BAYES SOLUTION OF SEQUENTIAL DECISION PROBLEM FOR MARKOV DEPENDENT OBSERVATIONS

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1. Summary. After discussing the sequential decision problem in the general case and its Bayes solution as given by Wald [11], LeCam [8] and others, in Section 2, we give the integral equation satisfied by the Bayes risk. In Section 3 we specialise to Markov dependent observations and in the last section we give an illustrative example.

2. Bayes solution in the general case.

(i) *Formulation.* Let $X = \{X_i\}$, $i = 1, 2, \cdots$ be a set of random elements, not necessarily real or vector-valued. Let $\mathfrak{X}$ be the space of values of X and $\mathfrak{A}$ be a σ-field on $\mathfrak{X}$ with respect to which all the X_i's are measurable. Let Ω be an arbitrary set of parameters, to each element ω of which there corresponds a probability distribution P_ω on $\mathfrak{A}$.

If $X_{i_1}, X_{i_2}, \cdots, X_{i_k}$ are observed in this order, we will say that $\lambda = \{i_1, i_2, \cdots, i_k\}$ is observed. Let $\{X_{i_1}, X_{i_2}, \cdots, X_{i_k}\}$ induce a σ-field $\mathfrak{A}_\lambda \subset \mathfrak{A}$ on $\mathfrak{X}$ and $\{X_0, X_1, \cdots, X_n\}$ induce $\mathfrak{A}_n$. At a certain stage λ of experimentation (including the start) one has to choose an element among a set Δ_λ of all actions available to the statistician after observing λ—either stopping experimentation and taking a terminal decision or continuing observation or taking some other action.

The above formulation is essentially due to Wald [10], [11]. LeCam [8] assumes that $\Delta_\lambda = T_\lambda \cup J_\lambda$ where T_λ is the set of terminal decisions after observing λ and J_λ, the set of decisions on how to continue experimentation. J_λ is associated with the set of indices of variables which could be observed in the next step. If J_λ consists of only one or no index, then we have the usual case of sequential analysis considered by Wald [11], Wald and Wolfowitz [12] and others. There is no loss of generality in this assumption if J_λ is the only set of indices of variables which will be observed in the next step.

The cost of observing λ is a non-negative $\mathfrak{A}_\lambda$-measurable function $c(\lambda, x, \omega)$ and if the terminal decision t is accepted by observing λ, the total amount paid is

$$c(\lambda, x, \omega) + W(\omega, t, \lambda)h(\lambda, x, \omega)$$

where W and h are extended numerical functions satisfying some assumptions. The formulation given above will include a situation discussed in [4] in connection with the inventory problem if the future joint distribution and loss are permitted to depend on the decision at the λth stage.

(ii) *Existence and completeness theorems.* Under certain assumptions on X,

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Δ_λ, cost and loss functions given below, LeCam [8] has proved the existence of Bayes solutions in the wide sense and under some more restrictions, Bayes solutions in the strict sense and has proved the completeness of the class of these Bayes solutions. In the former case ϵ-Bayes solutions will exist (for definition cf. Blackwell and Girshick [2]). Restrictions which he puts on X and cost function are (i) the set of variables which can be observed in not more than n steps is finite, (ii) for every λ, there exists a σ-finite measure μ_λ on $\mathcal{A}_\lambda$, such that whatever be $\omega \varepsilon \Omega$, P_ω is absolutely continuous with respect to μ_λ and (iii) $c(\lambda', x, \omega) \leqq c(\lambda, x, \omega)$ if λ' is an initial segment of λ and $c(\lambda, x, \omega) \to \infty$ as the number of co-ordinates of $\lambda \to \infty$. The restrictions on Δ_λ are of topological nature which ensure the existence of a measurable Bayes solution. They are, (iv) T_λ is metrisable and is a countable union of compact spaces and (v) Bayes risk after the stage λ is a $\mathcal{A}_\lambda$-measurable function. Assumption (i) is milder than the corresponding one imposed by Wald [11] and Kiefer [6], [7], who also have proved the existence and completeness theorems. Kiefer [6] establishes the existence of measurable Bayes solution in the stationary case, by first restricting himself to non-randomized decisions and then mixing with 'care'.

Wald [11] has proved the completeness theorems, under the restriction that the risk functions of the procedures is bounded. This restriction is removed by Ghosh [5] and Kiefer [7]. LeCam's conditions given above are milder than those considered by other authors. Evidently the above complete classes will be minimal complete, if they are admissible, which would be so, if for example Ω is compact and there exists a unique Bayes procedure associated with a given *a priori* probability distribution on Ω (cf. Wald [11]). Moreover, if the class of ϵ-Bayes solutions is complete, which would be so, if the class of Bayes solutions in the wide sense is complete, the class of Bayes solutions is ϵ-complete (cf. Wolfowitz [13]) and conversely.

(iii) *The integral equation.* Henceforward in this paper we shall assume that observations are taken sequentially one at a time. Let r_n be the minimum (Bayes) risk (assumed to exist) of terminating the procedure at the nth stage. We assume that r_n is $\mathcal{A}_n$-measurable, which will be so under the assumptions given in (ii). Let c_n be the cost of observing the first n variables and $z_n = r_n + c_n$. Then $\{z_n, \mathcal{A}_n, n \geqq 1\}$ is a stochastic process. Following the notion of a maximal regular generalised semi-martingale relative to a stochastic process $\{z_n\}$, Snell [9] proved that under some mild restrictions on z_n (which are satisfied in our case), such a semi-martingale $\{y_n\}$ exists and gives us the ϵ-Bayes solution, if we terminate at the mth stage, where

$$
\begin{aligned}
m &= 1 \quad \text{if} \quad y_1 = \infty \\
&= j \quad \text{if} \quad y_k < z_k - \epsilon \quad \text{for} \quad k < j \quad \text{and} \quad y_j \geqq z_j - \epsilon \qquad (1)\\
&= \infty \text{ if undefined by above.}
\end{aligned}
$$

Hence, if $\lim z_n = \infty$ a.s. a Bayes solution in the strict sense will exist.

Snell's [9] paper is an extension of a paper by Arrow, Blackwell and Girshick

[1] and of Wald and Wolfowitz [12] (cf. [11]). These papers give us a practical method of obtaining the Bayes (or ϵ-Bayes) solution in particular cases, at least in principle. Moreover, if y_n is the Bayes risk after n observations, then for all n,

$$y_n = \min(z_n , E(y_{n+1} \mid \mathcal{A}_n)) \text{ a.s.} \tag{2}$$

This integral equation is of great importance in the theory of sequential analysis and has been proved by truncation and passage to the limit in [1] and [12] and directly by Snell [9]. Evidently (2) needs modification if more than two choices are available for continuation at the nth stage (cf. [4]).

3. Determination of Bayes solution in the Markov case. Even though (1) and (2) determine the Bayes solution in the general case, it has not been possible to determine the stopping regions explicitly even in the case of dichotomy and of independent and identically distributed random variables. However some general properties of these Bayes solutions can be established. In this section we shall assume that $\{X_i\}$ form a Markov sequence with given initial observation X_0.

Let the cost function be

$$c(x_j , x_{j+1} , \cdots , x_{j+k} , \omega) = \sum_{i=1}^{k} c(x_{j+i-1} , x_{j+i} , \omega) \tag{3}$$

for all j and k and for any *a priori* distribution ξ,

$$E_\xi [c(x_i , x_{i+1} , \omega) \mid x_i] = c(\xi, x_i) > 0 \tag{4}$$

Let ξ_i denote the *a posteriori* probability distribution on Ω after i observations. We shall assume that the loss due to the acceptance of the terminal decision t depends only on ω and t and not on x. Under these assumptions, from (2), the *a posteriori* risk after n observations $(x_1 , \cdots , x_n)$ is (with obvious notation)

$$\begin{aligned} y_n(\xi_0 , x_0 , \cdots , x_n) &= E_{\xi_0}c(\omega, x_0 , \cdots , x_n) + \min(r_0(\xi_n), c(\xi_n , x_n) \\ &\quad + E(y_0(\xi_{n+1} , x_n , x_{n+1}) \mid x_n)) = E_{\xi_0}c(\omega , x_0 , \cdots , x_n), + \rho_n(\xi_n , x_n) \text{ say.} \end{aligned} \tag{5}$$

Hence we stop sampling at the nth stage if

$$r_0(\xi_n) < c(\xi_n , x_n) + E(y_0(\xi_{n+1} , x_n , x_{n+1}) \mid x_n), \tag{6}$$

or

$$\rho_n(\xi_n , x_n) = r_0(\xi_n),$$

for the first time, and continue sampling otherwise. This will give us the Bayes procedure.

It can be easily proved that $y_0(\xi, x)$ is concave and, in the case of finite number of alternatives $a_1 , a_2 , \cdots , a_k$, continuous in ξ. The latter property can be proved by truncation and following a method used in Blackwell and Girshick [2], for the case of independent identically distributed random variables. Hence the stopping region $\Xi_j \subset \Xi$, the space of all *a priori* distributions, can be split up into k closed convex regions which have at most boundary points in common, where we prefer one of the k actions. In particular, for a dichotomy with hypotheses 1 and 2, the region where we prefer hypothesis i at the jth stage, $\Xi_j(i)$, is an interval given by

$$\begin{aligned} \Xi_j(1) &= [\delta(x_j) \leqq \xi_j \leqq 1], \\ \Xi_j(2) &= [0 \leqq \xi_j \leqq \gamma(x_j)] \qquad \gamma(x_j) \leqq \delta(x_j), \end{aligned} \tag{7}$$

where ξ_0 is the *a priori* probability of hypothesis 1. By continuity in ξ of both sides of (5), the boundaries $\delta(x_j)$ and $\gamma(x_j)$ satisfy the equations

$$\begin{aligned} (1 - \gamma(x_j))w_{22} + \gamma(x_j)w_{12} &= \Phi(x_j, \gamma(x_j)) \\ \delta(x_j)w_{11} + (1 - \delta(x_j))w_{21} &= \Phi(x_j, \delta(x_j)), \end{aligned} \tag{8}$$

for all j and x_j, where w_{ij} denotes the loss of accepting j when i is true and

$$\Phi(x_j, \xi) = E(y_0(\xi, x_j, x_{j+1}) \mid x_j) + c(\xi, x_j) \tag{9}$$

Since $\Phi(x_j, \xi)$ is continuous and concave in ξ, (8) has a unique solution for $\xi \varepsilon (0, 1)$ if $w_{22} \leqq 0$ and $w_{11} \leqq 0$. Hence we have the Bayes solution at least in principle, even though the explicit form will involve the computation of $y_0(\xi, x_j, x_{j+1})$.

In the case of independent identically distributed random variables, by studying the associated SPR test we are able to get approximate Bayes solution, using Wald's approximation. The results of this section are proved in a less general manner in [3]. An illustrative example will be given in the next section.

4. Testing of hypothesis in a 2-state Markov chain. Let the transition probability matrices under H_1 and H_2 be

$$P_1 = \begin{pmatrix} 1 & 0 \\ p_1 & p_2 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1 & 0 \\ p_1' & p_2' \end{pmatrix}$$

respectively. Let w_{ij} be the loss incurred by accepting H_j when H_i is true, which is positive for $i \neq j$ and is zero for $i = j$, $(i, j = 1, 2)$. Let $c_1 > 0$ and $c_2 \geqq 0$ be the costs under H_1 and $c_1' > 0$ and $c_2' \geqq 0$ under H_2, of observations belonging to the first and second state respectively.

According to the general theory, if after r observations we are in State 1, sampling is continued if the *a posteriori* probability ξ_r of H_1 is such that

$$\gamma(1) < \xi_r < \delta(1), \tag{10}$$

sampling is discontinued and H_1 is accepted if $\xi_r > \delta(1)$ and sampling is discontinued and H_2 is accepted if $\xi_r < \gamma(1)$. Similarly if after r observations we are in State 2, sampling is continued if

$$\gamma(2) < \xi_r < \delta(2), \tag{11}$$

H_1 is accepted if $\xi_r > \delta(2)$ and H_2 is accepted if $\xi_r < \gamma(2)$. At the boundary points one is indifferent and may base his decision on an independent experiment such as tossing of an unbiased coin.

(i) For sequences whose 0th observation belongs to State 1, the Bayes solution is to make a decision without any observation. Thus

$$\gamma(1) = \delta(1) = w_{21}/(w_{21} + w_{12}). \tag{12}$$

(ii) For sequences whose 0th observation belongs to the second state, sampling is continued until it reaches State 1 or $\xi_r > \delta(2)$ or $\xi_r < \gamma(2)$, whichever happens earlier. If the last observation belongs to State 1, procedure given in (i) is followed with ξ replaced by ξ_r. Hence, if $\rho(2, \xi)$ is the loss incurred by the Bayes solution starting from State 2 with ξ as the *a priori* probability of H_1, $\gamma(2)$ and $\delta(2)$ are given by

$$\begin{aligned}\gamma(2)w_{12} &= \gamma(2)(c_1 p_1 + c_2 p_2) + (1 - \gamma(2))(c_1' p_1' + c_2' p_2') \\ &\quad + \min[(1 - \gamma(2))p_1' w_{21}, \gamma(2)p_1 w_{12}] \\ &\quad + (\gamma(2)p_2 + (1 - \gamma(2))p_2')\rho\left(2, \frac{\gamma(2)p_2}{\gamma(2)p_2 + (1 - \gamma(2))p_2}\right),\end{aligned} \tag{13}$$

$$\begin{aligned}(1 - \delta(2))w_{21} &= \delta(2)(c_1 p_1 + c_2 p_2) + (1 - \delta(2))(c_1' p_1' + c_2' p_2') \\ &\quad + \min[(1 - \delta(2))p_1' w_{21}, \delta(2)p_1 w_{12}] \\ &\quad + (\delta(2)p_2 + (1 - \delta(2))p_2')\rho\left(2, \frac{\delta(2)p_2}{\delta(2)p_2 + (1 - \delta(2)p_2}\right).\end{aligned} \tag{14}$$

Our main problem is to determine $\rho(2, \xi)$.

CASE (i): $p_2' < p_2$. In this case

$$\rho(2, \xi) = \xi[E_1(c \mid \xi) + w_{12}(1 - p_2)^{k_0}] + (1 - \xi)[E_2(c \mid \xi) + w_{21}(p_2')^{k_0}]. \tag{15}$$

$$\begin{aligned}\text{where } E_1(c \mid \xi) &= \text{expected cost of observations under } H_1 \text{ when } \xi \text{ is the} \\ &\qquad a \text{ priori probability of } H_1, \\ &= N_0 c_2 p_2^{N_0} + \sum_{k=0}^{N_0-1} (c_1 + kc_2)p_2^k p_1,\end{aligned} \tag{16}$$

where $N_0(\xi)$ is the smallest non-negative value of n for which

$$(p_2'/p_2)^n < \frac{\xi(1 - \delta(2))}{(1 - \xi)\delta(2)} \quad \text{or} \quad > \frac{\xi(1 - \gamma(2))}{(1 - \xi)\gamma(2)}. \tag{17}$$

Similarly

$$\begin{aligned}E_2(c \mid \xi) &= \text{expected cost of observations under } H_2, \\ &= N_0 c_2' (p_2')^{N_0} + \sum_{k=0}^{N_0-1} (c_1' + kc_2')(p_2')^k p_1'.\end{aligned} \tag{18}$$

Further $k_0(\xi)$ equals N_0 or the smallest integer k such that

$$\frac{\xi p_2^k p_1}{\xi p_2^k p_1 + (1 - \xi)p_2'^k p_1'} > \frac{w_{21}}{w_{21} + w_{12}}, \tag{19}$$

whichever is smaller. (15) follows from the fact, that, in this case, the probability of accepting H_1 ultimately when H_1 is true for sequences starting from State 2 equals $p_2^{k_0}$. Hence from (14), since

$$N_0(\xi_1(\xi, 2, 2)) = 0 = k_0(\xi_1(\xi, 2, 2)) = E_1(c \mid \xi_1) = E_2(c \mid \xi_1),$$

$$(20) \quad (1 - \delta(2))w_{21} = \delta(2)(c_1p_1 + c_2p_2) + (1 - \delta(2))(c_1'p_1' + c_2'p_2') + \min((1 - \delta(2))p_1'w_{21}, \delta(2)p_1w_{12}) + (1 - \delta(2))p_2'w_{21}$$

Thus $\delta(2)$ is given by

$$(21) \quad (1 - \delta(2))p_1'w_{21} = \delta(2)(c_1p_1 + c_2p_2) + (1 - \delta(2))(c_1'p_1' + c_2'p_2') + \delta(2)p_1w_{12}.$$

It may be noted that (21) gives the value of the upper boundary point for the Bayes solution truncated at the first observation.

Once we obtain $\delta(2)$ we can solve for $\gamma(2)$ from (13) as follows. Guess a value $\gamma_0(2)$ of $\gamma(2)$ and calculate $N_0(\xi_1(\gamma_0(2), 2, 2))$ and $k_0(\xi_1(\gamma_0(2)))$ and hence $E_1(c \mid \xi_1)$ and $E_2(c \mid \xi_1)$. Then (13) is a linear equation in $\gamma(2)$ and can be solved easily. If $\gamma_1(2)$ is the solution obtained, repeat the process of calculating $E_1(c \mid \xi_1)$ and $E_2(c \mid \xi_1)$, now from $\xi_1(\gamma_1(2), 2, 2)$, then from $\xi_1(\gamma_2(2), 2, 2)$ and so on until we get the same value of $k_0(\xi_1)$ and $N_0(\xi_1)$ for two consecutive iterations.

EXAMPLE 1. Let $w_{12} = 30$, $w_{21} = 60$, $c_1 = 1$, $c_2 = 3$, $c_1' = 2$, $c_2' = 5$ and

$$P_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

From (12), $\gamma(1) = \delta(1) = \frac{2}{3}$. Since $p_2' < p_2$, from (21) $\delta(2) = \frac{111}{148}$. Assume that $\gamma(2) = \gamma_0(2) = \frac{1}{2}$. Then $N_0(\xi_1) = k_0(\xi_1) = 1$, $E_1(c \mid \xi_1) = \frac{7}{3}$, $E_2(c \mid \xi_1) = 3$. Because for $\gamma(2) < \frac{2}{3}$, $\min[(1 - \gamma(2))40, 10\gamma(2)] = 10\gamma(2)$, from (13), $\gamma(2) = \frac{96}{181}$. Now assume that $\gamma(2) = \gamma_1(2) = \frac{96}{181}$. For this value of $\gamma(2)$, $N_0(\xi_1) = 1 = k_0(\xi_1)$. Hence $\gamma(2) = \frac{96}{181}$.

CASE (ii): $p_2' > p_2$. In this case the role of $\gamma(2)$ and $\delta(2)$ is the reverse of that of Case (i). Equation (13) does not involve $\delta(2)$ and hence can be solved for $\gamma(2)$. Substituting this value of $\gamma(2)$ in (14) we can solve for $\delta(2)$ in the same manner as for $\gamma(2)$ in case (i).

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