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A NOTE ON THE POISSON TENDENCY IN TRAFFIC DISTRIBUTION

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In a paper [1] by Leo Breiman it is shown that, under rather weak assumptions, the number of cars in an arbitrary interval I with the length $|I|$ will be asymptotically Poisson distributed with the mean $\sigma|I|$, as the time t tends to infinity. Here σ is a constant. Under the same assumptions as those of Breiman it will here be shown that the cars as the time tends to infinity will be distributed according to a Poisson process with the intensity σ. The assumptions and notations will be the same as those of [1]. By the well known representation of a Poisson process we will formulate our main theorem in the following way:

Theorem. *Let $I_1, I_2, \cdots I_n$ be n disjoint but otherwise arbitrary intervals on the space axis, with their respective lengths $|I_1|\ |I_2|, \cdots |I_n|$. Under the assumptions* (*a*), (*b*) *and* (*c*) *of* [1] *then*

$$\lim_{t\to\infty} P\{N_t(I_\nu) = j_\nu,\ \nu = 1, 2, \cdots n\} = \prod_{\nu=1}^{n} (\lambda_\nu^{j_\nu}/j_\nu!)e^{-\lambda_\nu},$$

where $\lambda_\nu = \sigma|I_\nu|$ and where $N_t(I_\nu)$ is the number of cars at time t in the interval I_ν.

For the proof we need a slight generalization of the theorem of Section 3 in [1]. Consider for every m an infinite sequence of trials $Z_1^{(m)}, Z_2^{(m)}, \cdots$, which are independent for fixed m and result in one of the outcomes "success of type ν" $= S_\nu$, $\nu = 1, 2, \cdots, n$ or failure F with the corresponding probabilities $P(Z_k^{(m)} = S_\nu) = P_{k\nu}^{(m)}$, $\nu = 1, 2, \cdots, n$, and $P(Z_k^{(m)} = F) = 1 - \sum_{\nu=1}^{n} P_{k\nu}^{(m)}$. Let the number of S_ν in the mth sequence be denoted by $N_{\nu m}$.

Lemma. *If*

(i) $\sum_{k=1}^{\infty} P_{k\nu}^{(m)} \to \lambda_\nu$ *as* $m \to \infty$ *for* $\nu = 1, 2, \cdots, n$,

(ii) $\sup_k P_{k\nu}^{(m)} \to 0$ *as* $m \to \infty$ *for* $\nu = 1, 2, \cdots, n$,

then for fixed $j_1, j_2, \cdots j_n$

$$\lim_{m\to\infty} P\{N_{\nu m} = j_\nu;\quad \nu = 1, 2, \cdots, n\} = \prod_{\nu=1}^{n} (\lambda_\nu^{j_\nu}/j_\nu!)e^{-\lambda_\nu}.$$

The lemma is very easily shown by using the technique of generating functions for n-dimensional random variables. For sake of completeness the proof is given in the appendix.

In the proof of the main theorem we have then only to show that

(i) $\lim_{t\to\infty} \sum_{k=1}^{\infty} P\{X_k(t)\ \varepsilon\ I_\nu \mid X_1, X_2, \cdots\} = \lambda_\nu$

(ii) $\lim_{t\to\infty} \sup_k P\{X_k(t)\ \varepsilon\ I_\nu \mid X_1, X_2, \cdots\} = 0$

for $\nu = 1, 2, \cdots, n$.

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The proof of these two relations can be done in exactly the same way as in Section 4 of [1].

APPENDIX

PROOF OF THE LEMMA: Denote the generating function of the random variable $(N_{1m}, N_{2m}, \cdots, N_{nm})$ by $H_m(s_1, s_2, \cdots s_n)$. If we can show that

$$\lim_{m\to\infty} H_m(s_1, s_2, \cdots, s_n) = \exp\left\{-\sum_{\nu=1}^{n} \lambda_\nu(1 - s_\nu)\right\},$$

then the lemma is proved. We have

$$H_m(s_1, s_2, \cdots, s_n) = \prod_{k=1}^{\infty}\left(1 - \sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right)$$

Taking the logarithms we get

$$\log H_m(s_1, s_2, \cdots, s_n) = \sum_{k=1}^{\infty} \log\left(1 - \sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right)$$

For fixed $\epsilon > 0$ and sufficiently large m, $\sum_{\nu=1}^{n} P_{k\nu}^{(m)} < \epsilon$ uniformly in k by Condition (ii). Thus for large m we have

$$\log\left(1 - \sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right) = -\sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu) + \vartheta\left(\sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right)^2$$

where $|\vartheta| \leqq 1$ and

$$\log H_m(s_1, s_2, \cdots, s_n) = -\sum_{\nu=1}^{n}\sum_{k=1}^{\infty} P_{k\nu}^{(m)}(1 - s_\nu) + \vartheta \sum_{k=1}^{\infty}\left(\sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right)^2,$$

where $|\vartheta| \leqq 1$. By Condition (i) the first term in the right member tends to $-\sum_{\nu=1}^{n} \lambda_\nu(1 - s_\nu)$. We now must show that the second term tends to 0.

For any numbers a_ν, such that $0 \leqq a_\nu < \infty$ the following simple inequality holds:

$$\left(\sum_{\nu=1}^{n} a_\nu\right)^2 \leqq n(\sup_{1\leqq\nu\leqq n} a_\nu) \sum_{\nu=1}^{n} a_\nu .$$

Using this inequality we get for $0 \leqq s_\nu \leqq 1$ that

$$\sum_{k=1}^{\infty}\left(\sum_{\nu=1}^{n} P_{k\nu}^{(m)}(1 - s_\nu)\right)^2 \leqq n(\sup_{1\leqq\nu\leqq n} \sup_k P_{k\nu}^{(m)}). \quad \sum_{\nu=1}^{n} \sum_{k=1}^{\infty} P_{k\nu}^{(m)} \to 0$$

by Condition (i) and (ii), as m tends to infinity. Thus the proof is complete.

REFERENCE

[1] BREIMAN, L. (1963). The Poisson tendency in traffic distribution. *Ann. Math. Statist.* **34** 308–311.