ON THE COMPLEX ANALOGUES OF T^2- AND R^2-TESTS[1]

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0. Introduction and Summary. Let ξ be a complex Gaussian random variable with mean $E(\xi) = \alpha$ and Hermitian positive definite complex covariance matrix $\Sigma = E(\xi - \alpha)(\xi - \alpha)^*$, where $(\xi - \alpha)^*$ is the adjoint of $(\xi - \alpha)$. Its probability density function is given by

$$(0.1) \qquad p(\xi \mid \alpha, \Sigma) = \pi^{-p}(\det \Sigma)^{-1} \exp[-(\xi - \alpha)^*\Sigma^{-1}(\xi - \alpha)],$$

with $E(\xi - \alpha)(\xi - \alpha)' = 0$. Write

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^* & \Sigma_{22} \end{pmatrix},$$

where Σ_{22} is the $(p - 1) \times (p - 1)$ lower right-hand submatrix of Σ.

Goodman (1963) found the maximum likelihood estimate of Σ and $\rho^2 = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^*/\Sigma_{11}$ when $\alpha = 0$ and also found the distributions of these estimates. The problems considered here are of

(i) testing the hypothesis $H_{01} : \alpha = 0$ that the mean of the vector ξ is 0 against the alternative $H_1 : \alpha^*\Sigma^{-1}\alpha > 0$ and

(ii) of testing the hypothesis $H_{02} : \Sigma_{12} = 0$ that the first component of ξ is independent of the others against the alternative $H_2 : \rho^2 > 0$.

Since likelihood ratio test has some optimum properties and has been found satisfactory for similar problems in the real case, we find the likelihood ratio tests of these problems and show that these tests possess certain optimum properties which are counterparts of the real case. These results will be presented in Sections 3 and 4. Section 1 deals with some known results of complex matrix algebra. In Section 2, we will prove some preliminary results which are useful for complex Gaussian statistical analysis. For an application of these results the reader is referred to Goodman (1963).

It may be remarked here that the likelihood ratio test is invariant under all transformations which leave the problem invariant and may be obtained from the densities of maximal invariant under the null hypothesis and the alternative.

1. Algebraic preliminaries. Our development relies on some results of complex matrix algebra and we will list them in this section, without any proof, in the form of lemmas. The materials summarized here can be found, for example, in MacDuffee (1946). In what follows, we will denote a diagonal matrix with diagonal elements $\lambda_1, \cdots, \lambda_p$ by $L(\lambda_1, \cdots, \lambda_p)$.

Lemma 1.1. *If H is a $p \times p$ Hermitian matrix, then there exists an unitary*

Received 22 October 1963; revised 2 November 1964.

[1] Research supported by ONR Contract No. Nonr-401(03) at Cornell University.

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$p \times p$ matrix U such that $U^*HU = L(\lambda_1, \cdots, \lambda_p)$ where λ_i $(i = 1, \cdots, p)$ are the characteristic roots of H.

LEMMA 1.2. *A Hermitian matrix is positive definite if all its characteristic roots are positive.*

LEMMA 1.3. *Every Hermitian positive definite matrix (semidefinite) H is uniquely expressible as $H = BB^*$ where B is Hermitian positive definite (semidefinite).*

LEMMA 1.4. *For every Hermitian positive definite matrix H, there exists a complex non-singular matrix B such that $BHB^* = I$ (identity matrix).*

2. Some theorems.

THEOREM 2.1. *Suppose $f(\Sigma) = C(\det \Sigma)^n \exp[-\mathrm{tr}\ \Sigma]$, where Σ is positive definite Hermitian and C is a positive constant. $f(\Sigma)$ is maximum at $\Sigma = \hat{\Sigma} = nI$.*

PROOF. Since Σ is positive definite Hermitian, by Lemmas 1.1, 1.2, there exists an unitary matrix U such that $U^*\Sigma U = L(\lambda_1, \cdots, \lambda_p)$ with $\lambda_i > 0$. Hence

$$\begin{aligned} f(\Sigma) &= C(\det (U^*\Sigma U))^n \exp[-\mathrm{tr}\ \{U^*\Sigma U\}] \\ &= C\textstyle\prod_{i=1}^{p} (\lambda_i \exp[-\lambda_i/n])^n, \end{aligned}$$

which is maximum if $\lambda_i = n$, $i = 1, \cdots, p$. Hence $f(\Sigma)$ is maximum at $\Sigma = \hat{\Sigma} = nI$.

THEOREM 2.2. *Let ξ be a p-variate Gaussian random variable with mean α and complex positive definite Hermitian covariance matrix Σ. Then $2\xi^*\Sigma^{-1}\xi$ is distributed as $\chi^2_{2p}(2\alpha^*\Sigma^{-1}\alpha)$, where $\chi^2_{2p}(\beta)$ is a non-central–chi-square with $2p$ degrees of freedoms and non-centrality parameter $\beta = E(\chi^2_{2p}(\beta)) - 2p$.*

PROOF. Let $\eta = C\xi$, where C is a $p \times p$ non-singular complex matrix, such that $C\Sigma C^* = I$. It is easy to check that η is distributed as a p-variate Gaussian random variable with mean $c\alpha = \beta$ and covariance matrix I. Writing $\eta = (\eta_1, \cdots, \eta_p)'$ with $\eta_j = X_j + iY_j$ and $\beta = (\beta_1, \cdots, \beta_p)'$ with $\beta_j = \beta_{jR} + i\beta_{jI}$, we obtain from (0.1) and above that the $2p$ random variables $X_1 - \beta_{1R}, \cdots, X_p - \beta_{pR}, Y_1 - \beta_{1I}, \cdots, Y_p - \beta_{pI}$ are independently and identically distributed normal random variables with mean 0 and variance $\frac{1}{2}$. Hence $\sum_{i=1}^{p2} [(X_i^2 + Y_i^2)] = 2\eta^*\eta = 2\xi^*\Sigma^{-1}\xi$ is distributed as $\chi^2_{2p}(\lambda)$ where $\lambda = 2\sum_{i=1}^{p} (\beta^2_{iR} + \beta^2_{iI}) = 2\beta^*\beta = 2\alpha^*\Sigma^{-1}\alpha$.

THEOREM 2.3. *Consider N independent identically distributed p-variate complex Gaussian random variables ξ_j, $j = 1, \cdots, N$ as a sample of size N from a population with* pdf *given by (0.1). The maximum likelihood estimates $\hat{\alpha}$, $\hat{\Sigma}$ of α, Σ respectively are given by*

$$\begin{aligned} N\hat{\alpha} &= \textstyle\sum_{i=1}^{N} \xi_i = N\bar{\xi}, \\ N\hat{\Sigma} &= \textstyle\sum_{i=1}^{N} (\xi_i - \bar{\xi})(\xi_i - \bar{\xi})^* = A. \end{aligned}$$

PROOF. From (0.1), the pdf $p(\xi_1, \cdots, \xi_N)$ of $\xi_1, \cdots, \xi_N$ is

$$p(\xi_1, \cdots, \xi_N) = \pi^{-Np}(\det \Sigma)^{-N} \exp[-\mathrm{tr}\ \Sigma^{-1}\{\textstyle\sum_{i=1}^{N} (\xi_i - \alpha)(\xi_i - \alpha)^*\}].$$

Now

$$\textstyle\sum_{i=1}^{N} (\xi_i - \alpha)(\xi_i - \alpha)^* = \sum_{i=1}^{N} (\xi_i - \bar{\xi})(\xi_i - \bar{\xi})^* + N(\bar{\xi} - \alpha)(\bar{\xi} - \alpha)^*.$$

Hence $\max_{\alpha,\Sigma} p(\xi_1, \cdots, \xi_N) = \max_\Sigma \pi^{-pN}(\det \Sigma)^{-N} \exp[-\operatorname{tr} \Sigma^{-1}A]$; and the maximum likelihood estimate of α is $\bar{\xi}$. Let us assume that A is positive definite Hermitian which we can do with probability 1. By Lemma 1.3,

$$\begin{aligned}\max_{\alpha,\Sigma} p(\xi_1, \cdots, \xi_N) &= \max_B \pi^{-pN}(\det \Sigma)^{-N} \exp[-\operatorname{tr} \Sigma^{-1}BB^*]\\ &= \max_B \pi^{-pN}(\det (BB^*))^{-N}[\det (B^*\Sigma^{-1}B)]^N \times \exp[-\operatorname{tr} (B^*\Sigma^{-1}B)],\end{aligned}$$

where B is a nonsingular $p \times p$ complex matrix such that $A = BB^*$. By Theorem 2.1, the maximum likelihood estimate of Σ is $\hat{\Sigma} = N^{-1}BB^* = N^{-1}A$.

THEOREM 2.4. *$N^{\frac{1}{2}}\bar{\xi}$, A are independent in distribution. $N^{\frac{1}{2}}\bar{\xi}$ has a p-variate complex Gaussian distribution with mean $N^{\frac{1}{2}}\alpha$ and complex covariance Σ; A is a complex Wishart $W_c(\Sigma, N, p)$ with* pdf.

$$p(A) = [\det (A)]^{N-p-1}/I(\Sigma) \exp[-\operatorname{tr} \Sigma^{-1}A], \tag{2.1}$$

where $I(\Sigma) = \pi^{p(p-1)/2}\prod_{i=1}^{p} \Gamma(N - i)(\det (\Sigma))^{N-1}$.

PROOF. Let $U = (U_{\alpha j})$ be a $N \times N$ unitary matrix such that the first row is $(N^{-\frac{1}{2}}, \cdots, N^{-\frac{1}{2}})$. Consider the transformation from $(\xi_1, \cdots, \xi_N)$ to $(\eta_1, \cdots, \eta_N)$ given by

$$\begin{aligned}\eta_1 &= N^{\frac{1}{2}}\bar{\xi},\\ \eta_\alpha &= \textstyle\sum_{j=1}^{N} U_{\alpha j}\xi_j, \qquad \alpha = 2, \cdots, N.\end{aligned}$$

Now η_α for each α, is being a complex valued linear function of ξ_α, $\alpha = 1, \cdots, N$; has complex Gaussian distribution. This follows from the definition of complex Gaussian distribution and the fact that the Jacobian of any nonsingular complex transformation: $\xi \rightarrow B\xi$ is $\det (BB^*)$. It is easy to check that

$$\begin{aligned}E(\eta_\alpha) &= 0, && \alpha = 2, \cdots, N;\\ E(\eta_1) &= N^{\frac{1}{2}}\alpha;\\ \operatorname{cov}(\eta_i\eta_j^*) &= 0, && \text{if } i \neq j;\\ &= \Sigma, && \text{if } i = j;\end{aligned}$$

and

$$\textstyle\sum_{i=1}^{N} (\xi_i - \bar{\xi})(\xi_i - \bar{\xi})^* = \sum_{i=2}^{N} \eta_i\eta_i^*.$$

Furthermore, it is easy to see that uncorrelatedness in the complex case also implies independence. Thus $N^{\frac{1}{2}}\xi$, A are independent in distribution. By Theorem 5.1, Goodman (1963), it follows that A has complex Wishart distribution with pdf given by (2.1). The mean and complex covariance matrix of $N^{\frac{1}{2}}\bar{\xi}$ are $N^{\frac{1}{2}}\alpha$, and Σ respectively. Hence the theorem.

REMARK.

$$p(\xi_1, \cdots, \xi_N) = \pi^{-pN}(\det \Sigma)^{-N} \exp[-\operatorname{tr} \{\Sigma^{-1}[A + N(\bar{\xi} - \alpha)(\bar{\xi} - \alpha)^*\}].$$

It thus follows from Neyman's criterion for sufficient statistic that $(\bar{\xi}, A)$ is sufficient for (α, Σ), (see Halmos and Savage (1949)).

3. Likelihood ratio test of H_{01} against H_1. Let $\xi_1, \cdots, \xi_N$ be a sample of N observations from $p(\xi \mid \alpha, \Sigma)$. We want to test the hypothesis $H_{01} : \alpha = 0$ against the alternative $H_1 : \alpha^*\Sigma^{-1}\alpha > 0$ on the basis of these observations. The likelihood ratio test consists in rejecting H_{01} if

$$\lambda = \max_{H_1} p(\xi_1, \cdots, \xi_N)/\max_{H_{01}} p(\xi_1, \cdots, \xi_N)$$

is greater than some predetermined constant depending on the size of the test. Applying Theorem 2.1, we obtain,

$$\lambda = (1 + N\bar{\xi}^*A^{-1}\bar{\xi})^N. \tag{3.1}$$

Thus, the likelihood ratio test of H_{01} against the alternative H_1 is given by $T_c^2 = N\bar{\xi}^*A^{-1}\bar{\xi} > k$, where k is a constant and is determined in such a way that the test has size α. To determine the constant k we now need the distribution of T_c^2 under H_{01}. Since we also need the distribution of T_c^2 under H_1 for later developments, we may, as well, find it, the distribution under H_{01} will follow from it immediately.

THEOREM 3.1. *T_c^2 under H_1 is distributed as the ratio $\chi^2_{2p}(2N\alpha^*\Sigma^{-1}\alpha)/\chi^2_{2(N-p)}$, where $\chi^2_{2p}(2N\alpha^*\Sigma^{-1}\alpha)$ is a non-central chi-square with $2p$ degrees of freedom and noncentrality parameter $2N\alpha^*\Sigma^{-1}\alpha$, $\chi^2_{2(N-p)}$ is a central chi-square with $2(N - p)$ degrees of freedom and is independent of $\chi^2_{2p}(2N\alpha^*\Sigma^{-1}\alpha)$ in distribution.*

PROOF. It may be checked that T_c^2 is a maximal invariant in the space of sufficient statistic $(\bar{\xi}, A)$ under the full linear group G of $p \times p$ non-singular complex matrices under multiplication which keep the problem of testing H_{01} against H_1 invariant in the usual fashion. The maximal invariant in the parametric space of (α, Σ) under this group is $\eta^2 = N\alpha^*\Sigma^{-1}\alpha$. Thus the distribution of T_c^2 depends on the parameters α and Σ, only through η^2. Hence we may without any loss of generality assume $\Sigma = I$ and redefine α such that $N\alpha^*\alpha = \eta^2$.

Let $Y = N^{\frac{1}{2}}\bar{\xi}$ and Q be an unitary matrix with first row $Y_1^*/(Y^*Y)^{\frac{1}{2}}, \cdots, Y_p^*/(Y^*Y)^{\frac{1}{2}}$ and other rows are defined arbitrary. Writing $U = QY$ and $B = QAQ^*$, we obtain $T_c^2 = U^*B^{-1}U = U_1U_1^*/(B_{11} - B_{12}B_{22}^{-1}B_{12}^*)$ where B is partitioned as $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$ with B_{22}, a $(p - 1) \times (p - 1)$ lower righthand submatrix of B. Furthermore, let A be similarly partioned into submatrices A_{ij} and $\xi_{[2]} = (\xi_2, \cdots, \xi_p)'$. Now,

$$\xi^*A\xi = (\xi_{[2]} + A_{22}^{-1}A_{12}^*\xi_1)^*A_{22}(\xi_{[2]} + A_{22}^{-1}A_{12}^*\xi_1) + \xi_1^*(A_{11} - A_{12}A_{22}^{-1}A_{12}^*)\xi_1.$$

Hence A is Hermitian positive definite iff. A_{22} and $A_{11} - A_{12}A_{22}^{-1}A_{21}$ are Hermitian positive definite. From (2.1), taking $\Sigma = I$ (the identity matrix), the joint pdf of A_{22}, A_{12} and $H = (A_{11} - A_{12}A_{22}^{-1}A_{21})$ is

$$I_0^{-1}(\det A_{22})^{N-p-1}(\det H)^{N-p-1} \exp [-\text{tr} \{H + A_{12}A_{22}^{-1}A_{12}^* + A_{22}\}]$$

where I_0 is the value of $I(\Sigma)$ with $\Sigma = I$. Thus it follows that H is independent of A_{22} and A_{12}, and is distributed as $W_c(1, N - p + 1, 1)$.

The conditional distribution of B, given Q, is that of $\sum_{\alpha=2}^{N} V_\alpha V_\alpha^*$ where, conditionally $V_\alpha = Q\eta_\alpha$ are independent and each has complex p-variate Gaussian distribution with mean 0 and covariance matrix I. Hence $B_{11} - B_{12}B_{22}^{-1}B_{12}^*$ is conditionally distributed as $\sum_{\alpha=1}^{N-p+1} W_\alpha W_\alpha$, where, conditionally W_α are independent and each has single variate complex Gaussian distribution with mean 0 and variance 1. By Theorem 2.2 and the fact that sum of independent chi-squares is a chi-square, it follows that $2(B_{11} - B_{12}B_{22}^{-1}B_{12}^*)$ is conditionally $\chi^2_{2(N-p)}$. Since this distribution does not depend on Q, $2(B_{11} - B_{12}B_{22}^{-1}B_{12}^*)$ is unconditionally $\chi^2_{2(N-p)}$. The quantity $2Y^*Y$, by Theorem 2.2, is $\chi^2_{2p}(2N\alpha^*\Sigma^{-1}\alpha)$. Hence the theorem.

THEOREM 3.2. *On the basis of observations* $\xi_1, \cdots, \xi_N$ *from the p-variate complex Gaussian distribution with mean* α *and complex covariance matrix* Σ; *of all level* β_0 *tests of* H_{01} *against the alternative* H_1, *which are invariant under the group G of transformations, the test based on* T_c^2 *is uniformly most powerful invariant.*

The proof of this theorem follows from Theorem 3.1 and the fact that we need only consider test functions based on the sufficient statistic $(\bar{\xi}, A)$.

THEOREM 3.3 *Of all level* β_0 *tests of* H_{01} *against* H_1 *with power functions depending on* η^2, *the test based on* T_c^2 *is uniformly most powerful.*

PROOF. As remarked in the preceding theorem, we may consider tests which are functions of $(\bar{\xi}, A)$ only. Let $\phi(\bar{\xi}, A)$ be any level α test of H_{01} against H_1 with power function depending on η^2 only. So $E_{H_1}\phi(\bar{\xi}, A) = E_{\alpha,\Sigma}\phi(\bar{\xi}, A) = E_{g^{-1}\alpha, g^{-1}\Sigma g^{-1*}}\phi(\bar{\xi}, A) = E_{\alpha,\Sigma}\phi(g\bar{\xi}, gAg^*)$ for $g \in G$. Thus,

$$E_{\alpha,\Sigma}[\phi(\bar{\xi}, A) - \phi(g\bar{\xi}, gAg^*)] = 0 \tag{3.2}$$

identically in α, Σ. Now, writing $\Sigma^{-1}\alpha = \theta = \theta_R + i\theta_I$, $\bar{\xi} = \bar{x} + i\bar{y}$, $A = A_R + iA_I$ and $\Sigma^{-1} = (I - \bar{\theta})$ where $\bar{\theta} = \bar{\theta}^*$, we obtain,

$$N \operatorname{tr} \Sigma^{-1}(\xi - \alpha)(\xi - \alpha)^*$$
$$= \operatorname{tr} N\Sigma^{-1}\{(\bar{x}\bar{x}' + \bar{y}\bar{y}') + i(\bar{y}\bar{x}' - \bar{x}\bar{y}') + \alpha\alpha^* - 2\theta_R\bar{x}' - 2\theta_I\bar{y}'\}.$$

Hence,

$$\exp[-\operatorname{tr} \Sigma^{-1}\{A + N(\bar{\xi} - \alpha)(\bar{\xi} - \alpha)^*]$$
$$= \exp[-\operatorname{tr}\{A + N(\bar{x}\bar{x}' + \bar{y}\bar{y}') + N\theta^*\Sigma\theta\}] \exp[\operatorname{tr}\{\bar{\theta}(A_R + N\bar{x}\bar{x}' + N\bar{y}\bar{y}') + i\bar{\theta}(A_I + N\bar{y}\bar{x}' - N\bar{x}\bar{y}') + 2\theta_R N\bar{x}' + 2\theta_I N\bar{y}'\}].$$

Let

$$g(\bar{\xi}, A) = [\phi(\bar{\xi}, A) - \phi(g\bar{\xi}, gAg^*)]$$
$$\cdot(\det(A))^{N-p-1} \exp[-\operatorname{tr}(A + N\bar{x}\bar{x}' + N\bar{y}\bar{y}')]$$
$$= h_1(\bar{x}, \bar{y}, A_R, A_I) + ih_2(\bar{x}, \bar{y}, A_R, A_I),$$

where h_1, h_2 are respectively the real and imaginary parts of g. Now from (3.2) we obtain,

$$(3.3) \quad \int h_j(\bar{x}, \bar{y}, A_R, A_I) \exp [\text{tr } \bar{\theta}\{(A_R + N\bar{x}\bar{x}' + N\bar{y}\bar{y}') + i\bar{\theta}(A_I + N\bar{y}\bar{x}' - N\bar{x}\bar{y}') + 2\theta_R N\bar{x}' + 2\theta_I N\bar{y}'\}] \, dA_R \, dA_I \, d\bar{x} \, d\bar{y} = 0, \quad j = 1, 2.$$

For each j, this is the Laplace transform of h_j with respect to the variables $A_R + N\bar{x}\bar{x}' + N\bar{y}\bar{y}'$, $A_I + N\bar{y}\bar{x}' - N\bar{x}\bar{y}'$, $N\bar{x}$ and $N\bar{y}$. Since this is zero, we get $h_j = 0$, $(j = 1, 2)$, except for a set of measure zero. Hence $\phi(\bar{\xi}, A) = \phi(g\bar{\xi}, gAg^*)$, $g \varepsilon G$ a.e., i.e. ϕ is almost invariant under G. It may be checked that a right invariant measure in G is $dg/[\det (gg^*)]^{p/2}$. Hence from Lehmann (1959), p. 226, ϕ is invariant under G.

4. Likelihood ratio test of H_{02} against H_2. On the basis of $\xi_1, \cdots, \xi_N$ the likelihood ratio for testing H_{02} against H_2 is

$$\begin{aligned}\lambda &= \max_{H_{02}} p(\xi_1, \cdots, \xi_N)/\max_{H_2} p(\xi_1, \cdots, \xi_N) \\ &= \max_{\Sigma_{12}=0} (\det (\Sigma))^{-N} \exp [-\text{tr } \Sigma^{-1}A]/\max (\det(\Sigma))^{-N} \exp [-\text{tr } \Sigma^{-1}A] \\ &= [(A_{11} - A_{12}A_{22}^{-1}A_{12}^*)/A_{11}]^N \qquad \text{(By Theorem 2.1)}, \\ &= (1 - R_c^2)^N,\end{aligned}$$

where $R_c^2 = A_{12}A_{22}^{-1}A_{12}^*/A_{11}$. Hence the likelihood ratio test of H_{02} against H_2, at significance level β_0, is defined by the critical region $R_c^2 \geqq k$, where k is a constant and is chosen so that the probability of $R_c^2 \geqq k$ under the null hypothesis is equal to β_0. It is clear that the transformations $(\alpha, \Sigma, \bar{\xi}, A) \rightarrow (\alpha + c, \Sigma, \bar{\xi} + c, A)$, with c a complex number, leave this problem invariant. The action of these transformations is to reduce the problem to that where $\alpha = 0$ (known) and $A = \sum_{i=1}^{N} \xi_i\xi_i^*$ is sufficient, where N has been reduced by one from what it was originally. We therefore treat this latter formulation and consider $\xi_1, \cdots, \xi_N$ to have zero mean.

Let G_1 be the group of $p \times p$ nonsingular complex matrices whose first column and first row contain only zeroes except for the first element. It is easily seen that this group, operating as $(A, \Sigma) \rightarrow (gAg^*, g\Sigma g^*)$, leaves the problem invariant and a maximal invariant in the space of A under G_1 is R_c^2. From Goodman (1963), the probability density function of R_c^2 under H_2 is given by

$$(4.1) \quad p(R_c^2) = \{\Gamma(N - 1)/\Gamma(p - 1)\Gamma(N - p)\}(1 - \rho^2)^{N-1} \cdot (R_c^2)^{p-2}(1 - R_c^2)^{N-p-1}F(N - 1, N - 1; p - 1; R_c^2\rho^2);$$

where $F(,;;)$ denotes the hypergeometric series.

The development now parallels that of Section 3. Proceeding along the same line and using (4.1), we may get the following:

THEOREM 4.1. *The test based on R_c^2 is uniformly most powerful invariant for* *testing* H_{02} against H_2 *among the class of level* β_0 *tests, which are invariant with respect to* G_1.

THEOREM 4.2. *Of all level* β_0 *tests of* H_{02} *against* H_2 *with power function depending on* ρ^2, *the test based on* R_c^2 *is uniformly most powerful.*

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