LIMITING DISTRIBUTIONS OF RESPONSE PROBABILITIES

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1. Introduction and summary. In an earlier paper [3], one of us considered certain limiting distributions of response probabilities arising from the two experimenter-controlled events learning model of Bush and Mosteller [1]. There, as here, the probabilistic model took the form of a Markov process $p_0, p_1 \cdots$ satisfying the following conditions:

(i) p_0 has an arbitrary distribution on $(0, 1)$;

(ii) if p_n is given, then $p_{n+1} = a_1 + \alpha_1 p_n$ with probability π_1 and $p_{n+1} = a_2 + \alpha_2 p_n$ with probability π_2;

(iii) $\pi_1 + \pi_2 = 1$, $0 \leqq a_j \leqq 1$ and $0 \leqq \alpha_j \leqq 1 - a_j$, $(j = 1, 2)$.

The random variable p_n is called "the response probability on trial n."

It has been shown by Karlin [2] that a limiting distribution exists as $n \to \infty$. If p is the random variable of this limiting distribution, it can be shown ([1], p. 98) that the distribution of p is concentrated on $[\min(\lambda_1, \lambda_2), \max(\lambda_1, \lambda_2)]$, where $\lambda_1 = a_1/(1 - \alpha_1)$ and $\lambda_2 = a_2/(1 - \alpha_2)$.

In the present note, it is shown that for the case $\alpha_1 = \alpha_2 = \alpha$, $\pi_1 = \pi_2 = \frac{1}{2}$, the characteristic function of the distribution of the random variable

$$x = (\lambda_1 + \lambda_2 - 2p)/(\lambda_2 - \lambda_1), \qquad (\lambda_1 \neq \lambda_2), \tag{1.1}$$

when suitably standardized, tends to the characteristic function of the standardized normal distribution as $\alpha \to 1$.

2. Asymptotic normality as $\alpha \to 1$. For the case $\alpha_1 = \alpha_2 = \alpha$, $\pi_1 = \pi_2 = \frac{1}{2}$, it follows from results of Bush and Mosteller [1] that $E(p) = (\lambda_1 + \lambda_2)/2$, and $\text{Var}(p) = (1 - \alpha)(\lambda_2 - \lambda_1)^2/4(1 + \alpha)$. Using these in conjunction with equation (1.1), we obtain $E(x) = 0$ and $\text{Var}(x) = (1 - \alpha)/(1 + \alpha)$. Thus the random variable $z = x[(1 + \alpha)/(1 - \alpha)]^{\frac{1}{2}}$ is standardized.

McGregor and Hui [3] have shown that the characteristic function of the distribution of x is

$$\varphi_x(t) = \prod_{n=0}^{\infty} \cos[(1 - \alpha)\alpha^n t] = \cos[(1 - \alpha)t]\varphi_x(\alpha t).$$

Thus the characteristic function of the distribution of z satisfies

$$\varphi_z(t) = \cos[(1 - \alpha^2)^{\frac{1}{2}} t]\varphi_z(\alpha t). \tag{2.1}$$

Since $\varphi_z(t)$ is an even function of t, we may write

$$\varphi_z(t) = \sum_{r=0}^{\infty} \mu_{2r}(-1)^r t^{2r}/(2r)!, \tag{2.2}$$

where $\mu_{2r} = E(z^{2r})$. Combining equations (2.1) and (2.2), we have

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$$\sum_{r=0}^{\infty} \mu_{2r}(-1)^r t^{2r}/(2r)!$$
$$= \{\sum_{r=0}^{\infty} \mu_{2r}(-1)^r(\alpha t)^{2r}/(2r)!\}\{\sum_{k=0}^{\infty} (-1)^k t^{2k}(1-\alpha^2)^k/(2k)!\}$$
$$= \sum_{r=0}^{\infty} (-1)^r t^{2r}\{\sum_{k=0}^{r} \mu_{2(r-k)}\alpha^{2(r-k)}(1-\alpha^2)^k/(2r-2k)!(2k)!\}.$$

Thus

$$\mu_{2r} = (2r)!\sum_{k=0}^{r} \mu_{2(r-k)}\alpha^{2(r-k)}(1-\alpha^2)^k/(2r-2k)!(2k)!,$$

or

$$\mu_{2r} = (2r)!\sum_{k=1}^{r} \mu_{2(r-k)}\alpha^{2(r-k)}(1-\alpha^2)^k/(2r-2k)!(2k)!(1-\alpha^{2r}).$$

Let $\mu_{2r}^* = \lim_{\alpha\to1}\mu_{2r}$. Then $\mu_{2r}^* = (2r-1)\mu_{2r-2}^* = \prod_{k=0}^{r-1}(2r-2k-1) = (2r)!/r!2^r$. Thus $\mu_{2r}^*/(2r)! = [r!2^r]^{-1}$ and the characteristic function of the distribution of z obtained by letting $\alpha \to 1$ is, therefore,

$$\sum_{r=0}^{\infty} (-t^2/2)^r/r! = \exp(-t^2)/2,$$

which is the characteristic function of the standardized normal distribution.

REFERENCES

[1] Bush, Robert R. and Mosteller, Fredrick (1955). *Stochastic Models for Learning.* Wiley, New York.

[2] Karlin, Samuel (1953). Some random walks arising in learning models I. *Pacific J. Math.* **3** 725–756.

[3] McGregor, John R. and Hui, Y. Y. (1962). Limiting distributions associated with certain stochastic learning models. *Ann. Math. Statist.* **33** 1281–1285.