INCREASING PROPERTIES OF PÓLYA FREQUENCY FUNCTIONS

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1. Introduction. A probability density function on the real line $r(x)$ is said to be a Pólya frequency function of order 2 (PF_2) if $x_2 \geqq x_1$, $z_2 \geqq z_1$ implies

$$\begin{vmatrix} r(x_1 - z_1) & r(x_1 - z_2) \\ r(x_2 - z_1) & r(x_2 - z_2) \end{vmatrix} \geqq 0.$$

The following properties of PF_2 functions are well known [3]:

Property 1. Many of the common densities of probability theory are PF_2. In particular the normal density

$$r(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}[(x - m)/\sigma]^2\},$$

if PF_2 for every choice of $\sigma > 0$.

Property 2. A PF_2 function is bounded, non-zero on an interval and zero outside this interval (which may be infinite).

Property 3. A PF_2 function is logarithmically concave, and hence has first and second derivatives existing almost everywhere (with respect to Lebesgue measure).

Property 4. Let $r(x)$ be PF_2. Then

$$\begin{aligned} r_{ab}(x) &= r(x)/\textstyle\int_a^b r(z)\,dz & a \leqq x \leqq b \\ &= 0 & \text{otherwise} \end{aligned}$$

is also PF_2. (Here a or b may be infinite.)

Property 5. Let $r(x)$ and $q(x)$ be PF_2. Then their convolution

$$p(x) = \int_{-\infty}^{\infty} r(t)q(x - t)\,dt$$

is also PF_2.

2. The main theorem. The theorem presented below was originally suggested by Dr. S. Karlin. (The special case in which the underlying random variables are binomial has been previously verified in unpublished work of F. Proschan and R. E. Barlow.) Its proof hinges on a simple geometric property of PF_2 density functions, which will be pointed out explicitly following the body of the argument.

Theorem. *Let $X_1, X_2, \cdots, X_n$ be n independent random variables with PF_2 densities $r_1(x), r_2(x), \cdots, r_n(x)$ respectively, let $S = \sum_1^n X_i$ be their sum, and let $\Phi(x_1, x_2, \cdots, x_n)$ be a real measurable function on Euclidean n-space which is*

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non-decreasing in each of its arguments. Then

$$E(\Phi(X_1, X_2, \cdots, X_n) \mid S = s)$$

is a non-decreasing function of s.

PROOF. One version of the conditional expectation is

$$E(\Phi \mid S = s) = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi\left((x_1, \cdots x_{n-1}, s - \sum_1^{n-1} x_i\right) r_1(x_1) \cdots r_{n-1}(x_{n-1}) r_n\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1(x_1) \cdots r_{n-1}(x_{n-1}) r_n\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}.$$

In the sequel, the symbol $E(\Phi \mid S = s)$ will always refer to the expression on the right, above. It will be shown that whenever the denominator of this expression is non-zero for two values of the argument, $t \geqq s$, and the numerator exists at the points s and t, either as a finite number or as $\pm \infty$, then

$$E(\Phi \mid S = t) \geqq E(\Phi \mid S = s).$$

It is sufficient to verify the theorem for bounded Φ. For if Φ_c is defined to be

$$\Phi_c = \Phi \quad \text{for} \quad |\Phi| \leqq c$$

$$\Phi_c = c \quad \text{for} \quad \Phi > c$$

$$\Phi_c = -c \quad \text{for} \quad \Phi < -c,$$

then the monotone convergence theorem applied to the positive and negative parts of Φ implies

$$\lim_{c \to \infty} E(\Phi_c \mid S = s) = E(\Phi \mid S = s),$$

whenever the right side exists.

For every $\sigma > 0$ consider the frequency functions

$$r_{i\sigma}(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} r_i(x - t) \exp\left[-\tfrac{1}{2}(t^2/\sigma^2)\right] dt, \qquad i = 1, 2, \cdots, n.$$

The $r_{i\sigma}(x)$ are positive for all values of x and are PF_2 by Properties 1 and 5. Moreover for Φ bounded, using obvious notation,

$$\lim_{\sigma \to 0} E_\sigma(\Phi \mid S = s) = E(\Phi \mid S = s).$$

(This follows from an improved version of the Helly-Bray theorem. Although Φ is not necessarily continuous here, the limiting distribution is absolutely continuous, which is sufficient.) Therefore in addition to assuming that Φ is bounded, it can be assumed without loss of generality that the $r_i(x)$ are everywhere positive. Together these two conditions imply that the expression given previously for $E(\Phi \mid S = s)$ exists finitely everywhere.

Assume the theorem has been verified for $2, 3, \cdots, n - 1$ random variables, and define

$$g(t, u) \equiv E\left(\Phi \mid \sum_1^{n-1} X_i = t, X_n = u\right)$$

$$= \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi\left(x_1, \cdots x_{n-2}, t - \sum_1^{n-2} x_i, u\right) r_1(x_1) \cdots r_{n-1}\left(t - \sum_1^{n-2} x_i\right) dx_1 \cdots dx_{n-2}}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1(x_1) \cdots r_{n-2}(x_{n-2})\, r_{n-1}\left(t - \sum_1^{n-2} x_i\right) dx_1 \cdots dx_{n-2}}.$$

$g(t, u)$ is non-decreasing in t by the induction hypothesis, and non-decreasing in u by the non-decreasing nature of Φ in its last argument.

The random variable $\sum_1^{n-1} X_i$ has density

$$r_*(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1(x_1) \cdots r_{n-2}(x_{n-2}) r_{n-1}(t - \sum_1^{n-2} x_i)\, dx_1 \cdots dx_{n-2},$$

which is PF_2 by Property 5 (applied $n - 2$ times successively). Thus, applying the main theorem with $n = 2$, the function

$$\int_{-\infty}^{\infty} g(t, s - t) r_*(t) r_n(s - t)\, dt / \int_{-\infty}^{\infty} r_*(t) r_n(s - t)\, dt$$

is increasing in s. Let T be the random variable $\sum_1^{n-1} X_i$. Then the expression above can be rewritten as

$$E(g(T, X_n) \mid T + X_n = s) = E(E(\Phi \mid \textstyle\sum_1^{n-1} X_i, X_n) \mid \sum_1^{n-1} X_i + X_n = s)$$
$$= E(\Phi \mid \textstyle\sum_1^n X_i = s)$$

which verifies the result for n random variables.

It remains to demonstrate the theorem for $n = 2$ random variables. The inequality below, which will be referred to as $I*$, follows directly from the definition of a PF_2 density $r(x)$.

$$I*: r(s + \Delta - x_2)/r(s + \Delta - x_1) \geqq r(s - x_2)/r(s - x_1).$$

($I*$ is valid foı all values of s, $x_2 \geqq x_1$, and $\Delta \geqq 0$.)

For $0 < \alpha < 1$, let $(x_{\alpha,s}, z_{\alpha,s} \equiv s - x_{\alpha,s})$ be the unique point on the line $x + z = s$ satisfying

$$\int_{-\infty}^{x_{\alpha,s}} r_1(x) r_2(s - x)\, dx / \int_{x_{\alpha,s}}^{\infty} r_1(x) r_2(s - x)\, dx = \alpha/1 - \alpha.$$

(Then

$$\int_{-\infty}^{x_{\alpha,s}} r_1(x) r_2(s - x)\, dx / \int_{-\infty}^{\infty} r_1(x) r_2(s - x)\, dx = \alpha,$$

i.e., $(x_{\alpha,s}, z_{\alpha,s})$ is the 100α percentile point for the conditional distribution on the line $x + z = s$, the zero percentile point being taken at the upper left.)

LEMMA. *For every value of* $\Delta \geqq 0$, *and every value of* α, $0 \leqq \alpha \leqq 1$, $x_{\alpha,s+\Delta} \geqq x_{\alpha,s}$ *and* $z_{\alpha,s+\Delta} \geqq z_{\alpha,s}$.

PROOF.

$$\frac{\int_{-\infty}^{x_{\alpha,s}} r_1(x)r_2(s+\Delta-x)\,dx}{\int_{x_{\alpha,s}}^{\infty} r_1(x)r_2(s+\Delta-x)\,dx} = \frac{\int_{-\infty}^{x_{\alpha,s}} r_1(x)[r_2(s+\Delta-x)/r_2(s+\Delta-x_{\alpha,s})]\,dx}{\int_{x_{\alpha,s}}^{\infty} r_1(x)[r_2(s+\Delta-x)/r_2(s+\Delta-x_{\alpha,s})]\,dx}$$

$$\leqq \frac{\int_{-\infty}^{x_{\alpha,s}} r_1(x)[r_2(s-x)/r_2(s-x_{\alpha,s})]\,dx}{\int_{x_{\alpha,s}}^{-\infty} r_1(x)[r_2(s-x)/r_2(s-x_{\alpha,s})]\,dx} = \frac{\alpha}{1-\alpha},$$

the inequality following from $I*$ applied to both the numerator and denominator of the right hand expression of the equality. This implies $x_{\alpha,s+\Delta} \geqq x_{\alpha,s}$, and by symmetry, $z_{\alpha,s+\Delta} \geqq z_{\alpha,s}$.

Making a transformation of variables, (the "probability integral transformation" $\alpha = F(x \mid s+\Delta)$, where F is the conditional cumulative distribution function on the line $x+z = s+\Delta$, and likewise for the line $x+z = s$),

$$E(\Phi \mid S = s+\Delta) = \int_0^1 \Phi(x_{\alpha,s+\Delta}, z_{\alpha,s+\Delta})\,d\alpha$$

$$\geqq \int_0^1 \Phi(x_{\alpha,s}, z_{\alpha,s})\,d\alpha = E(\Phi \mid S = s).$$

This completes the proof of the main theorem.

It is possible to show that the curve $(x_{\alpha,s}, z_{\alpha,s})$ traced out in the $(x\text{-}z)$ plane as s varies from $-\infty$ to ∞ is continuous and differentiable with respect to s almost everywhere. Since $x_{\alpha,s}$ and $z_{\alpha,s}$ are both increasing functions of s, this curve must have positive slope at every point. Remembering that $(x_{\alpha,s}, z_{\alpha,s})$ is the conditional 100α percentile point for each s, the following loose geometric interpretation is informative: the conditional probability mass given $x+z = s$ flows in lines of positive slope (with respect to both the x and the z axis) as s is increased.

To make this statement more precise, and at the same time extend it to higher dimensions, let A be any set in n-space such that if $(x_1, x_2 \cdots x_n) \,\varepsilon\, A$ and $x_1' \geqq x_1$, $x_2' \geqq x_2, \cdots, x_n' \geqq x_n$, then $(x_1', x_2' \cdots x_n') \,\varepsilon\, A$. The generalization of the statement "the conditional probability mass flows in lines of positive slope" is $P(A \mid S = t) \geqq P(A \mid S = s)$ for every pair of numbers $t \geqq s$ and every such set A. That this condition is satisfied follows immediately from the theorem applied to the indicator functions of the sets A. The converse implication is also valid, and follows by applying the inequality above to sets A of the form

$$A_c = \{\Phi(x_1, x_2 \cdots x_n) > c\} \qquad -\infty < c < \infty.$$

Now let $\Phi_t(x_1, x_2, \cdots, x_n)$ be a family of measurable functions increasing in every argument for each fixed value of t, and increasing in t for each fixed value of $x_1, x_2, \cdots, x_n$.

COROLLARY 1. *Under the conditions of the theorem, the function*

$$g(a, b) = E(\Phi_{a+b-S}(X_1, X_2 \cdots X_n) \mid a \leqq S \leqq a + b), \qquad b \geqq 0,$$

is increasing in both a and b.

PROOF.

$$g(a, b) = \int_a^{a+b} E(\Phi_{a+b-s} \mid S = s) r^*(s)\, ds / \int_a^{a+b} r^*(s)\, ds$$

where

$$r^*(s) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1(x_1) \cdots r_{n-1}(x_{n-1}) r_n(s - \textstyle\sum_1^{n-1} x_i)\, dx_1 \cdots dx_{n-1}.$$

(So $r^*(s)$ is PF_2 by Property 5.) This can be written

$$g(a, b) = \frac{\int_{-\infty}^{\infty} E(\Phi_{a+b-s} \mid S = s) r^*(s)[I_{(-b,0)}(a - s)/b]\, ds}{\int_{-\infty}^{\infty} r^*(s)[I_{(-b,0)}(a - s)/b]\, ds}$$

where

$$I_{(-b,0)}(x) = 1 \quad \text{if} \quad -b \leqq x \leqq 0$$
$$= 0 \quad \text{otherwise.}$$

The function $I_{(-b,0)}(x)/b$ is easily shown to be PF_2. Hence the main theorem applied to the function $E(\Phi_{b+t} \mid S = s)$ shows that $g(a, b)$ is increasing in a.

Likewise, $g(a, b)$ may be written

$$g(a, b) = \frac{\int_{-\infty}^{\infty} E(\Phi_{a+b-s} \mid S = s) r_{a\infty}^*(s)[I_{(-a,B-a)}(b - s)/B]\, ds}{\int_{-\infty}^{\infty} r_{a\infty}^*(s)[I_{(-a,B-a)}(b - s)/B]\, ds}$$

for all b less than some arbitrary positive constant B. Here

$$r_{a\infty}^*(s) = [r^*(s)]/[\textstyle\int_a^{\infty} r^*(t)\, dt] \quad \text{for} \quad s \geqq a$$
$$= 0 \quad \text{otherwise,}$$

a function which is PF_2 by Property 4. The theorem then implies $g(a, b)$ is increasing in b.

Let $\Phi(x_1, x_2, \cdots, x_n)$ be an increasing function as before, and let $I_A(x_1, x_2, \cdots, x_n)$ be the indicator function of the first orthant,

$$I_A(x_1, x_2, \cdots, x_n) = 1 \quad \text{if} \quad x_i \geqq 0 \qquad i = 1, 2, \cdots, n,$$
$$= 0 \quad \text{otherwise.}$$

The function $I_A(x_1, x_2, \cdots, x_n)\Phi(x_1, x_2, \cdots, x_n)$ will be increasing in every argument if $\Phi(0, 0, \cdots, 0) \geqq 0$. Assuming this is the case, the theorem implies that both $E(I_A\Phi \mid S = s)$ and $E(I_A \mid S = s) \equiv P(A \mid S = s)$ are increasing functions of s. Their ratio,

$$\frac{E(I_A\,\Phi \mid S = s)}{P(A \mid S = s)} = \frac{\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-\Sigma_1^{n-2} x_i} \Phi\left(x_1, x_2 \cdots s - \sum_1^{n-1} x_i\right) \cdot r_1(x_1) r_2(x_2) \cdots r_n\left(s - \sum_1^{n-1} x_i\right) dx_{n-1}\, dx_{n-2} \cdots dx_1}{\int_0^s \int_0^{s-x_1} \cdots \int_0^{s-\Sigma_1^{n-2} x_i} r_1(x_1) r_2(x_2) \cdots \cdot r_n\left(s - \sum_1^{n-1} x_i\right) dx_{n-1}\, dx_{n-2} \cdots dx_1}$$

is equal to the conditional expectation $E(\Phi \mid S = s, (X_1, X_2, \cdots, X_n) \,\varepsilon$ first orthant). Corollary 2 shows that this quantity is also increasing in s.

COROLLARY 2. *Let A be a rectangle set in n-space*

$$A = \{(x_1, x_2 \cdots x_n) \mid a_1 \leqq x_1 \leqq b_1, a_2 \leqq x_2 \leqq b_2 \cdots a_n \leqq x_n \leqq b_n\},$$

and $\Phi(x_1, x_2, \cdots, x_n)$ a function defined on A and increasing in every argument. Let $X_1, X_2, \cdots, X_n$ be n independent random variables with respective PF_2 densities $r_1(x_1), r_2(x_2) \cdots r_n(x_n)$. Then

$$E(\Phi(X_1, X_2 \cdots X_n) \mid \textstyle\sum_1^n X_i = s, \qquad (X_1, X_2 \cdots X_n) \,\varepsilon\, A)$$

is an increasing function of s.

PROOF. Define

$$\begin{aligned} \Phi*(x_1, x_2 \cdots x_n) &= \Phi(b_1, b_2 \cdots b_n) && \text{if } x_i > b_i \text{ for any } i \\ &= \Phi(x_1, x_2 \cdots x_n) && \text{if } (x_1, x_2 \cdots x_n) \,\varepsilon\, A \\ &= \Phi(a_1, a_2 \cdots a_n) && \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} r_i^*(x) &= (r_i(x))/(\textstyle\int_{a_i}^{b_i} r_i(x)\,dx) && a_i \leqq x \leqq b_i \\ & && i = 1, 2, \cdots, n \\ &= 0 && \text{otherwise.} \end{aligned}$$

The $r_i^*(x)$ are PF_2 by Property 4, $\Phi*$ is defined over all of n-space and is increasing in each argument, and hence, by the theorem,

$$g(s) \equiv \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi*\left(x_1 \cdots x_{n-1}, s - \sum_1^{n-1} x_i\right) \cdot r_1^*(x_1) \cdots r_n^*\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1^*(x_1) \cdots r_n^*\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}$$

is increasing in s. It is easily verified that

$$g(s) = E(\Phi \mid S = s, (X_1, X_2, \cdots, X_n) \varepsilon A).$$

NOTE. In two dimensions, Corollary 2 can be given the following interpretation: Let $f(x; s)$ represent the conditional density function on the line $x + z = s$, evaluated at the point $(x, s - x)$, and let $F(x; s)$ be the corresponding cumulative distribution function, $F(x; s) = \int_{-\infty}^{x} f(y; s)\, dy$.

Applying the corollary to the set $A = \{(x, z) : x \leqq d\}$ and the function

$$\begin{aligned} \Phi(x, z) &= 0 \quad \text{if} \quad x < 0 \\ &= 1 \quad \text{if} \quad x \geqq 0, \end{aligned}$$

yields

$$\frac{F(d; s) - F(c; s)}{F(d; s)} \leqq \frac{F(d; t) - F(c; t)}{F(d; t)}$$

for $s \leqq t$ and $c \leqq d$. This is equivalent to the statement that $F(x; s)$ is totally positive of order two.

In the lemma to the main theorem, it was shown that the conditional α-percentile line, $\{(x, z) : F(x; x + z) = \alpha\}$ has a positive slope everywhere. Let $m(x; s)$ represent this slope, measured with respect to the x-axis, for the line which passes through the point $(x, s - x)$. Additional information on the behavior of $m(x; s)$ can now be obtained by letting t approach s in the inequality above. It is then easily seen that the function

$$\frac{f(x; s)}{1 - F(x; s)} \frac{1}{(m(x; s) + 1)}$$

must be non-decreasing in x for each value of x. (The factor $f/(1 - F)$ is itself non-decreasing: it is the hazard rate of f, which is directly verified to be PF_2 [1].)

A symmetric argument shows that

$$\frac{f(s - z; s)}{F(s - z; s)} \frac{m(s - z; s)}{m(s - z; s) + 1}$$

is non-increasing in z. These two restrictions on $m(x, s)$ indicate the ways in which the property of positive flow of the conditional probability mass (as expressed in the lemma) is weaker than the assumption of independent PF_2 marginal distributions.

3. Some additional remarks.

1. An integer-valued random variable is said to be PF_2 if its saltuses satisfy

$$\begin{vmatrix} p(m_1 - n_1) & p(m_1 - n_2) \\ p(m_2 - n_1) & p(m_2 - n_2) \end{vmatrix} \geqq 0$$

for all pairs of integers $m_2 \geqq m_1$, $n_2 \geqq n_1$. The proofs and theorems presented here extend without difficulty to such random variables.

2. A density function which is a member of the exponential family, $r(x) = g(x)h(w)e^{wx}$, will be PF_2 for each value of the parameter w if and only if $g(x)$ is PF_2. Given an observation of n independent random variables $X_1, X_2, \cdots, X_n$, each with such a density function $r_i(x) = g_i(x)h_i(w)e^{wx}$, $i = 1, 2, \cdots, n$, (where each $g_i(x)$ is PF_2), it is desired to make some decision concerning w. A decision rule d for this situation consists of a set of conditional distributions on the action space given the observations, $d(a \mid x_1, x_2, \cdots, x_n)$. For each such rule there is a corresponding rule $d*$ depending only on the sufficient statistic $\sum_1^n X_i$, yielding the same risk as d for every value of the parameter w:

$$d*\left(a \mid \sum_1^n x_i = s\right) = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\left(a \mid x_1 \cdots s - \sum_1^{n-1} x_i\right) r_1(x_1) \cdots r_n\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1(x_1) \cdots r_n\left(s - \sum_1^{n-1} x_i\right) dx_1 \cdots dx_{n-1}}.$$

In the case where $d(a \mid x_1, x_2, \cdots, x_n)$ is increasing in each of the x_i, for some value of a, it follows from the theorem that $d*(a \mid s)$ will be an increasing function of s.

3. There are other multivariate distributions which share the increasing properties discussed above. For instance, given any multivariate normal distribution F, the conditional distribution of F in the plane $S = s$ is normal with covariance independent of s, and mean sV (where V is a vector such that sV always lies in the plane $S = s$). Thus F will satisfy the conclusion of the main theorem if and only if V lies in the positive orthant.

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