584	0.78
0.8317	.494	0.80
0.8619	.395	0.76
0.9134	.296	0.75
0.9281	.247	0.68
0.9698	.197	0.68
1.0033	.148	0.61
1.0605	.0987	0.56
1.0850	.0740	0.48
1.1381	.0576	0.49
1.1495	.0494	0.45
1.2271	.0329	0.45
1.2288	.0247	0.34

Example 2. Let the pooled ordered sample have the form

$$1\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 2\ 2.$$

Here $m = 5$, $n = 6$. Since the sample can be transformed, by one transposition of a pair of adjacent members, into one in which there is no overlap, W^2 has the second largest attainable value, and so also has U^2. Therefore we need not evaluate W^2 or U^2. The formulae of Section 3 yield $P(W^2) = 4/462$ and $P(U^2) = 33/462$. This $P(U^2)$ could also have been read from Table 1.

Example 3. Let the pooled ordered sample have the form

$$1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1.$$

Since this is a cyclic permutation of the sample of Example 2, U^2 has the same value as before and $P(U^2) = 33/462$.

REFERENCES

[1] ANDERSON, T. W. and DARLING, D. A. (1952). Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

[2] ANDERSON, T. W. (1962). On the distribution of the two-sample Cramér-von Mises criterion. *Ann. Math. Statist.* **33** 1148–1159.

[3] BURR, E. J. (1963). Distribution of the two-sample Cramér-von Mises criterion for small equal samples. *Ann. Math. Statist.* **34** 95–101.

[4] WATSON, G. S. (1961). Goodness-of-fit tests on a circle. *Biometrika* **48** 109–114.

[5] WATSON, G. S. (1962). Goodness-of-fit tests on a circle. II. *Biometrika* **49** 57–63.