A UNIQUENESS THEOREM FOR STATIONARY MEASURES OF ERGODIC MARKOV PROCESSES[1]

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1. Summary. If β_1 and β_2 are not identically zero σ-finite invariant measures for a measurable invertible ergodic transformation S on a measure space, and $\beta_1(E) > 0$ implies $\beta_2(E) > 0$ for measurable sets E, then $\beta_2 = c\beta_1$ for some constant $c \neq 0$ ([4], p. 35). In this paper a corresponding result will be proved for stationary measures of a Markov process (Theorem 1). Theorem 1 is a generalization of the corollary of [6], p. 863. In that paper, the authors impose conditions ensuring that the shift transformation has no wandering sets of positive measure, and then they use Hopf's theorem. In Section 3, some new and known results are seen to follow readily from Theorem 1. The recurrence condition introduced by Harris [5] is discussed, and Theorem 1 is used to give a new proof of the uniqueness theorem of [5] independent of the existence of stationary measures, and generalizing the theorem to σ-fields which are not necessarily separable.

2. Introduction. We consider $(X_n, n \geqq 0)$, a Markov process on (Ω, Σ) having stationary transition probabilities. α is called a *stationary measure* for the process if α is not identically zero, is σ-finite and $\int P(t, \cdot)\alpha(dt) = \alpha(\cdot)$ on Σ. Let Ω_0 be the set of unilateral sequences $\omega = (\omega_0, \omega_1, \cdots)$, $\omega_n \varepsilon \Omega$ and Ω_1 the set of bilateral sequences $\omega = (\cdots \omega_{-1}, \omega_0, \omega_1, \cdots)$. The X_n process may be represented on Ω_0 by giving X_0 the distribution α where α is stationary ([2], p. 190). The stationarity of α will permit the finiteness assumption to be relaxed. See, e.g., [6]. The process is strictly stationary [2] since α is stationary, and the induced measure on Ω_0 cylinders, which we call α_0, can be extended to a σ-field of Ω_0 sets, to be called Σ_0.

The strictly stationary process $(X_n, n \geqq 0)$ may be embedded in a process $(X_n, -\infty < n < \infty)$, the *extended process* ([2], p. 456). The extended process may be represented on Ω_1; the measure induced by α on Ω_1 cylinders will be denoted by α_1 and can be extended to a σ-field of Ω_1 sets denoted by Σ_1. Notice that if $E \subseteq \Omega_1$ is defined only by restrictions on ω_n, $n \geqq 0$, then $\alpha_1(E) = \alpha_0(E)$.

Let T be the shift transformation: if $\omega \varepsilon \Omega_0$ or Ω_1, then $(T\omega)_n = \omega_{n+1}$. Then it is easy to see that α_0 and α_1 defined above are T-invariant: $\alpha_0(T^{-1}E) = \alpha_0(E)$, $\alpha_1(T^{-1}E) = \alpha_1(TE) = \alpha_1(E)$ for $E \varepsilon \Sigma_0$, $E \varepsilon \Sigma_1$ respectively. A stationary α is called *ergodic* for the process $(X_n, n \geqq 0)$ if the σ-field of invariant events is trivial, that is, if $V \varepsilon \Sigma_0$, and $T^{-1}V$ and V differ at most by an α_0-null set, then $\alpha_0(V) = 0$ or $\alpha_0(V') = 0$. If the extended process $(X_n, -\infty < n < \infty)$ is

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considered instead, the definition of ergodicity changes every "0" subscript above to "1". We will make use of the fact ([2], p. 458) that *if* α *is ergodic for the process* $(X_n, n \geqq 0)$, *then* α *is ergodic for the extended process* $(X_n, -\infty < n < \infty)$, *and conversely*. This theorem will be tacitly invoked in the sequel.

For any measures m and n on Σ, $m \gg n$ means that $m(E) > 0$ whenever $n(E) > 0$ on Σ. $m \equiv n$ means $m \gg n$ and $n \gg m$. For any set U in any space, U' is its complement.

Throughout this paper, if α is any stationary measure or n is any finite measure, the subscript "0" appended to α or n always indicates the measure induced by α or n on Σ_0, the subscript "1" appended to α always indicates the measure induced by α on Σ_1, as described above. It will be understood that the subscript "1" refers to the extended process.

3. The uniqueness theorem. In this section α and β are assumed to be stationary measures for the process $(X_n, n \geqq 0)$.

LEMMA 1. *For each* $U \varepsilon \Sigma_0$,

$$\alpha_0(U) = \int P(U \mid X_0 = t)\alpha(dt) \tag{1}$$

where the integrand is the conditional probability of U *starting at* $X_0 = t$.

PROOF. It is easily verified that the integral is countably additive on Σ_0. The proof is concluded by observing that for cylinders the integral *defines* α_0.

LEMMA 2. $\alpha \gg \beta$ *implies* $\alpha_0 \gg \beta_0$.

PROOF. $\alpha_0(U) = 0$ implies by (1) that $P(U \mid X_0 = t) = 0$ a.e. (α) and so $P(U \mid X_0 = t) = 0$ a.e. (β). Hence $\beta_0(U) = 0$.

Notice that $\alpha + \beta$ is also a stationary measure.

LEMMA 3. *If* $\alpha \gg \beta$ *and* α *is ergodic, then* β *and* $\alpha + \beta$ *are ergodic.*

PROOF. Let V and V' be a decomposition of Ω_0 into two invariant sets. Then, say, $\alpha_0(V) = 0$, so by Lemma 2, $\beta_0(V) = 0$. Since $\alpha \gg \alpha + \beta$, the same reasoning shows that $(\alpha + \beta)_0(V) = \alpha_0(V) + \beta_0(V) = 0$.

LEMMA 4. *If* $\alpha \gg \beta$ *and* α *is ergodic, then* $\alpha_1 \gg \beta_1$.

PROOF. Suppose that there exists a set $N \varepsilon \Sigma_1$, $\alpha_1(N) = 0$ and $\beta_1(N) > 0$. Then $\bigcup_{n=-\infty}^{\infty} T^n N = V$ is an invariant set and $\alpha_1(V) = 0$, $\beta_1(V) > 0$. Now, we have that $(\alpha + \beta)_1 = \alpha_1 + \beta_1$ on Σ_1. But then $(\alpha + \beta)_1(V) > 0$ and $(\alpha + \beta)_1(V') > 0$. This contradicts the ergodicity of $\alpha + \beta$ proved by Lemma 3.

THEOREM 1. *If* $\alpha \gg \beta$ *and* α *is ergodic, then* $\beta = c\alpha$ *for some constant* $c > 0$.

PROOF. α_1 and β_1 are invariant under the invertible shift T which, in the language of [4], is ergodic. Moreover $\alpha_1 \gg \beta_1$. We may now use the known result on invariant measures ([4], p. 35) but for completeness we sketch the brief remainder of the proof. By the Radon-Nikodym theorem, the invariance of α_1 and β_1 and a well-known change of variable formula ([7], p. 342) we obtain

$$\beta_1(E) = \int_E f(\omega)\alpha_1(d\omega) = \int_{T^{-1}E} f(T\omega)\alpha_1(d\omega) \tag{2}$$

$$\beta_1(T^{-1}E) = \int_{T^{-1}E} f(\omega)\alpha_1(d\omega). \tag{3}$$

(2) and (3) imply that $f(T\omega) = f(\omega)$ a.e. (α_1) and so f is invariant and by ergodicity reduces to a constant $c > 0$ a.e. (α_1). Thus $\beta_1 = c\alpha_1$ and so $\beta = c\alpha$.

EXAMPLES. Consider the unrestricted random walk on the integers with transition probabilities

$$p_{i,i-1} = \tfrac{1}{3}$$

$$p_{i,i+1} = \tfrac{2}{3}$$

$$p_{ij} = 0 \quad \text{otherwise.}$$

Then $\alpha(\{i\}) = 1$ for all i is a stationary measure as well as $\beta(\{i\}) = 2^i$ for all i. Thus neither α nor β are ergodic and there are invariant sets which are not trivial. This ties in with Blackwell's result ([1], p. 656) that there are non-trivial invariant sets if and only if there is a non-constant bounded solution to the equation: $k_i = \frac{2}{3} k_{i+1} + \frac{1}{3}k_{i-1}$. Since $k_i = 0$ for $i \leqq 0$, and for $i \geqq 1$, $k_i = 2\,(1 - (\frac{1}{2})^i)$ is a bounded non-constant solution, there are non-trivial invariant sets.

If $p_{i,i+1} = 1$ for all i, $p_{ij} = 0$ otherwise, we have an example of a process with a unique stationary measure $\alpha(\{i\}) = 1$ for all i. α is ergodic for the process, but the process is non-recurrent.

4. Applications. To apply Theorem 1, we need to have probabilistic sufficient conditions for ergodicity. "i.o." below means infinitely often. Consider

(C): there exists a σ-finite measure m on Σ such that $m(E) > 0$ implies

$$P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 1 \qquad \text{for all } t \varepsilon \Omega.$$

(C) was introduced in [5] where, in case Σ is separable, it implies the existence and uniqueness of a stationary measure α.

The next theorem is a generalization of Theorem 2 of [6], the proof of which is only roughly sketched there. Our proof follows those ideas.

THEOREM 2. *Let (C) hold, and let α be stationary for the process $(X_n, n \geqq 0)$. Then α is ergodic.*

PROOF. Let $V \subseteq \Omega_0$ be invariant, and let $K = \{t: P(V \mid X_0 = t) > 0\}$. If $m(K) > 0$, we show $P(V \mid X_0 = t) = 1$ for almost every $(m)t \varepsilon K$. For, assume that there exists a set $L \subseteq K$, with $P(V \mid X_0 = t) \leqq \delta < 1$ for $t \varepsilon L$, $m(L) > 0$. Without loss of generality, we may suppose that for some $\epsilon > 0$ we also have $\epsilon \leqq P(V \mid X_0 = t)$ for $t \varepsilon L$. For each fixed $x_0 \varepsilon \Omega$, almost every (with respect to conditional probability starting at x_0) sequence $\omega = (x_0, x_1, x_2, \cdots)$ has infinitely many elements of L appearing, by (C). We then obtain

$$P(V \mid X_0 = x_0, x_1 = X_1, \cdots, X_n = x_n) = P(V \mid X_n = x_n) = P(V \mid X_0 = x_n),$$

using the Markov property and the stationarity of transition probabilities. By the martingale 0-1 law ([2], p. 324) the left-hand side and thus $P(V \mid X_0 = x_n)$ converges to 0 or 1 with probability one as $n \to \infty$. For $x_n \varepsilon L$, we have $P(V \mid X_0 = x_n) \geqq \epsilon$ and this is true for infinitely many n. Thus $\lim_{n\to\infty} P(V \mid X_0 = x_n) = 1$, a contradiction since $P(V \mid X_0 = t) \leqq \delta$ for $t \varepsilon L$. We have therefore shown that $P(V \mid X_0 = t) = 1$ for almost every $t \varepsilon K$. Since $m(K) > 0$, it is easy to see, using (C), that $P(V \mid X_0 = t) = 1$ for *every* $t \varepsilon \Omega$. Thus $\alpha_0(V') = 0$. If, on the contrary, $m(K) = 0$, then a repetition of the foregoing argument for the set $K_1 = \{t : P(V' \mid X_0 = t) > 0\}$ proves that $P(V' \mid X_0 = t) = 1$ for every $t \varepsilon \Omega$, and so $\alpha_0(V) = 0$.

COROLLARY 1. *Assume only that (C) is satisfied for almost all $(m)t$, and that m is stationary. Then m is ergodic.*

PROOF. $m_0(V) = \int P(V \mid X_0 = t)m(dt)$ and the integrand is 0 or 1 a.e. (m) if V is invariant.

Corollary 1 is essentially the result of Theorem 2 of [6].

COROLLARY 2. *Under (C), $P(X_n \varepsilon E$ i.o. $\mid X_0 = t) = 0$ for all t or 1 for all t, for each fixed $E \varepsilon \Sigma$.*

PROOF. The events considered are invariant.

As an application of Theorem 1, we derive a result in [6].

THEOREM 3. *Let α be stationary such that $\alpha(E) > 0$ implies $P(X_n \varepsilon E$ i.o. $\mid X_0 = t) = 1$ a.e. (α), and let $\alpha \gg \beta$ where β is another stationary measure. Then $\beta = c\alpha$ for some constant $c > 0$.*

PROOF. By Corollary 1 above, α is ergodic. Apply Theorem 1.

DEFINITION. If m is any measure on Σ, the process is called *m-irreducible* if $m(E) > 0$ implies $P(X_n \varepsilon E$ for some $n \mid X_0 = t) > 0$ for almost all $(m)t$.

THEOREM 4. *Let α be a finite stationary measure and let the process be α-irreducible. Then α is ergodic.*

PROOF. Let $\alpha(E) > 0$. Suppose there exists a set $F \subseteq E$, $\alpha(F) > 0$ with $P(X_n \varepsilon E$ for at most finitely many $n \mid X_0 = t) > 0$ for each $t \varepsilon F$. Let $B = \{\omega : \omega \varepsilon \Omega_0 ; X_n(\omega) \varepsilon E$ for at most finitely many n; $X_0(\omega) \varepsilon F\}$. Then

$$\alpha_0(B) = \int P(B \mid X_0 = t)\alpha(dt) \geqq \int_F P(B \mid X_0 = t)\alpha(dt > 0.$$

Let $E_0 = \{\omega : \omega \varepsilon \Omega_0 ;\ X_0(\omega) \varepsilon E\}$. Then $\alpha_0(E_0) = \alpha(E) > 0$, and $B \subseteq E_0$, $\alpha_0(B) > 0$, but for each $\omega \varepsilon B$, $T^n\omega \varepsilon E_0$ for at most finitely many n. This contradicts the strong recurrence theorem of ergodic theory ([4], p. 10). Thus we have shown that if $\alpha(E) > 0$, $P(X_n \varepsilon E$ i.o. $\mid X_0 = t) = 1$ for almost every $(\alpha)t \varepsilon E$. Let $\tilde{E} = \{t : P(X_n \varepsilon E$ i.o. $\mid X_0 = t) = 1\}$. $\alpha(\tilde{E}) \geqq \alpha(E) > 0$. Notice also that $\tilde{E}$ is a closed set, that is, $P(t, \tilde{E}) = 1$ for every $t \varepsilon \tilde{E}$. The assumption of α-irreducibility proves that $\alpha(\tilde{E}) = 1$. Indeed, if $\alpha(\tilde{E}') > 0$, we must have $P(X_n \varepsilon \tilde{E}'$ for some $n \mid X_0 = t) > 0$ for some $t \varepsilon \tilde{E}$, a contradiction because $\tilde{E}$ is closed. By Corollary 1, α is ergodic.

The next sequence of results will investigate condition (C) which will be assumed henceforth. Theorem 7 generalizes the uniqueness result of [5]. The proof of uniqueness in [5] as well as the indication that Theorem 6 holds in separable σ-fields used the existence theorem and the "process on A" approach whereas the proofs given here use neither.

Without loss of generality, we may assume that the measure m in (C) is a probability measure, since, given a σ-finite measure, a finite measure with the same sets of measure zero may always be constructed. Define $(S^k m)(E) = \int P^k(t, E)\, m(dt)$, $k \geqq 1$, and $(S^0 m)(E) = m(E)$.

THEOREM 5. *$P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 0$ for all t or $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 1$ for all t according as $(S^k m)(E) = 0$ for all $k \geqq 0$ or $(S^k m)(E) > 0$ for some $k \geqq 0$.*

PROOF. If $(S^k m)(E) > 0$ for some k, then $P^k(t, E) \geqq \epsilon$, say, on the set K, $m(K) > 0$. By (C), $P(X_n \varepsilon K \text{ i.o.} \mid X_0 = t) = 1$ for all $t \varepsilon \Omega$, and it then easily follows that $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 1$ for all $t \varepsilon \Omega$. If $(S^k m)(E) = 0$ for all k, $P^k(t, E) = 0$ a.e. (m) for each k and $P(X_n \text{ is never in } E \mid X_0 = t) = 1$ a.e. (m). By the preceding Corollary 2, $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 0$ for all t.

Let us notice that $\alpha \gg m$ for any stationary α, since $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 1$ for all t if $m(E) > 0$, whereas if $\alpha(E) = 0$, $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 0$ a.e. (α). Let $\{a_n\}$ be a sequence of positive numbers with sum 1 and define $n(\cdot) = \sum_{k=0}^{\infty} a_k (S^k m)(\cdot)$ on Σ. Then, by the preceding theorem, $n(E) = 0$ or $n(E) > 0$ according as $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 0$ or 1. Also remark that $\alpha \gg n$ if α is any stationary measure.

THEOREM 6. *Let α be stationary. Then $\alpha(E) > 0$ implies $P(X_n \varepsilon E \text{ i.o.} \mid X_0 = t) = 1$ for all t.*

PROOF. Consider the extended process $(X_n, -\infty < n < \infty)$. Let $V \subseteq \Omega_1$, $V = (X_n \varepsilon E' \text{ for all } n)$. V is invariant and so differs from the event $W = (X_n \varepsilon E' \text{ for all } n \geqq 0)$ by an α_1-null event ([2], p. 459). By the remarks about n, we may assume that $n(E) = 0$, otherwise the conclusion of the theorem follows. Theorem 5 then makes it clear that $n_0(W) = 1$. Hence, since $\alpha \gg n$, as in Lemma 2 it is clear that $\alpha_0 \gg n_0$, so $\alpha_0(W) > 0$. Since W depends only on $n \geqq 0$, $\alpha_0(W) = \alpha_1(W) > 0$. This proves that $\alpha_1(V) > 0$ and by ergodicity $\alpha_1(V') = 0$. Thus $\alpha_0(W') = 0$. But since $W' = (X_n \varepsilon E \text{ for some } n \geqq 0)$, $\alpha_0(W') \geqq \alpha(E) > 0$, a contradiction. Thus the hypothesis that $n(E) = 0$ must be rejected and the theorem is proved.

COROLLARY 1. *For any stationary α, $\alpha \equiv n$.*

COROLLARY 2. *If α and β are stationary, then $\alpha \equiv \beta$.*

THEOREM 7. *There exists at most one stationary measure α for the process.*

PROOF. By Theorem 2 any stationary measure α is ergodic. If β is another stationary measure, $\alpha \equiv \beta$ by Corollary 2 above, and, *a fortiori*, $\alpha \gg \beta$. Now apply Theorem 1.

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