ON ASYMPTOTIC MOMENTS OF EXTREME STATISTICS

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1. Introduction. Let Y_n be the maximum (or minimum) of n independent observations of a random variable X. On pp. 87–89 of his book, Gumbel [1] stated that it was still not known for general cases how the mean and standard deviation of Y_n depend on the sample size n and that such knowledge would have important applications. A later paper by Sen [2] contains some results of this type, but only for cases in which the distribution of X has a finite upper (or lower) end point.

A method for establishing the asymptotic behavior (as $n \to \infty$) of moments of Y_n is illustrated in this paper. It also implicitly provides formulas that can be used to compute lower and upper bounds on these moments, for finite values of n. The method involves the introduction of an auxiliary variable y and a double limit, first on n and then on y.

Of the three general cases treated here, only one requires that the distribution of X have a finite upper end point. Although none of them permit X to have a normal or general gamma distribution, the author believes and hopes to show later that some of the results can be extended to include these important cases. The results are presented in the next section and are proved in the last section.

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2. The results. Let F be the right-continuous probability distribution function of X, and let a_2 be the value, either finite or $+\infty$, for which $F(a_2) = 1$ and $F(x) < 1$ for $x < a_2$. For simplicity, we assume throughout that F has a continuous derivative f on some open interval (a_1, a_2). No detailed knowledge of $F(x)$ for $x < a_1 < a_2$ is required, since the asymptotic behavior of a moment of Y_n (if it exists) depends only on the properties of F in a neighborhood of a_2.

Hereafter, μ_n and σ_n^2 denote the mean and variance of Y_n, and $\lambda_k = E(|X - a|^k)$, where a is a constant to be specified. As usual, E and Γ denote the expectation operator and the gamma function. Of course, $x \to \alpha-$ indicates that x approaches α always with $x < \alpha$. Also, $g(x) \sim h(x)$ as $x \to \alpha-$, with finite α or $\alpha- = +\infty$, signifies that $g(x)/h(x)$ possesses a limit as $x \to \alpha-$ and that this limit is 1. Similarly, $A_n \sim B_n$ signifies that A_n/B_n possesses the limit 1 as $n \to \infty$.

Theorem 1. *If there are real constants a, $b > 0$, and $c > 0$ such that $F(a) = 1$, $F(x) < 1$ for $x < a$, and, as $x \to a-$,*

$$1 - F(x) \sim b(a - x)^c, \tag{2.1}$$

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$$f(x) \sim bc(a - x)^{c-1}, \tag{2 2}$$

then

$$E[(a - Y_n)^k] \sim (bn)^{-k/c}\Gamma[1 + (k/c)]; \qquad (k > -c, \lambda_k < \infty), \tag{2.3}$$

whence

$$\sigma_n^2 \sim (bn)^{-2/c}\{\Gamma[1 + (2/c)] - \Gamma^2[1 + (1/c)]\}; \qquad (\lambda_2 < \infty). \tag{2.4}$$

THEOREM 2. *If there are real constants a, $b > 0$, and $c > 0$ such that $F(x) < 1$ for $x < \infty$ and, as $x \to \infty$,*

$$1 - F(x) \sim b(x - a)^{-c}, \tag{2.5}$$

$$f(x) \sim bc(x - a)^{-c-1}, \tag{2.6}$$

then

$$E[(Y_n - a)^k] \sim (bn)^{k/c}\Gamma[1 - (k/c)]; \qquad (k < c, \lambda_k < \infty), \tag{2.7}$$

whence

$$\sigma_n^2 \sim (bn)^{2/c}\{\Gamma[1 - (2/c)] - \Gamma^2[1 - (1/c)]\}; \qquad (c > 2, \lambda_2 < \infty). \tag{2.8}$$

THEOREM 3. *If there are real constants a, $b > 0$, $c > 0$, and $r > 0$ such that $F(x) < 1$ for $x < \infty$ and, as $x \to \infty$,*

$$1 - F(x) \sim r \exp[-b(x - a)^c], \tag{2.9}$$

$$f(x) \sim bcr(x - a)^{c-1} \exp[-b(x - a)^c], \tag{2.10}$$

then

$$E[(Y_n - a)^k] \sim (b^{-1} \log rn)^{k/c}; \qquad (\lambda_k < \infty). \tag{2.11}$$

The constant a obviously has different meanings in these theorems; it is an upper bound in Theorem 1, while it may be related to a lower limit in the other two theorems. Although a could be replaced by zero with no real loss in generality, it is worthwhile to retain it and to see clearly that the variances (2.4), (2.8), and (2.15) are independent of this location parameter.

When $k > 0$, it is noteworthy that the moments of (2.3) and (2.4) approach zero, while those of (2.7), (2.8), and (2.11) approach infinity. Of course, with $k = 1$, relations (2.3), (2.7), and (2.11) provide the behavior of the mean μ_n.

We remark that the random variable X of Theorem 1 has the limited type and that X of Theorem 2 has the Cauchy type of initial distributions. When $c = 1$, X of Theorem 3 may have any initial distribution of the exponential type that satisfies (2.9); but this requirement excludes most important distributions of this type, namely, normal and gamma distributions. For Theorems 1 and 2 and for Theorem 3 when $c = 1$, the initial distribution is stable with respect to the largest value Y_n, but is not stable for Theorem 3 when $c \neq 1$.

If the characteristic largest value u_n, defined by

$$1 - F(u_n) = 1/n.$$

is introduced, then (2.3), (2.7), and (2.11) can be restated as

$$E[(a - Y_n)^k] \sim (a - u_n)^k \Gamma[1 + (k/c)] \qquad \text{as } u_n \to a-, \tag{2.12}$$

$$E[(Y_n - a)^k] \sim (u_n - a)^k \Gamma[1 - (k/c)] \qquad \text{as } u_n \to \infty, \tag{2.13}$$

$$E[(Y_n - a)^k] \sim (u_n - a)^k \qquad \text{as } u_n \to \infty. \tag{2.14}$$

Reference to pp. 281 and 264 of [1] will show that relations (2.12) and (2.13) are closely related to known results for the asymptotic moments of order k for the limited type and the Cauchy type of initial distributions. But the author believes that (2.14) is new and interesting; it implies that $\mu_n \sim u_n$, independently of a, b, c, and r, which is also a special case of (2.12) with $c = 1$.

If A_{nk} denotes the right side of (2.3), then $A_{n2} - A_{n1}^2 > 0$. Because of this, (2.4) follows at once from (2.3). A similar remark applies to (2.7) and (2.8).

However, for (2.11), $A_{n2} - A_{n1}^2 = 0$; and therefore a more complex procedure will be required in this case to obtain the asymptotic behavior of σ_n. In fact, this behavior has not been established, although some attempts have suggested the *conjecture* that

$$\sigma_n^2 \sim (\tfrac{1}{6})(\pi/bc)^2(b^{-1}\log rn)^{(2/c)-2}; \qquad (\lambda_2 < \infty), \tag{2.15}$$

when at least (2.9) holds. This relation is especially interesting because it suggests that even the qualitative behavior of σ_n depends critically on c. Thus, when (2.15) applies, σ_n approaches ∞, $\pi/6^{\frac{1}{2}}b$, or 0 when, respectively, $c < 1$, $c = 1$, or $c > 1$.

To facilitate a definitive investigation of the validity of (2.15) by some other author as well as to provide some insight into the limitations of the method of this paper, a proof is given at the end of Section 3 that (2.11) and (2.15) are satisfied when

$$F(x) = \exp(-r \exp[-b(x - a)^c]) \tag{2.16}$$

for all large x. The author has also shown that (2.11) and (2.15) are satisfied by $F(x) = 1 - \exp[-b(x - a)^2]$, $(x \geqq a)$. Finally, note that the exponential and logistic distributions satisfy (2.9) and (2.10) and that the results on pp. 116 and 128 of [1] show that (2.15) reduces to the correct expressions for these cases.

3. Proofs. Since only obvious modifications of the proof of (2.7) are required to obtain a proof of (2.3), the latter proof is left for the reader.

To prove (2.7), first choose any $x_0 \geqq a$ such that F has a continuous derivative f on $x_0 \leqq x < \infty$. For each $y \geqq x_0$, let

$$J_{nk}(y) = n \int_y^\infty (x - a)^k F^{n-1}(x) f(x)\, dx, \tag{3.1}$$

if the integral exists. Then, since the distribution function of Y_n is F^n,

$$E[(Y_n - a)^k] = J_{nk}(y) + n \int_{-\infty}^{y} (x - a)^k F^{n-1}(x)\, dF(x),$$

whence

$$|E[(Y_n - a)^k] - J_{nk}(y)| \leqq \lambda_k n F^{n-1}(y), \tag{3.2}$$

where this bound is finite if and only if $\lambda_k < \infty$.

To proceed toward certain upper and lower bounds on $J_{nk}(y)$, first let $G(x)$ and $g(x)$ denote the right sides of (2.5) and (2.6), and define functions u and v on $x_0 \leqq x < \infty$ by

$$\begin{aligned} u(x) &= [1 - F(x)]/G(x), \\ v(x) &= f(x)/g(x). \end{aligned} \tag{3.3}$$

For each $y \geqq x_0$, let

$$\begin{aligned} \alpha_1(y) &= \max\,[u(x)/F(x)], \qquad & \alpha_2(y) &= \min u(x), \\ \beta_1(y) &= \min v(x), & \beta_2(y) &= \max v(x), \end{aligned} \tag{3.4}$$

where the maxima and minima are taken over all x on $y \leqq x \leqq \infty$, with $u(\infty) = v(\infty) = F(\infty) = 1$. Then, (2.5) and (2.6) imply that

$$\alpha_i(y) \to 1 \quad \text{and} \quad \beta_i(y) \to 1 \quad \text{as} \quad y \to \infty \qquad (i = 1, 2). \tag{3.5}$$

A short computation will now show that (3.1), (3.3), (3.4), and the well-known relation $e^{-t/(1-t)} \leqq 1 - t \leqq e^{-t}$, $(0 \leqq t \leqq 1)$ imply that

$$J_{nk1}(y) \leqq J_{nk}(y) \leqq J_{nk2}(y), \tag{3.6}$$

where

$$\begin{aligned} J_{nki}(y) &= \int_y^{\infty} (x - a)^k h_{ni}(x, y)\, dx \qquad (i = 1, 2), \\ h_{ni}(x, y) &= \beta_i(y) n g(x) \exp\,[-\alpha_i(y)(n - 1)G(x)]. \end{aligned} \tag{3.7}$$

From (3.2) and (3.6),

$$J_{nk1}(y) - \lambda_k n F^{n-1}(y) \leqq E[(Y_n - a)^k] \leqq J_{nk2}(y) + \lambda_k n F^{n-1}(y). \tag{3.8}$$

To appraise (3.7), choose any $y_0 \geqq x_0$ such that $0 < F(y_0) < 1$ and $\alpha_2(y) > 0$ and $\beta_1(y) > 0$ on $y_0 \leqq y < \infty$. Then, by setting $\int_y^\infty = \int_a^\infty - \int_a^y$ and by using properties of the gamma function, we find that, if $k < c$, $y \geqq y_0$, and $n > 2$,

$$J_{nki}(y) = A_{nk} \gamma_{ki}(y) P_{nk}[1 - Q_{nki}(y)], \tag{3.9}$$

where

$$
\begin{aligned}
A_{nk} &= (bn)^{k/c}\Gamma(1 - k/c), \\
\gamma_{ki}(y) &= \beta_i(y)[\alpha_i(y)]^{-(1-k/c)}, \\
P_{nk} &= [n/(n - 1)]^{1-k/c}, \\
Q_{nki}(y) &= [A_{nk}\gamma_{ki}(y)P_{nk}]^{-1}\int_a^y (x - a)^k h_{ni}(x, y)\,dx \\
&\qquad\qquad \leqq (n - 1)^{1-k/c} \exp[-(n - 2)\delta_i(y)], \\
\delta_i(y) &= \alpha_i(y)b(y - a)^{-c}.
\end{aligned}
$$

Now, put (3.9) into (3.8), divide the result by A_{nk}, and let $n \to \infty$. Since $k < c$, $\lambda_k < \infty$, $Q_{nki}(y) > 0$, and $F(y) < 1$ and $\delta_i(y) > 0$ for $y < \infty$, even though $F(\infty) = 1$ and $\delta_i(\infty) = 0$, we see that $Q_{nki}(y) \to 0$ and $A_{nk}^{-1}\lambda_k nF^{n-1}(y) \to 0$ as $n \to \infty$, for $y_0 \leqq y < \infty$. Since also $P_{nk} \to 1$,

$$\gamma_{k1}(y) \leqq \lim_{n\to\infty} A_{nk}^{-1}E[(Y_n - a)^k] \leqq \gamma_{k2}(y). \tag{3.10}$$

Letting $y \to \infty$ in (3.10) finally yields (2.7), because (3.5) implies that $\gamma_{ki}(y) \to 1$.

To prove (2.11), we first proceed exactly as before to derive relations (3.1) through (3.8) and also choose the constant y_0 as before. For $y \geqq y_0$, we now have $h_{ni}(x, y) = \beta_i(y)ng(x)\exp[-z_{ni}(x, y)]$, where

$$z_{ni}(x, y) = \alpha_i(y)r(n - 1)\exp[-b(x - a)^c] \tag{3.11}$$

and $g(x)$ is the right side of (2.10). If the variable x in the integrand of (3.7) is replaced by the new variable z given by the right side of (3.11), we find that, for every real k,

$$J_{nki}(y) = A_{nk}\gamma_i(y)P_{nki}(y)K_{nki}(y), \tag{3.12}$$

where

$$
\begin{aligned}
A_{nk} &= (b^{-1}\log rn)^{k/c}, \\
\gamma_i(y) &= \beta_i(y)/\alpha_i(y), \\
P_{nki}(y) &= (n/(n - 1))(\log q_{ni}(y)/\log rn)^{k/c}, \\
K_{nki}(y) &= \int_0^{z_{ni}(y)} [1 - (\log z/\log q_{ni}(y))]^{k/c}e^{-z}\,dz, \\
q_{ni}(y) &= \alpha_i(y)r(n - 1), \\
z_{ni}(y) &= z_{ni}(y, y).
\end{aligned} \tag{3.13}
$$

The desired result (2.11) follows *via* relation (3.10), with γ_{ki} replaced by γ_i, from (3.8) and (3.12), by the same general argument used for (2.7). The only new non-trivial fact required is that, for large finite values of y,

$$K_{nki}(y) \to 1 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

To prove (3.14), we first let

$$(3.15)\qquad w_{ni}(z, y) = \log z/\log q_{ni}(y),$$

and substitute the finite Taylor series

$$(1 - w)^{k/c} = \sum_{j=0}^{m} \binom{k/c}{j} (-1)^j w^j + R_m(w)$$

into (3.13) to find that

$$(3.16)\quad K_{nki}(y) = \sum_{j=0}^{m} \binom{k/c}{j} (-1)^j \Gamma^{(j)}(1) [\log q_{ni}(y)]^{-j} - L_{nkim}(y) + M_{nkim}(y),$$

where $\Gamma^{(j)}(1) = \int_0^\infty (\log z)^j e^{-z}\, dz$,

$$(3.17)\quad L_{nkim}(y) = \sum_{j=0}^{m} \binom{k/c}{j} (-1)^j [\log q_{ni}(y)]^{-j} \int_{z_{ni}(y)}^{\infty} (\log z)^j e^{-z}\, dz,$$

$$(3.18)\quad M_{nkim}(y) = \int_0^{z_{ni}(y)} R_m(w_{ni}(z, y)) e^{-z}\, dz.$$

In (3.16), $\Gamma^{(j)}(1)$ denotes the jth derivative of the gamma function evaluated at 1 and equals 1, $-C$, $C^2 + \pi^2/6$ when $j = 0, 1, 2$, where C is Euler's constant.

Because $q_{ni}(y) \to \infty$ as $n \to \infty$ for $y \geqq y_0$, the finite series in (3.16) approaches 1, for every integer $m \geqq 0$. Therefore, (3.14) will be proved when we show that, for large finite y,

$$(3.19)\qquad L_{nkim}(y) \to 0 \quad \text{and} \quad M_{nkim}(y) \to 0 \qquad \text{as } n \to \infty.$$

To prove (3.19), we hereafter let $y \geqq y_0$ have any fixed finite value and let $n \geqq n_0(y)$, where $n_0(y)$ is any fixed integer such that $z_{ni}(y) > 1$ for all $n \geqq n_0(y)$ and $i = 1, 2$. Then, because $z_{ni}(y) \to \infty$ as $n \to \infty$, the first part of (3.19) follows at once from the following appraisal of the integral in (3.17):

$$\int_{z_{ni}(y)}^{\infty} (\log z)^j e^{-z}\, dz \leqq e^{-\frac{1}{2}z_{ni}(y)} \int_0^\infty z^j e^{-z/2}\, dz = j!\, 2^{j+1} e^{-\frac{1}{2}z_{ni}(y)}.$$

If k/c equals a non-negative integer, we can set $m = k/c$; then $R_m(w)$ and $M_{nkim}(y)$ are identically zero, and (3.19) has been proved. Otherwise, we use in (3.18) Cauchy's form of the remainder, viz.,

$$R_m(w) = \binom{k/c}{m+1} (m + 1)(-1)^{m+1} w^{m+1} [(1 - \xi/w)/(1 - \xi)]^m (1 - \xi)^{(k/c)-1},$$

in which ξ is some number between 0 and w. To bound this remainder, we first note that, as z in (3.18) varies from its lower to its upper limit, w of (3.15) varies from $-\infty$ to the positive value

$$(3.20)\quad w_{ni}(y) = \log z_{ni}(y)/\log q_{ni}(y) = 1 - [b(y - a)^c/\log q_{ni}(y)].$$

Since $w_{ni}(y) < 1$, an examination of the two cases in which $-\infty < w \leqq \xi \leqq 0$ and $0 \leqq \xi \leqq w \leqq w_{ni}(y)$ shows that

$$|R_m(w)| \leqq C_{km}|w|^{m+1}|1 - \xi|^{(k/c)-1} \tag{3.21}$$

for $-\infty < w \leqq w_{ni}(y)$, where $C_{km} = (m + 1)\left|\binom{k/c}{m+1}\right|$.

The appraisal of (3.18) must now be divided into two cases, depending on whether k/c is smaller or larger than 1. If $-\infty < k/c < 1$, $|1 - \xi|^{(k/c)-1} \leqq [1 - w_{ni}(y)]^{-(1-k/c)}$ for $-\infty < w \leqq w_{ni}(y)$. Substitution of this result, (3.15), (3.20), (3.21), and the fact that

$$\int_0^{z_{ni}(y)} |\log z|^{m+1}e^{-z}\,dz \leqq \int_0^1 (-\log z)^{m+1}\,dz + \int_0^\infty z^{m+1}e^{-z}\,dz = 2(m+1)! \tag{3.22}$$

into (3.18) shows that, if $k/c < 1$,

$$|M_{nkim}(y)| \leqq 2C_{km}(m+1)![b(y-a)^c]^{-(1-k/c)}[\log q_{ni}(y)]^{-(m+k/c)},$$

which $\to 0$ as $n \to \infty$, if $m + k/c > 0$. If $k < 0$, note that m must be chosen large enough so that $m > -k/c$.

If $k/c > 1$, we note that $(1 - \xi)^{(k/c)-1} \leqq 1$ when $0 \leqq \xi \leqq w < 1$. When $w \leqq \xi \leqq 0$, we let h be the smallest integer such that $h \geqq (k/c) - 1$, and note that

$$(1-\xi)^{(k/c)-1} \leqq (1-w)^{(k/c)-1} \leqq \sum_{j=0}^{h} \binom{h}{j}(-w)^j.$$

Putting these facts, (3.15), (3.21), and (3.22) into (3.18) shows that, if $k/c > 1$,

$$|M_{nkim}(y)| \leqq \left(\int_1^{z_{ni}(y)} + \int_0^1\right)|R_m(w_{ni}(z, y))|e^{-z}\,dz$$

$$\leqq C_{km}\,[\log q_{ni}(y)]^{-(m+1)}\left\{(m+1)! + \sum_{j=0}^{h}\binom{h}{j}(m+j+1)!\,[\log q_{ni}(y)]^{-j}\right\},$$

which $\to 0$ as $n \to \infty$.

Finally, we indicate how (2.15) can be obtained when F has the form of (2.16), which clearly satisfies the hypotheses and therefore the conclusion of Theorem 3. In this case, evaluate $J_{nk}(y)$ of (3.2) by direct substitution into (3.1). The result is (3.12), (3.13), and (3.16) through (3.18) with $\gamma_i(y)$, $P_{nki}(y)$, $q_{ni}(y)$, and $z_{ni}(y)$ replaced respectively by 1, 1, rn, and rn $\exp[-b(x-a)^c]$. Since y now appears only in the error terms (3.17) and (3.18), the series in (3.16) with $m \geqq 2$ can be used to show that $J_{n2}(y) - J_{n1}^2(y)$ equals the right side of (2.15) plus other terms of no relative importance for large n. If $\lambda_2 < \infty$, then (2.15) follows from this result and (3.2). Unfortunately, the method of this paper does not seem to yield (2.15) in such a simple manner from merely the hypotheses of Theorem 3, because of complications caused by $\gamma_i(y)$ and $P_{nki}(y)$ in (3.12).

REFERENCES

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