BOUNDS FOR THE NUMBER OF COMMON TREATMENTS BETWEEN ANY TWO BLOCKS OF CERTAIN PBIB DESIGNS

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0. Introduction and summary. In an earlier paper [4], the author has given upper bounds for the number of disjoint blocks in (i) Semi-regular GD designs, (ii) certain PBIB designs with two associate classes having triangular association scheme, (iii) certain PBIB designs with two associate classes having L_2 association scheme and (iv) certain PBIB designs with three associate classes having rectangular association scheme. In this paper, we give bounds for the number of common treatments between any two blocks of the above-mentioned PBIB designs. The main tools used to establish the results of this paper are the theorems proved by (i) Bose and Connor [1], (ii) Raghavarao [3], and (iii) Vartak [6].

1. Semi-regular GD designs. An incomplete block design with v treatments, each replicated r times in b blocks of size k is said to be group divisible (GD) [2], if the treatments $v = mn$ can be divided into m groups, each with n treatments, so that treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks ($\lambda_1 \neq \lambda_2$). The primary parameters of such a design are $v = mn$, b, r, k, λ_1, λ_2, $n_1 = (n - 1)$, $n_2 = n(m - 1)$. They obviously satisfy the relations $bk = vr$, $(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1)$, $r \geqq \lambda_1$, $r \geqq \lambda_2$. Semi-regular GD designs [1] are characterised by $rk - v\lambda_2 = 0$ and $r - \lambda_1 > 0$. Bose and Connor [1] proved the following theorem for semi-regular GD designs.

THEOREM 1.A. *For a semi-regular* GD *design, k is divisible by m. If $k = cm$, then every block must contain c treatments from every group.*

We use Theorem 1.A to obtain bounds for the number of common treatments between any two blocks of semi-regular GD designs. The result is given in Theorem 1.

THEOREM 1. *If x be the number of treatments common between any two blocks of a semi-regular* GD *design, then* $\max(0, T_1) \leqq x \leqq \min(k, T_2)$, *where*

$$T_1 = k(r - 1)/(b - 1) - (b - 2)^{\frac{1}{2}} \cdot A,$$

$$T_2 = k(r - 1)/(b - 1) + (b - 2)^{\frac{1}{2}} \cdot A$$

and

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$$A^2 = \left[\frac{k^2\{(v-k)(b-r)-(v-rk)(v-m)\}}{v(v-m)} - \frac{k^2(r-1)^2}{(b-1)}\right] \Big/ (b-1)$$
$$= k^2(v-k)(b-r)(b-v+m-1)/v(v-m)(b-1)^2.$$

PROOF. Let the blocks be denoted by $B_1, B_2, \cdots, B_b$. Denote the number of treatments common between B_1 and B_i by x_i, $i = 2, 3, \cdots, b$. Let $x_2 = x$. Considering the treatments of the block B_1 singly, we obtain

$$\sum_{i=3}^{b} x_i = k(r-1) - x. \tag{1.1}$$

The block B_1, by virtue of Theorem 1.A, contains k/m treatments from each group which form pairs of first associates. Hence, considering the treatments of the block B_1 pairwise, we get

$$\begin{aligned}\sum_{i=3}^{b} x_i(x_i-1) \\ = \{k[(k-m)\lambda_1 + k(m-1)\lambda_2 - m(k-1)]/m\} - x(x-1).\end{aligned} \tag{1.2}$$

Following the method of proving author's [4] result (2.3) from (2.1) and (2.2), we can show from (1.1) and (1.2) that

$$\begin{aligned}\sum_{i=3}^{b}(x_i - \bar{x})^2 = \frac{k^2[(v-k)(b-r)-(v-rk)(v-m)]}{v(v-m)} \\ - x^2 - \frac{[k(r-1)-x]^2}{(b-2)} \geqq 0.\end{aligned} \tag{1.3}$$

Then, Theorem 1 follows from (1.3).

COROLLARY 1.1. *If in a semi-regular* GD *design* $b = v - m + 1$, *then there are* $k(r-1)/(v-m)$ *treatments common between any two blocks of this design.*

This result is also proved in [4].

2. PBIB designs with two associate classes having a triangular association scheme. A PBIB design with two associate classes is said to have a triangular association scheme [2], if the number of treatments is $v = n(n-1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) the positions in the principal diagonal are blank,

(b) the $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n-1)/2$, corresponding to the treatments,

(c) the array is symmetric about the principal diagonal,

(d) for any treatment θ, the first associates are exactly those treatments which lie in the same row and same column as θ.

The primary parameters of this design are $v = n(n-1)/2$, b, r, k, λ_1, λ_2, $n_1 = 2n - 4$, $n_2 = (n-3)(n-2)/2$. The following theorem is proved by Raghavarao [3].

THEOREM 2.A. *If in a* PBIB *design with two associate classes having a triangular association scheme,* $rk - v\lambda_1 = n(r - \lambda_1)/2$, *then* $2k$ *is divisible by* n. *Further,* *every block of this design contains $2k/n$ treatments from each of the n rows of the association scheme.*

We use Theorem 2.A to obtain bounds for the number of treatments common between any two blocks for this design in which $rk - v\lambda_1 = n(r - \lambda_1)/2$. The result is given in Theorem 2.

THEOREM 2. *If x be the number of treatments common between any two blocks of a* PBIB *design with two associate classes having a triangular association scheme and with $rk - v\lambda_1 = n(r - \lambda_1)/2$, then* $\max(0, T_1) \leqq x \leqq \min(k, T_2)$, *where*

$$T_1 = [k(r-1)/(b-1)] - (b-2)^{\frac{1}{2}} \cdot A, \qquad T_2 = [k(r-1)/(b-1)] + (b-2)^{\frac{1}{2}} \cdot A$$

and

$$A^2 = \left[\frac{k^2 \cdot \{n(b+1-2r) - (v-rk)(n-2)\}}{n \cdot (v-n)} - \frac{k^2(r-1)^2}{(b-1)}\right] \Big/ (b-1)$$

$$= \frac{k^2(v-k)(b-r)(b-v+n-1)}{v(v-n)(b-1)^2}.$$

PROOF. Using notation as in Theorem 1, we again get

$$\sum_{i=3}^{b} x_i = k(r-1) - x. \tag{2.1}$$

Also, by virtue of Theorem 2.A and considering treatments of the block B_1 pairwise, we get

$$\begin{aligned}\sum_{i=3}^{b} x_i(x_i - 1) &= n \cdot (2k/n) \cdot ((2k/n) - 1)(\lambda_1 - 1) \\ &\quad + [k(k-1) - n \cdot (2k/n)((2k/n) - 1)](\lambda_2 - 1) - x(x-1).\end{aligned} \tag{2.2}$$

Following the method of proving author's [4] result (3.4) from (3.1) and (3.2), we can show from (2.1) and (2.2) that

$$\begin{aligned}\sum_{i=3}^{b} (x_i - \bar{x})^2 &= \frac{k^2 \cdot \{n(b+1-2r) - (v-rk)(n-2)\}}{n(v-n)} \\ &\quad - x^2 - \frac{[k(r-1) - x]^2}{(b-2)} \geqq 0.\end{aligned} \tag{2.3}$$

Theorem 2 follows from (2.3)

COROLLARY 2.1. *If in a* PBIB *design with two associate classes having a triangular association scheme and $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b = v - n + 1$, then there are $k(r-1)/(v-n)$ treatments common between any two blocks of this design.*

This result is also proved in [4].

3. PBIB designs with two associate classes having a L_2 association scheme. A PBIB design with two associate classes is said to have a L_2 association scheme [2], if the number of treatments is $v = s^2$, where s is a positive integer and the treatments can be arranged in an $s \times s$ square such that treatments in the same row or the same column are first associates, while others are second associates. The primary parameters of this design are $v = s^2$, b, r, k, $n_1 = 2(s - 1)$, $n_2 = (s - 1)^2$, λ_1 and λ_2. The following theorem is proved by Raghavarao [5].

THEOREM 3.A. *If in a* PBIB *design with two associate classes having a L_2 association scheme, $rk - v\lambda_1 = s(r - \lambda_1)$, then k is divisible by s. Further, every block of this design contains k/s treatments from each of the s rows (or columns) of the association scheme.*

We use Theorem 3.A to obtain bounds for the number of treatments common between any two blocks of this design with $rk - v\lambda_1 = s(r - \lambda_1)$. The result is given in Theorem 3.

THEOREM 3. *If x be the number of treatments common between any two blocks of a* PBIB *design with two associate classes having a L_2 association scheme with $rk - v\lambda_1 = s(r - \lambda_1)$, then* $\max(0, T_1) \leqq x \leqq \min(k, T_2)$, *where*

$$T_1 = \{[k(r - 1)/(b - 1)]\} - (b - 2)^{\frac{1}{2}} \cdot A,$$

$$T_2 = \{[k(r - 1)/(b - 1)]\} + (b - 2)^{\frac{1}{2}} \cdot A$$

and

$$A^2 = \left[\frac{k^2\{(b - r)(v - k) - (s - 1)^2(v - rk)\}}{v(s - 1)^2} - \frac{k^2(r - 1)^2}{(b - 1)}\right] \Big/ (b - 1)$$

$$= \frac{k^2(v - k)(b - r)(b - v + 2s - 2)}{v(s - 1)^2 (b - 1)^2}.$$

PROOF. Using notation as in Theorem 1, we again get

$$\sum_{i=3}^{b} x_i = k(r - 1) - x. \tag{3.1}$$

Also, by virtue of Theorem 3.A and considering treatments of the block B_1 pairwise we get

$$\begin{aligned} &\sum_{i=3}^{b} x_i(x_i - 1) \\ &\quad = (k/s)[2(k - s)\lambda_1 + (sk + s - 2k)\lambda_2 - s(k - 1)] - x(x - 1). \end{aligned} \tag{3.2}$$

Following the method of proving author's [4] result (4.4) from (4.1) and (4.2), we can show from (3.1) and (3.2) that

$$\begin{aligned} \sum_{i=3}^{b} (x_i - \bar{x})^2 = {} & \frac{k^2[(b - r)(v - k) - (s - 1)^2(v - rk)]}{v(s - 1)^2} \\ & - x^2 - \frac{[k(r - 1) - x]^2}{(b - 2)} \geqq 0. \end{aligned} \tag{3.3}$$

Theorem 3 follows from (3.3).

COROLLARY 3.1. *If in a* PBIB *design with two associate classes having a L_2 association scheme and $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + 2$, then there are $k(r - 1)/(s - 1)^2$ treatments common between any two blocks of this design.*

This result is also proved in [4].

4. PBIB designs with three associate classes having a rectangular association scheme. A PBIB design with three associate classes is said to have a rectangular association scheme [5], if the number of treatments is $v = v_1 \cdot v_2$ and the treatments can be arranged in the form of a rectangle of v_1 rows and v_2 columns, so that the first associates of any treatment are the other $(v_2 - 1)$ treatments of the same row, the second associates are the other $(v_1 - 1)$ treatments of the same column; while the remaining $(v_1 - 1) \cdot (v_2 - 1)$ treatments are the third associates. The primary parameters of this design are $v = v_1 \times v_2$, b, r, k, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$, $n_3 = n_1 n_2$, λ_1, λ_2 and λ_3. Vartak [5] has proved that the characteristic roots of NN' of this design are $\theta_0 = rk$, $\theta_1 = r - \lambda_1 + (v_1 - 1) \cdot (\lambda_2 - \lambda_3)$, $\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3)$, $\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3$. In this paper, we consider this design with $\theta_1 = 0 = \theta_2$. The following theorems were proved by Vartak [6].

THEOREM 4.A. *If in a* PBIB *design with three associate classes having a rectangular association scheme,* $\theta_1 = 0$, *then* k *is divisible by* v_2 *and every block of this design contains* k/v_2 *treatments from every column of the association scheme.*

THEOREM 4.B. *If in a* PBIB *design with three associate classes having a rectangular association scheme,* $\theta_2 = 0$, *then* k *is divisible by* v_1 *and every block of this design contains* k/v_1 *treatments from every row of the association scheme.*

We use Theorems 4.A and 4.B to obtain bounds for the number of treatments common between any two blocks of the above design with $\theta_1 = 0 = \theta_2$.

The result is given in Theorem 4.

THEOREM 4. *If* x *be the number of treatments common between any two blocks of a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$, *then* $\max(0, T_1) \leqq x \leqq \min(k, T_2)$, *where*

$$T_1 = [k(r - 1)/(b - 1)] - (b - 2)^{\frac{1}{2}} \cdot A,$$

$$T_2 = [k(r - 1)/(b - 1)] + (b - 2)^{\frac{1}{2}} \cdot A$$

and

$$A^2 = \left[\frac{k\{r(v - k)^2 - kp(v - rk)\}}{vp} - \frac{k^2(r - 1)^2}{(b - 1)}\right] \Big/ (b - 1)$$
$$= k^2(v - k)(b - r)(b - p - 1)/vp(b - 1)^2,$$

p being equal to $(v_1 - 1)(v_2 - 1)$.

PROOF. Using notation as in Theorem 1, we again get

$$\sum_{i=3}^{b} x_i = k(r - 1) - x. \tag{4.1}$$

Now using Theorems 4.A and 4.B and considering treatments of the block B_1 pairwise, we get

$$\sum_{i=3}^{b} x_i(x_i - 1) = (k/v)[v_2(k - v_1)(\lambda_1 - \lambda_3) + v_1(k - v_2)(\lambda_2 - \lambda_3) + v(k - 1)(\lambda_3 - 1)] - x(x - 1). \tag{4.2}$$

Following the method of proving author's [4] result (5.7) from (5.1) and (5.3), we get from (4.1) and (4.2)

$$\sum_{i=3}^{b} (x_i - \bar{x})^2 = \frac{k\,[r(v - k)^2 - kp(v - rk)]}{vp} - x^2 - \frac{[k(r - 1) - x]^2}{(b - 2)} \geqq 0. \tag{4.3}$$

Theorem 4 follows from (4.3).

COROLLARY 4.1. *If in a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$, $b = p + 1$, *then there are* $k(r - 1)/p$ *treatments common between any two blocks of this design.*

This result is also proved in [4].

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