A RECURRENCE FOR PERMUTATIONS WITHOUT RISING OR FALLING SUCCESSIONS

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1. Introduction. For n elements, the rising successions in question are $12, 23, \cdots, \overline{n-1n}$; the falling successions are $21, 32, \cdots, \overline{nn-1}$. The enumeration of the permutations of the title has been considered by Irving Kaplansky [1] in the form of what he calls the "n-kings problem": in how many ways may n kings be placed on an n by n chessboard so that no two attack each other? In a later paper [2], he has treated the more general problem of enumerating permutations of n elements by the number of successions of either kind (more briefly, by the number of instances in which i is next to $i+1$, $i = 1, 2, \cdots, n-1$). If S_{nk} is the typical number of such an enumeration, $S_n(t) = \sum S_{nk}t^k$ is called the enumerator (of permutations by number of successions); $S_n(t)$ is a polynomial in t of degree $n-1$.

It will be shown that

$$(1)\quad \begin{aligned} S_n(t) &= (n+1-t)S_{n-1}(t) - (1-t)(n-2+3t)S_{n-2}(t) \\ &\quad - (1-t)^2(n-5+t)S_{n-3}(t) + (1-t)^3(n-3)S_{n-4}(t), \quad n > 3 \end{aligned}$$

with $S_0(t) = S_1(t) = 1$, $S_2(t) = 2t$, $S_3(t) = 4t + 2t^2$. Recurrence (1) has the particular virtue of reducing to the following pure recurrence for the numbers of the title, $S_n = S_n(0)$:

$$(2)\quad \begin{aligned} S_n &= (n+1)S_{n-1} - (n-2)S_{n-2} \\ &\quad - (n-5)S_{n-3} + (n-3)S_{n-4}, \quad n > 3. \end{aligned}$$

2. Preliminary résumé. The results of [1] and [2] needed for present purposes are as follows:

$$(3)\qquad S_n(t) = \textstyle\sum_{k=0}^n A_{nk}(n-k)!(t-1)^k,$$

where

$$(4)\qquad A_{nk} = A_{n-1,k} + A_{n-1,k-1} + A_{n-2,k-1}, \qquad n > 1$$

or

$$(5)\qquad A_n(x) = \textstyle\sum_{k=0}^n A_{nk}x^k = (1+x)A_{n-1}(x) + xA_{n-2}(x)$$

where, by convention, $A_0(x) = A_1(x) = 1$. It following at once from (3) and (4) that (primes denote derivatives)

$$(6)\quad \begin{aligned} S_n(t) &= (n-1+t)S_{n-1}(t) \\ &\quad + (1-t)S'_{n-1}(t) - (n-1)(1-t)S_{n-2}(t) - (1-t)^2S'_{n-2}(t), \quad n > 1 \end{aligned}$$

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with $S_0(t) = S_1(t) = 1$, as above. Equation (6) implies

$$(7) \quad S_{nk} = S_{n-1,k-1} + (n-1-k)S_{n-1,k} + (k+1)S_{n-1,\,k+1} + (n-k)S_{n-2,\,k-1} - (n-1-2k)S_{n-2,k} - (k+1)S_{n-2,\,k+1}\,.$$

In particular ($S_n = S_{n0}$), $S_n - (n-1)S_{n-1} + (n-1)S_{n-2} = S_{n-1,1} - S_{n-2,1}$ which is not a pure recurrence. It is convenient to introduce "associated" polynomials of $A_n(x)$, namely $a_n(x) = x^n A_n(-x^{-1})$, in terms of which $S_n(t)$ may be written compactly as

$$(3a) \qquad S_n(t) = (1-t)^n a_n[E(1-t)^{-1}]0!, \qquad E^k 0! = k!.$$

3. Generating function and recurrence of associated polynomials. By definition and by equation (5), $a_0 = 1$, $a_1(x) = x$, and

$$(8) \qquad a_n(x) = (x-1)a_{n-1} - xa_{n-2}\,, \qquad n > 1.$$

Hence the generating function $a(x, y) = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n + \cdots$ is given by

$$(9) \qquad (1 + y - xy + xy^2)a(x, y) = 1 + y.$$

It follows that, indicating partial derivatives by suffixes,

$$(10) \qquad (1 + y - xy + xy^2)a_x(x, y) - y(1-y)a(x, y) = 0,$$
$$(1 + y - xy + xy^2)a_y(x, y) + (1 - x + 2xy)a(x, y) = 1.$$

Combining the latter with (9) leads to

$$(11) \quad (1+y)(1 + y - xy + xy^2)a_y(x, y) = x(1 - 2y - y^2)a(x, y),$$

and (11) and the first of (10) lead to

$$(12) \qquad x(1 - 2y - y^2)a_x(x, y) = y(1 - y^2)a_y(x, y).$$

Now rewrite the first of (10) as $(1 + y)a_x(x, y) = y(1-y)a(x\ y) + xy(1-y)a_x(x, y)$ then, multiplying throughout by $(1 - 2y - y^2)$, it is found that

$$(13) \quad (1+y)(1-2y-y^2)a_x(x, y) = y(1-y)(1-2y-y^2)a(x, y) + y^2(1-y)(1-y^2)a_y(x, y),$$

a relation with coefficients free of x.

Equating coefficients of y^n leads to the final recurrence (again, primes denote derivatives)

$$(14) \quad na_{n-1}(x) - (n+1)a_{n-2}(x) - (n-4)a_{n-3}(x) + (n-3)a_{n-4}(x) = a_n'(x) - a_{n-1}'(x) - 3a_{n-2}'(x) - a_{n-3}'(x).$$

4. Recurrence for $S_n(t)$. The left-hand side of (14) may be translated into an expression in $S_n(t)$ by use of (3a); for the right-hand side there is a similar expression, namely

$$(15)\qquad (1-t)^{n-1}a_n'[E(1-t)^{-1}]0! = \sum_{k=0}^{n-1} A_{nk}(n-k)!(t-1)^k = S_n(t) - (1-t)^n a_n(0)$$

and it is easy to see from (8) and its boundary condition that $a_n(0) = \delta_{n0}$. Hence, omitting functional arguments,

$$(16)\qquad \begin{aligned} &nS_{n-1} - (1-t)(n+1)S_{n-2} - (1-t)^2(n-4)S_{n-3} \\ &+ (1-t)^3(n-3)S_{n-4} = S_n - (1-t)S_{n-1} - 3(1-t)^2S_{n-2} \\ &- (1-t)^3S_{n-3} - (1-t)^n(\delta_{n0} - \delta_{n-1,0} - 3\delta_{n-2,0} - \delta_{n-3,0}). \end{aligned}$$

After simplification, (16) is (1).

Professor Carlitz (private communication) has noticed that (1) implies

$$(17)\qquad S(t, u) = \sum_{n=0}^{\infty} S_n(t)u^n = \sum_{k=0}^{\infty} k!u^k\{[1-(1-t)u]/[1+(1-t)u]\}^k.$$

Equation (17) in its turn implies a new expression for A_{nk} namely

$$A_{nk} = \sum_{j=0}^{k} \binom{n-k}{k-j}\binom{n+j-k-1}{j}.$$

REFERENCES

[1] KAPLANSKY, IRVING (1944). Symbolic solution of certain problems in permutations. *Bull. Amer. Math. Soc.* **50** 906–914.

[2] KAPLANSKY, IRVING (1945). The asymptotic distribution of runs of consecutive elements. *Ann. Math. Statist.* **16** 200–203.