ON THE ASYMPTOTIC BEHAVIOR OF DENSITIES WITH APPLICATIONS TO SEQUENTIAL ANALYSIS[1]

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Summary. $\{X_n\}$ is a sequence of random variables defined on some probability space, $\mathcal{P} = \{P_\theta, \theta \varepsilon \Theta\}$ is the family of distributions of $\{X_n\}$ and $\mathcal{A}_n$ is the subfield generated by $(X_j, j \leqq n)$. It is assumed that Θ is a real interval, $X_n \to \theta$ a.e. P_θ and that, for each n, X_n is sufficient for $\mathcal{P}$ on $\mathcal{A}_n$ while $\mathcal{P}$ is a homogeneous monotone likelihood ratio family on $\mathcal{A}_n$. Let $p_{\theta n}$ denote the density of P_θ with respect to some σ-finite measure on $\mathcal{A}_n$ and consider the sequence $\{R_n\}$, where $R_n = p_{\theta_2 n}/p_{\theta_1 n}$ and $\theta_1 < \theta_2$. Conditions are given for the occurrence of the following limiting behavior of R_n: (1) there exists a θ_0, $\theta_1 < \theta_0 < \theta_2$, such that $R_n \to 0$ or ∞ a.e. P_θ according as $\theta <$ or $> \theta_0$ and (2) $P_{\theta_0}(b < R_n < a) \to 0$ for $0 < b < a < \infty$. This limiting behavior of R_n guarantees the termination with probability one of sequential probability ratio tests based on $\{X_n\}$. Let $q_{\theta n}$ denote the density of P_θ on the Borel field generated by X_n. In order to describe the results of this paper we introduce Condition A_1 which states, essentially, that $n^{-1} \ln [q_{\theta n}(x)/K(n)] \to h(\theta, x)$ for some functions K and h satisfying very mild conditions. Condition A_2 requires $h(\theta, x)$ to possess, for each fixed x, a unique maximum at $\theta = x$. It is shown that under condition A_1, part (1) of the above limiting behavior of R_n is equivalent to the statement that $h(\theta, x)$ satisfies Condition A_2 with θ_0 being the solution of $h(\theta_2, x) = h(\theta_1, x)$. Moreover, under Condition A_1, part (2) is implied by Conditions A_3 and A_4, where Condition A_3 states that $q_{\theta_2 n}(\theta_0 + c/n)/q_{\theta_1 n}(\theta_0 + c/n) \to \alpha e^{c\beta}$ for some $\alpha, \beta > 0$ and all $c \neq 0$, while Condition A_4 requires $n^{\frac{1}{2}}(X_n - \theta_0)$ to possess under P_{θ_0} a limiting distribution which is continuous at 0. For the purpose of checking the above conditions we study the asymptotic behavior of generalized Laplace integrals. The results enable us to assert that a certain Condition B (stronger than Condition A_1) obtains, where B states, essentially, that $q_{\theta n}(x) \sim K(n)C(\theta, x)e^{nh(\theta, x)}$ for some functions K, C, and h satisfying mild conditions. The same techniques enable us to verify Condition A_3 also. The applications include the sequential t-, F-, χ^2-, T^2- and other tests.

1. Introduction.

Let $(\Omega, \mathcal{A}', P')$ be a probability space, where $\mathcal{A}'$ is a σ-field of subsets of Ω and $\mathcal{P}'$ is a family of probability measures on $\mathcal{A}'$ indexed by δ: $\mathcal{P}' = \{P_\delta', \delta \varepsilon \Delta\}$. $\{Z_j\}$ is a sequence of random variables defined on Ω where each Z_j takes values in a k-dimensional Euclidean space. Let $\{X_n\}$ be another sequence of variables where $X_n = f_n(Z_1, \cdots, Z_n)$ and f_n is a Baire function

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mapping kn-space into the real line. It will be understood throughout that $n \geqq n_0$, where n_0 is some appropriately chosen positive integer. Let $\mathfrak{A}_n$ denote the subfield generated by $(X_j, j \leqq n)$ and let $\mathfrak{A}$ denote the minimal subfield containing $\mathfrak{A}_n$ for all n. Suppose that, for each n, the joint distribution of $(X_j, j \leqq n)$ depends on δ only through a certain real valued function of δ, say $\theta = \theta(\delta)$, and that $\theta(\Delta) = \Theta$ is an interval. The family of distributions of $\{X_n\}$ is denoted by $\mathcal{P} = \{P_\theta, \theta \varepsilon \Theta\}$. Let P_{θ_1} and P_{θ_2} be two members of $\mathcal{P}$ and let $p_{\theta_i n}$ be the density of P_{θ_i} with respect to some σ-finite measure μ_n on $\mathfrak{A}_n$ for $i = 1$, 2 and all n. Finally, we let $R_n = p_{\theta_2 n}/p_{\theta_1 n}$ which makes R_n defined a.e. P_{θ_1}.

The limiting behavior of R_n first considered by Wirjosudirdjo [27] is important for its implications in sequential analysis, where it is desired to know whether a sequential probability test (SPRT) [24] based on $\{X_n\}$ terminates with probability one. Since a SPRT has a positive lower stopping bound and a finite upper stopping bound, the test terminates with probability one with respect to any P_θ, if either lim inf $R_n = 0$ a.e. P_θ or lim sup $R_n = \infty$ a.e. P_θ. The same conclusion holds for a generalized SPRT if the stopping bounds are bounded below by a positive number and above by a finite number.

It was remarked in [27] that if the X_n are identically and independently distributed, $R_n \to 0$ or ∞ a.e. P_θ according as $\theta <$ or $> \theta_0$, where $E_{\theta_0} \ln R_1 = 0$, whereas lim inf $R_n = 0$ and lim sup $R_n = \infty$ a.e. P_{θ_0}. Sequences $\{X_n\}$ of neither independent nor identically distributed random variables arise in tests of composite hypotheses, in the presence of nuisance parameters, when the principle of invariance is invoked. Examples of this situation include the sequential t-, F-, χ^2-, T^2-, ordinary and multiple correlation coefficient tests. We shall see in a later section that, in each of these examples, the situation regarding the limiting behavior of R_n is similar to that of the case of identically and independently distributed random variables: for any $\theta_1 < \theta_2$ there exists a θ_0, $\theta_1 < \theta_0 < \theta_2$, such that

(1.1) $\qquad R_n \to 0$ or a.e. P_θ according as $\theta <$ or $> \theta_0$,

(1.2) $\qquad P_{\theta_0}(b < R_n < a) \to 0$ for $0 < b < a < \infty$, and

(1.3) $\qquad$ lim inf $R_n = 0$ and lim sup $R_n = \infty$ a.e. P_{θ_0}.

The preceding result for the sequential t-test is implicit in a paper by David and Kruskal [4] and is obtained in the stated form by Wirjosudirjo [27]. The sequential F-test is treated by Johnson [15], but the asymptotic formula used is not justified. A different approach for the sequential F-test is given by Hoel [13] using weighting functions [24] rather than invariance for testing composite hypotheses. A special case of the sequential F-test, when the hypothesis is null ($\theta_1 = 0$), is treated by Ray [23]. Jackson and Bradley [14] have a treatment of the sequential χ^2- and T^2-tests. However, their termination proof is given in detail only for the χ^2-test, while it is claimed that the termination proof for the T^2-test can be reduced to that of the χ^2-test. This seems unjustified to us. In fact, it will be seen in Section 4 that the behavior of R_n in the T^2-test is identical to its behavior in the F-test. The correlation coefficient tests (V and VI of Section 4) as well as special cases of VII are mentioned in [10] while Application VIII is also treated in [16].

The foregoing examples fall under the following general problem. Let the sequence $\{X_n\}$ and its family of distributions $\mathcal{P} = \{P_\theta, \theta \varepsilon \Theta\}$ satisfy the following assumptions:

$$
\begin{aligned}
&\text{(i) The index set } \Theta \text{ is a real interval and } \theta_1 < \theta_2 .\\
&\text{(ii) For each } n, X_n \text{ is a sufficient statistic for } \mathcal{P} \text{ on } \mathcal{A}_n .\\
&\text{(iii) For each } n, \mathcal{P} \text{ is a monotone likelihood ratio family on } \mathcal{A}_n .\\
&\text{(iv) For each } n, \mathcal{P} \text{ is a homogeneous family on } \mathcal{A}_n .\\
&\text{(v) } X_n \to \theta, \text{ a.e. } P_\theta, \text{ for each } \theta \varepsilon \Theta.
\end{aligned}
\tag{1.4}
$$

We recall that $\mathcal{P}$ is said to be homogeneneous [11] on $\mathcal{A}_n$ if for every θ' and θ'' $\varepsilon\, \Theta$, $P_{\theta'}$ is absolutely continuous with respect to $P_{\theta''}$ on $\mathcal{A}_n$. By virtue of Assumption (iv) of (1.4), R_n becomes defined and finite a.e. P_θ for all $\theta \varepsilon \Theta$. Assumption (ii) (Lemma 2.1 of [27]) implies that $R_n = q_{\theta_2 n}/q_{\theta_1 n}$, where $q_{\theta n}$ denotes the density of P_θ with respect to μ_n on the Borel field generated by X_n. Thus, there exists a Baire function r_n mapping the real line into itself such that $R_n = r_n(X_n)$. In view of (ii), Assumption (iii) states that $r_n(x) = q_{\theta_2 n}(x)/q_{\theta_1 n}(x)$ is a nondecreasing function of x [19]. We shall see in the sequel that Assumptions (iii) and (v) enable us to consider $r_n(x)$, rather than R_n itself, for studying the limiting behavior of R_n.

Wirjosudirjo [27] has treated this general problem under the first four assumptions with f_n restricted to be a symmetric function of $(z_1, \cdots, z_n)$ and the Z_j restricted to be identically and independently distributed real random variables. Actually, the treatment and the conclusions in [27] need no change if the Z_j are allowed to be vector valued and possibly dependent while the symmetry assumption of the f_n is replaced by the weaker assumption that $\bigcap_n \mathcal{B}_n$ is the trivial subfield (Ω, ϕ), where $\mathcal{B}_n$ denotes the subfield generated by $(X_j, j \geqq n)$. In particular, the a.e. convergence of R_n to a constant continues to hold under the new assumptions. Wirjosudirjo's results are close to (1.1) and (1.2), but not quite as strong.

It has come to our attention that R. Berk, in a recent and as yet unpublished dissertation [2], has obtained results which overlap those presented here. Proofs of termination of SPRT's follow as a by-product of his different methods and approach.

The almost sure termination of SPRT's is only a consequence of the results given in this paper on the asymptotic behavior of densities. These results could be used for other purposes such as an investigation of the ASN function (average sample number [24]) of SRPT's. Moreover, the aymptotic behavior of generalized Laplace integrals could be applied in non-sequential inference problems also.

2. Conditions guaranteeing the desired limiting behavior of R_n. In this section we characterize, under Condition A_1, the desired limiting behavior of R_n given in (1.1) with respect to any P_θ, $\theta \neq \theta_0$. A sufficient condition is given guaranteeing the desired behavior of R_n with respect to P_{θ_0}. From now on we assume that $q_{\theta n}(x) > 0$ for $x \varepsilon \Theta$ and 0 for $x \not\varepsilon \Theta$. Thus, for each (θ_1, θ_2) $r_n(x)$ will be considered as a function on Θ.

THEOREM 2.1. *Let Assumptions* (1.4) *hold and let* $\theta_0 \varepsilon \Theta$. *Then,* (1) $R_n \to 0$ a.e. P_θ, $\theta < \theta_0$, *if and only if* $r_n(x) \to 0$, $x < \theta_0$, *and* (2) $R_n \to \infty$ a.e. P_θ, $\theta > \theta_0$, *if and only if* $r_n(x) \to \infty$, $x > \theta_0$.

PROOF. We prove the equivalence in (1) only since that in (2) can be obtained from (1) by interchanging the roles of θ_1 and θ_2. We have seen in Section 1 that $R_n(\omega) = r_n(X_n(\omega))$, for $\omega \varepsilon \Omega$. Let $r_n(x) \to 0$ for all $x < \theta_0$ and let $\theta < \theta_0$. Suppose ω is not in an exceptional null set so that $X_n(\omega) \to \theta$, by (v) of (1.4), and choose x such that $\theta < x < \theta_0$. Thus, there exists N_ω such that $X_n(\omega) \leqq x$ for all $n \geqq N_\omega$. By (iii), $r_n(X_n(\omega)) \leqq r_n(x)$ for $n \geqq N_\omega$. Since $r_n(X_n(\omega)) \geqq 0$ for all n and $r_n(x) \to 0$, we conclude that $r_n(X_n(\omega)) \to 0$. We assume next that $R_n \to 0$ a.e. P_θ, $\theta < \theta_0$. Let $x < \theta_0$ and choose θ such that $x < \theta < \theta_0$. For all ω not lying in an exceptional null set, $X_n(\omega) \to \theta$ and $R_n(\omega) \to 0$. Thus, there exists N_ω such that $X_n(\omega) \geqq x$ for $n \geqq N_\omega$. By (iii), $r_n(X_n(\omega)) \geqq r_n(x)$ for $n \geqq N_\omega$. Since $r_n(X_n(\omega)) \to 0$ and $r_n(x) \geqq 0$ for all n, we conclude that $r_n(x) \to 0$.

In the following it is understood that all functions are measurable and that all limits are taken as $n \to \infty$. We introduce the following definitions:

CONDITION A_1. $\{q_{\theta n}(x)\}$ *is said to satisfy Condition* A_1 *if:*

(i) *there exist functions* $K > 0$ *and* h *such that* $n^{-1} \ln[q_{\theta n}(x)/K(n)] \to h(\theta, x)$ *for each* θ, $x \varepsilon \Theta$. *This implies that* $n^{-1} \ln r_n(x) \to g(x) \equiv h(\theta_2, x) - h(\theta_1, x)$ *for any* $\theta_1 < \theta_2$.

(ii) $g(x)$ *is strictly increasing and continuous on* Θ *for any* $\theta_1 < \theta_2$.

CONDITION A_2. *A measurable function of two real arguments* $h(\theta, x)$ *is said to satisfy Condition* A_2 *if, for each fixed* x, $h(\theta, x)$ *has a unique maximum at* $\theta = x$.

CONDITION A_3. $\{q_{\theta n}(x)\}$ *is said to satisfy Condition* A_3 *if for any* θ_0 *such that* $n^{-1} \ln r_n(\theta_0) \to 0$, $r_n(\theta_0 + c/n) \to \alpha e^{c\beta}$ *for some* $\alpha, \beta > 0$ *and all* $c \neq 0$.

CONDITION A_4. $\{q_{\theta n}(x)\}$ *is said to satisfy Condition* A_4 *if for any* θ_0 *such that* $n^{-1} \ln r_n(\theta_0) \to 0$, $n^{\frac{1}{2}}(X_n - \theta_0)$ *has under* P_{θ_0} *a limiting distribution (not necessarily a probability distribution)* Q, *say, which is continuous at the origin.*

Note that Condition B given below is stronger than Condition A_1.

CONDITION B. $\{q_{\theta n}(x)\}$ *is said to satisfy Condition* B *if:*

(i) *there exist functions* K, $C > 0$ *and* h *such that* $q_{\theta n}(x) \sim K(n)C(\theta, x) \cdot e^{nh(\theta,x)}$ *for each* θ, $x \varepsilon \Theta$. *This implies that* $n^{-1} \ln r_n(x) \to g(x) \equiv h(\theta_2, x) - h(\theta_1, x)$ *for any* $\theta_1 < \theta_2$.

(ii) $g(x)$ *is strictly increasing and continuous for any* $\theta_1 < \theta_2$.

THEOREM 2.2. *Let Assumptions* (1.4) *hold. The following three statements are equivalent under Condition* A_1 *(and therefore under Condition* B *also):*

(1) *For any* $\theta_1 < \theta_2$, *there exists* θ_0, $\theta_1 < \theta_0 < \theta_2$, *such that* $r_n(x) \to 0$ *or* ∞ *according as* $x <$ *or* $> \theta_0$.

(2) *For any* $\theta_1 < \theta_2$, $g(x) = 0$ *has a (necessarily unique) solution strictly between* θ_1 *and* θ_2.

(3) $h(\theta, x)$ *satisfies Condition* A_2.

PROOF. (1) implies (2): If $x > \theta_0$ and $g(x) \leqq 0$, then for x' such that $\theta_0 < x' < x$, $g(x') < 0$ by (ii) of Condition A_1 so that $r_n(x') \to 0$ which contradicts (1). Hence $g(x) > 0$ for $x > \theta_0$. By the continuity of g, $g(\theta_0) = 0$.

(2) implies (1): Take θ_0 equal to the solution of $g(x) = 0$.

(3) implies (2): $g(\theta_2) > 0$ and $g(\theta_1) < 0$ by (3). Thus, (2) follows by Condition A_1 (ii).

(2) implies (3): By Condition A_1, (2) implies that $g(\theta_2) > 0$ and $g(\theta_1) < 0$ for all $\theta_1 < \theta_2$. This implies that $h(\theta, x) < h(x, x)$ for $x < \theta$ and $x > \theta$. Thus $h(\theta, x)$ satisfies Condition A_2.

LEMMA 2.1. *Let Assumptions* (1.4) *hold. If Conditions* A_3 *and* A_4 *are satisfied, then* $P_{\theta_0}(b \leqq R_n \leqq a) \to 0$ *for all* $0 < b < a < \infty$. *Moreover,* $P_{\theta_0}(R_n \geqq a) \to Q([0, \infty))$, $P_{\theta_0}(R_n \leqq b) \to Q((-\infty, 0])$ *and* Q *is necessarily a probability distribution.*

PROOF. (The following proof is as given in [27] for a special case.)

$$(2.1)\quad P_{\theta_0}(\theta_0 - c/n < X_n < \theta_0 + c/n) = P_{\theta_0}(c/n^{\frac{1}{2}} < n^{\frac{1}{2}}(X_n - \theta_0) < c/n^{\frac{1}{2}}.$$

Since the interval $(-c/n^{\frac{1}{2}}, c/n^{\frac{1}{2}})$ can be made arbitrarily small by taking n large enough and since Q is continuous at the origin, it follows that

$$(2.2)\qquad P_{\theta_0}(\theta_0 - c/n < X_n < \theta_0 + c/n) \to 0.$$

Given $0 < b < a < \infty$, there exists $c > 0$ such that $\alpha e^{-c\beta} < b < a < \alpha e^{c\beta}$. By Condition A_3, there exists N such that $r_n(\theta_0 - c/n) < b$ and $r_n(\theta_0 + c/n) > a$ for $n \geqq N$. Thus, for $n \geqq N$, $P_{\theta_0}(b \leqq r_n(X_n) \leqq a) \leqq P_{\theta_0}(r_n(\theta_0 - c/n) < r_n(X_n) < r_n(\theta_0 + c/n)) \leqq P_{\theta_0}(\theta_0 - c/n < X_n < \theta_0 + c/n)$, by (iii) of (1.4). Using (2.2) we find that $P_{\theta_0}(b \leqq R_n \leqq a) \to 0$. Similarly, $P_{\theta_0}(\theta_0 + c/n \leqq X_n) = P_{\theta_0}(c/n \leqq n(X_n - \theta_0)) \to Q([0, \infty))$ and the same is true if c is replaced by $-c$. There exists N such that $r_n(\theta_0 - c/n) < a < r_n(\theta_0 + c/n)$ and, therefore, $P_{\theta_0}(\theta_0 + c/n \leqq X_n) \leqq P_{\theta_0}(a \leqq r_n(X_n)) \leqq P_{\theta_0}(\theta_0 - c/n \leqq X_n)$ for $n \geqq N$. Thus, $P_{\theta_0}(a \leqq r_n(X_n)) \to Q([0, \infty))$ and $P_\theta(r_n(X_n) \leqq b) \to Q((-\infty, 0])$. The last statement of the theorem follows from the fact that $Q((-\infty, \infty)) = \lim P_{\theta_0}(R_n \varepsilon [-\infty, \infty]) = 1$.

COROLLARY 2.1. *Let Assumptions* (1.4) *hold. If Conditions* A_1, A_2, A_3 *and* A_4 *are satisfied, then* R_n *possesses the limiting behavior* (1.1) *and* (1.2) *where* θ_0 *is the solution of* $h(\theta_2, x) = h(\theta_1, x)$.

PROOF. By Lemma 2.1 and Theorems 2.1 and 2.2.

THEOREM 2.3. *Let Assumptions* (1.4) *hold and let Conditions* A_1, A_3, *and* A_4 *be satisfied. Denote the solution of* $h(\theta_2, x) = h(\theta_1, x)$ *by* θ_0 *and assume that* $0 < Q([0, \infty)) < 1$. *If* $\bigcap_n \mathcal{B}_n$ *is the trivial subfield* (Ω, ϕ) *where* $\mathcal{B}_n$ *denotes the subfield generated by* $(X_j, j \geqq n)$, *then* R_n *possesses the limiting behavior* (1.1)–(1.3). *Moreover,* $h(\theta, x)$ *necessarily satisfies Condition* A_2.

PROOF. Since we assume the triviality of the tail field of the $\{X_n\}$, we can use the results of [27]: $R_n \to 0$ a.e. P_θ, $\theta \leqq \theta_1$ and $R_n \to$ a.e. $P_\theta > \theta_2$. Using Theorem

2.1(1) with θ_0 replaced by θ_1 we deduce that $g(x) \leqq 0$ for $x \leqq \theta_1$. Similarly, $g(x) \geqq 0$ for $x \geqq \theta_2$. This implies that $\theta_0 \varepsilon [\theta_1, \theta_2]$. Notice that the triviality of the tail field gives: lim inf $R_n = c$ a.e. P_{θ_0} and lim sup $R_n = c'$ a.e. P_{θ_0} for some constants c and c' with $0 \leqq c \leqq c' \leqq \infty$. If $c > 0$ or $c' < \infty$, the conclusions of Lemma 2.1 are contradicted since $P_{\theta_0}(r_n(X_n) \geqq a)$ and $P_{\theta_0}(r_n(X_n) \leqq b)$ have positive limits for all $0 < b < a < \infty$. Thus, (1.3) follows. Since we have established that R_n does not converge a.e. with respect to P_{θ_0}, $\theta_0 \neq \theta_1$ and $\theta_0 \neq \theta_2$. Hence $\theta_0 \varepsilon (\theta_1, \theta_2)$ so that (2) of Theorem 2.2 is satisfied and this, in turn, implies (3) of that theorem.

We remark that the triviality of $\bigcap_n \mathcal{B}_n$ will be checked in our applications by using the Hewitt-Savage zero-one law [12] with f_n being a symmetric function of $(z_1, \cdots, z_n)$ for all n.

3. The asymptotic behavior of integrals of a certain form. Let R be a subset of Euclidean r-space E^r and let

$$I(n) = \int_R f(z)k_n(z)e^{n\psi(z)}dz, \tag{3.1}$$

where $z = (z_1, \cdots, z_r)$ is an r-dimensional variable, $dz = dz_1 \cdots dz_r$, the domain of integration R will be specified as the need arises and all the functions used are measurable. We shall study the asymptotic behavior of $I(n)$, as $n \to \infty$, under some conditions on f, ψ, and the k_n, which we call Condition (b). We remark that the treatment and the results need no change if n is assumed to be real valued. However, since the applications of Section 4 involve integral n, we shall think of n as being integer valued. In order to state Condition (b) we shall require R to contain the origin 0 and be such that its intersection with any neighborhood of 0 has positive r-dimensional Lebesgue measure. As a result of this requirement, continuity and differentiability of functions at 0 are to be understood in terms of sequences of points belonging to R and converging to zero (but 0 may be on the boundary of R). It is assumed in Condition (b) that $\psi(z)$ has a unique maximum at zero. The underlying idea in the results is that the major contribution to $I(n)$, when n is large, comes from a neighborhood of 0. This idea is due to Laplace [5] and the procedure of Laplace's method is to approximate ψ in that neighborhood by the first few terms in its expansion. This topic is discussed by De Bruijn [5], Erdélyi [7] and Wilf [26]. In fact, some general results are given in [5]. However, we shall derive some still more general results. In the following we denote $\sum_{i=1}^r z_i^2$ by $|z|^2$.

CONDITION (b). *The functions f, ψ, and the k_n appearing in* (3.1) *are said to satisfy Condition* (b) *on R if*:

(i) $\psi(z) < \psi(0)$ *for all z in R, $z \neq 0$.*

(ii) *There exist $\alpha_1, \beta_1 > 0$ such that $\psi(z) \leqq \psi(0) - \alpha_1$ for $|z| > \beta_1$ while $\psi(z)$ is continuous for $|z| \leqq \beta_1$.*

(iii) *There exists a neighborhood of* 0 *in which $\psi(z)$ has continuous partial derivatives of the first order ($\psi_i'(z)$, $i = 1, \cdots, r$) with $\psi_i'(0) = 0$ for $1 \leqq i \leqq r_0$ and $\psi_i'(0) < 0$ for $r_0 < i \leqq r$, where r_0 is a non-negative integer $\leqq r$. In case $r_0 > 0$, $\psi(z)$ possesses, in addition, continuous partial derivatives of the second* *order* $(\psi''_{ij}(z),\ i, j = 1, \cdots, r)$ *with the* $r_0 \times r_0$ *matrix* $[\psi''_{ij}(0)],\ i, j = 1, \cdots, r_0$, *being negative definite.*

(iv) $f(z) \geqq 0$ *for* $z \in R$ *and* $f(z) > 0$ a.e. (*with respect to the r-dimensional Lebesgue measure*) *in some neighborhood of* 0.

(v) $k_n(z) \to k(z)$ *uniformly in some neighborhood of* 0, $k(z)$ *is continuous at* 0 *and* $0 < k \leqq k_n(z) \leqq k' < \infty$ *for some constants* k *and* k' *independent of* n.

(vi) *For some positive integer* n_0, $I(n_0) < \infty$.

REMARKS ON CONDITION (b). (i) and (iv)–(vi) imply that $I(n) < \infty$ for all $n \geqq n_0$. In the following we let E_0^r denote the positive orthant of E^r, i.e., $E_0^r = \{z : z_i \geqq 0 \text{ for } i = 1, \cdots, r\}$.

THEOREM 3.1. *Let* $I(n)$ *be given by* (3.1) *where* f, ψ, *and the* k_n *satisfy Condition* (b) *on* $R = E_0^r$. *Suppose* $f(z) = \bar{f}(z)\prod_{i=1}^r z_i^{l_i}$, *where* $\bar{f}(z)$ *is continuous at* 0, $\bar{f}(0) > 0$ *and the* $l_i \geqq 0$. *Assume that* $[\psi''_{ij}(0)],\ i, j = 1, \cdots, r_0$, *is in the diagonal form. Then*

$$I(n) \sim \bar{f}(0)k(0)Cn^{-c}e^{n\psi(0)},$$

where

$$c = \sum_{i=1}^{r_0}\tfrac{1}{2}(l_i + 1) + \sum_{i=r_0+1}^{r}(l_i + 1)$$

and

$$C = \prod_{i=1}^{r_0}\Gamma(\tfrac{1}{2}(l_i + 1))2^{-1}(-\tfrac{1}{2}\psi''_{ii}(0))^{-\frac{1}{2}(l_i + 1)} \cdot \prod_{i=r_0+1}^{r}\Gamma(l_i + 1)[-\psi_i'(0)]^{-(l_i+1)}.$$

PROOF. (0) Without loss of generality the neighborhoods in (iii)–(v) can be assumed to coincide and be $|z| \leqq \gamma$ where $0 < \gamma \leqq \beta_1$.

(1) Without loss of generality, we assume that $\psi(0) = 0$, $n_0 = 0$, $\bar{f}(0) = 1$ and $k(0) = 1$. We show that $I(n)n^cC^{-1} \to 1$.

(2) Taylor's theorem and continuity of the partial derivatives imply: given $\epsilon > 0$ there exists δ $(0 < \delta < \gamma)$ such that for $|z| < \delta$ (some of the following sums might be vacuous for $r_0 = 0$ or r)

$$\sum_{i=r_0+1}^{r}\psi_i'(0)z_i + \tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(\psi''_{ij}(0) - 2\epsilon)z_iz_j \leqq \psi(z) \leqq \sum_{i=r_0+1}^{r}\psi_i'(0)z_i + \tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(\psi''_{ij}(0) + 2\epsilon)z_iz_j. \tag{3.2}$$

The terms in the double sums with i or $j > r_0$ can be absorbed in the single sums. Noting that $\sum_{i\neq j}z_iz_j < r_0^2\delta^2$ for $|z| < \delta(\bar{\epsilon} = r_0^2\delta^2\epsilon)$,

$$\sum_{i=r_0+1}^{r}(\psi_i'(0) - \eta)z_i + \tfrac{1}{2}\sum_{i=1}^{r_0}(\psi''_{ii}(0) - 2\bar{\epsilon})z_i^2 \leqq \psi(z) \leqq \sum_{i=r_0+1}^{r}(\psi_i'(0) + \eta)z_i + \tfrac{1}{2}\sum_{i=1}^{r_0}(\psi''_{ii}(0) + 2\bar{\epsilon})z_i^2 \tag{3.3}$$

for $|z| < \delta$, where δ and η may be chosen so small that $\psi_i'(0) + \eta < 0$ for $r_0 < i \leqq r$ and $\frac{1}{2}\sum_{i=1}^{r_0}(\psi''_{ij}(0) + 2\bar{\epsilon})z_i^2 < 0$ for $z \neq 0$, $z \in E_0^r$, in view of (iii).

(3) There exists $\rho > 0$ such that $\psi(z) \leqq -\rho$ for $|z| \geqq \delta$. This is trivial if $\delta = \beta_1$, by (ii). If $\delta < \beta_1$ we observe that $\psi(z)$ is continuous for $\delta \leqq |z| \leqq \beta_1$,

and thus attains its maximum at a point ξ in this closed set. Take $\rho = \min(-\psi(\xi), \alpha_1) > 0$.

(4) Let $I_1(n)$ and $I_2(n)$ denote the integral (3.1) taken over the subregions $|z| \leqq \delta$ and $|z| > \delta$ of E_0^r respectively. Thus $I(n) = I_1(n) + I_2(n)$. We use the symbols $\int_{\leqq}$ and $\int_{>}$ to denote integrals over the subregions $|z| \leqq \delta$ and $|z| > \delta$, respectively. By (iv) of Condition (b) and (3) of the proof, $0 < I_2(n) \leqq e^{-n\rho}\int_{>}f(z)k_n(z)dz$. Using (v) and (vi) together with (1) we conclude that for some K,

$$0 < I_2(n) \leqq Ke^{-n\rho}. \tag{3.4}$$

(5) Let Q' and Q'' stand for the left and right hand members of (3.3) respectively. (3.3) implies

$$\int_{\leqq}f(z)k_n(z)e^{nQ'}dz \leqq I_1(n) \leqq \int_{\leqq}f(z)k_n(z)e^{nQ''}dz. \tag{3.5}$$

Let $F(Q')$ denote $\prod_{i=1}^{r} z_i^{l_i} e^{nQ'}$ and let $F(Q'')$ be similarly defined. Notice that Q' and Q'' satisfy all the properties of ψ in Condition (b) and that $\int F(Q'')dz$ is a special case of $I(n)$. Thus, there exist ρ', $K' > 0$ such that $0 < \int_{>}F(Q')dz < \int_{>}F(Q'')dz < K'e^{-n\rho'}$.

(6) Denote by $\inf_{\leqq}$ ($\sup_{\leqq}$) the infimum (supremum) taken over the subregion $|z| \leqq \delta$. Let $\varphi_n(z) = \bar{f}(z)k_n(z)$ and $\varphi(z) = \bar{f}(z)k(z)$. Using (5) we obtain

$$\int_{\leqq}f(z)k_n(z)e^{nQ'}dz \geqq \inf_{\leqq}\{\varphi_n(z)\}[\int_{\leqq}F(Q')dz - K'e^{-n\rho'}], \tag{3.6}$$

$$\int_{\leqq}f(z)k_n(z)e^{nQ''}dz \leqq \sup_{\leqq}\{\varphi_n(z)\}\int F(Q'')dz. \tag{3.7}$$

(7) It follows from (3.4)–(3.7) that $\inf_{\leqq}\{\varphi_n(z)\}[\int F(Q')dz - K'e^{-n\rho'}] \leqq I(n) \leqq \sup_{\leqq}\{\varphi_n(z)\}[\int F(Q'')dz + Ke^{-n\rho}]$.

(8) Let Q be the expression obtained from Q' and Q'' by replacing ϵ and η by 0. For l, λ, $m > 0$,

$$\int_0^\infty u^l e^{-\lambda u^m}du = \Gamma((l+1)/m)m^{-1}\lambda^{-(l+1)/m} \tag{3.8}$$

and

$$\int F(Q)dz = \prod_{i=1}^{r_0} \int_0^\infty z_i^{l_i}\exp[\tfrac{1}{2}n\psi''_{ii}(0)z_i^2]dz_i \cdot \prod_{i=r_0+1}^{r} \int_0^\infty z_i^{l_i}\exp[n\psi_i'(0)z_i]dz_i = Cn^{-c}, \tag{3.9}$$

where c and C are as given in the theorem. A similar expression is obtained for $\int F(Q')dz$ and $\int F(Q'')dz$ with $\psi''_{ii}(0)$ replaced by $\psi''_{ii}(0) \mp 2\bar{\epsilon}$ and $\psi_i'(0)$ replaced by $\psi_i'(0) \mp \eta$. We denote the values of these integrals by $C'n^{-c}$ and $C''n^{-c}$ respectively.

(9) It follows from (7) that

$$\inf_{\leqq}\{\varphi_n(z)\}[C' - K'n^c e^{-n\rho'}] \leqq I(n)n^c \leqq \sup_{\leqq}\{\varphi_n(z)\}C'' + Kn^c e^{-n\rho}. \tag{3.10}$$

As $n \to \infty$, $n^c e^{-n\rho}$ and $n^c e^{-n\rho'} \to 0$. The hypothesis of the theorem and (v) of Condition (b) imply that $\varphi_n(z) \to \varphi(z)$ uniformly in a neighborhood of 0, so that $\inf_{\leqq}\varphi_n(z) \to \inf_{\leqq}\varphi(z)$ and $\sup_{\leqq}\varphi_n(z) \to \sup_{\leqq}\varphi(z)$. In (3.10), let $n \to$

∞ first and then let ϵ and $\delta \to 0$. Using the continuity of φ at $z = 0$, we obtain $I(n)n^c \to C$. By (1), this is the desired result.

In Theorem 3.2 given below we shall take R to be the Cartesian product $E^{r_0} \times E_0^{r-r_0} = \{z : z_i \varepsilon (-\infty, \infty) \text{ for } i = 1, \cdots, r_0 \text{ and } z_i \varepsilon [0, \infty) \text{ for } i = r_0 + 1, \cdots, r\}$. The determinant of the matrix $[a_{ij}]$ is denoted by $|[a_{ij}]|$.

THEOREM 3.2. *Let $I(n)$ be given by* (3.1) *where f, ψ, and the k_n satisfy Condition* (b) *on $R = E^{r_0} \times E_0^{r-r_0}$. Suppose $f(z) = \bar{f}(z) \prod_{i=r_0+1}^{r} z_i^{l_i}$ where $\bar{f}(z)$ is continuous at* 0, *$\bar{f}(0) > 0$, and the $l_i > 0$. Then $I(n) \sim \bar{f}(0)k(0)Dn^{-d}e^{n\psi(0)}$, where $d = r_0/2 + \sum_{i=r_0+1}^{r} (l_i + 1)$ and $D = [-(2\pi)^{r_0}/|[\psi''_{ij}(0)]|]^{\frac{1}{2}} \prod_{i=r_0+1}^{r} \Gamma(l_i + 1)[-\psi_i'(0)]^{-(l_i+1)}$.*

PROOF. There exists an $r_0 \times r_0$ orthogonal matrix P such that $P'[\psi''_{ij}(0)]P$ is diagonal. Denote this diagonal matrix by $\Lambda = [\lambda_{ij}]$. Since $[\psi''_{ij}(0)]$ is negative definite, $\lambda_{ii} < 0$ for $i = 1, \cdots, r_0$. We change the variables in (3.1) from the z_i to the y_i, where $y_i = z_i$ for $i = r_0 + 1, \cdots, r$ and $(y_1, \cdots, y_{r_0})' = P(z_1, \cdots, z_{r_0})'$. This change of variables and the particular form of R and f allow us to assume that $I(n)$ is as given in the statement of the theorem with $[\psi''_{ij}(0)] = \Lambda$. We may write $I(n)$ as the sum of 2^{r_0} integrals over E_0^r by partitioning R and changing the sign of some variables if necessary. It is easily seen, by (iv) of Condition (b), that the hypothesis of Theorem 3.1 is satisfied for each of these integrals with $l_i = 0$ for $1, \cdots, r_0$. It follows from Theorem 3.1 that $I(n) \sim \bar{f}(0)k(0)\, 2^{r_0}Cn^{-c}e^{n\psi(0)}$, where C and c are given in Theorem 3.1. Because $\prod_{i=1}^{r_0} \lambda_{ii} = |[\psi''_{ij}(0)]|$ and $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, it follows that $2^{r_0}Cn^{-c} = Dn^{-d}$ where D and d are given in Theorem 3.2.

NOTE. If in Theorem 3.1 the $l_i = 0$ and $k_n(z) = 1$, then the conclusion gives the result in [5] on p. 65 for the cases $r_0 = r = 1$, and $r_0 = 0$, $r = 1$ and on p. 71 for the case $r_0 = r$.

We shall make use of the preceding results in establishing some of the conditions of Section 2. For this purpose we consider a sequence of densities $\{q_{\theta n}(x)\}$, θ, $x \,\varepsilon\, \Theta$, such that

$$q_{\theta n}(x) = K(n)I(n; \theta, x) \tag{3.11}$$

with $K > 0$ and

$$I(n; \theta, x) = \int_R f(z; \theta, x)k_n(z; \theta, x)e^{n\psi(z;\theta,x)}dz, \tag{3.12}$$

and for each (θ, x), $I(n; \theta, x)$ satisfies the hypothesis of Theorem 3.2. We give the following definition:

CONDITION (c). *The functions f, ψ, and the k_n appearing in* (3.12) *are said to satisfy Condition* (c) *on R at $x = \theta_0$ if*:

(i) *For each (z, θ), $\psi(z; \theta, x)$ has a continuous derivative $\psi'(z; \theta, x)$ with respect to x in a neighborhood of $x = \theta_0$.*

(ii) *$f(z; \theta, x)$ and $\psi'(z; \theta, x)$ are continuous in the pair (z, x) for each fixed θ.*

(iii) *$k_n(z; \theta, x) \to k(z; \theta, x)$ as $n \to \infty$ uniformly in (z, x) for (z, x) in a neighborhood of $(0, \theta_0)$ and each fixed θ.*

THEOREM 3.3. *Let the sequence of densities $\{q_{\theta n}(x)\}$ be given by* (3.11) *and* (3.12) *where, for each* (θ, x), $I(n; \theta, x)$ *satisfies the hypothesis of Theorem* 3.2. *If* $\partial^2\psi(0; \theta, x)/\partial\theta\partial x$ *exists and is* > 0 *for all* $\theta, x \in \Theta$, *then Condition* B (*and therefore Condition* A_1) *is satisfied with* $h(\theta, x) = \psi(0; \theta, x)$. *Let* θ_0 *be the solution of* $h(\theta_2, x) = h(\theta_1, x)$, *where* $\theta_1 < \theta_2$. *If* f, ψ *and the* k_n *satisfy also Condition* (c) *at* $x = \theta_0$ *for all* $\theta_1 < \theta_2$, *then Condition* A_3, *is satisfied with* $\beta = g'(\theta_0)$, *where* $g(x) = h(\theta_2, x) - h(\theta_1, x)$.

Proof. Theorem (3.2) implies that Condition B(i) is satisfied with $h(\theta, x) = \psi(0; \theta, x)$. Condition B(ii), i.e., g is strictly increasing and continuous, follows from the assumption that $\partial^2\psi(0; \theta, x)/\partial\theta\partial x > 0$. Condition (c)(i) implies that for each (z, θ), and $c \neq 0$ and n large enough:

$$(3.13) \quad \psi(z; \theta, \theta_0 + c/n) = \psi(z; \theta, \theta_0) + (c/n)\psi'(z; \theta, \theta_0 + (c/n)\alpha_n),$$

where $0 < \alpha_n < 1$. (3.11)–(3.13) give, after suppressing θ on the right hand side,

$$(3.14) \quad q_{\theta n}(x) = \int_R f(z; \theta_0 + c/n) \exp [c\psi'(z; \theta_0 + (c/n)\alpha_n)]k_n(z; \theta_0 + c/n)e^{n\psi(z;\theta_0)}dz.$$

Notice that (ii) and (iii) of Condition (c) imply that, as $n \to \infty$, $\psi'(z; \theta_0 + (c/n)\alpha_n)$, $f(z; \theta_0 + c/n)$ and $k_n(z; \theta_0 + c/n)$ converge uniformly in a neighborhood of $z = 0$ to $\psi'(z; \theta_0)$, $f(z; \theta_0)$ and $k(z; \theta_0)$ respectively. Thus, these functions appearing in (3.14) satisfy the same conditions as $k_n(z)$ in (v) of Condition (b). Theorem 3.2 and the fact that $\psi(0; \theta_2, \theta_2) = \psi(0, \theta_1, \theta_0)$ imply:

$$(3.15) \quad r_n(\theta_0 + c/n) \to \alpha e^{c\beta},$$

where $\beta = g'(\theta_0) > 0$ and $\alpha = \{\bar{f}(0; \theta_2, \theta_0)k(0; \theta_2, \theta_0)D(\theta_2, \theta_0)/\bar{f}(0; \theta_1, \theta_0) \cdot k(0; \theta_1, \theta_0)D(\theta_1, \theta_0)\} > 0$. Thus, Condition A_3 is satisfied.

Remark on Theorem 3.3. In most of the applications that we shall study in Section 4 the densities are representable as integrals of the form (3.11)–(3.12) with $\psi(z; \theta, x)$ achieving its maximum at $z = z_0(\theta, x)$ instead of the origin. A linear transformation taking the origin to z_0 will make the densities satisfy the hypothesis of Theorem 3.3. However, this linear transformation need not be performed for the purpose of obtaining $h(\theta, x)$, since $\max_z \psi(z; \theta, x) = \psi(z_0; \theta, x) = h(\theta, x)$.

Condition (a). *The functions* f, ψ *and the* k_n *appearing in* (3.1) *are said to satisfy Condition* (a) *on* R *if*:

(i) $\psi(z) \leqq \psi(0)$ *for all* $z \in R$.

(ii) $\psi(z)$ *is continuous at* 0.

(iii) $f(z) \geqq 0$ *for* $z \in R$ *and* $f(z) > 0$ a.e. (*with respect to the r-dimensional Lebesgue measure*) *in some neighborhood of* 0.

(iv) *There exist constants* k *and* k' *independent of* n, $0 < k \leqq k' < \infty$, *such that* $k \leqq k_n(z) \leqq k'$ *for all* $z \in R$.

(v) *For some positive integer* n_0, $I(n_0) < \infty$.

Remarks on Condition (a). (i) and (iii)–(v) imply that $I(n) < \infty$ for $n \geqq n_0$. Notice that Condition (b) implies Condition (a).

THEOREM 3.4. *Let $I(n)$ be given by* (3.1) *where f, ψ and the k_n satisfy Condition* (a) *on R. Then, $n^{-1} \ln I(n) \to \psi(0)$.*

PROOF. (1) We assume without loss of generality that $\psi(0) = 0$ and $n_0 = 0$. It is sufficient, by (iv), to prove the theorem for the case $k_n(z) = 1$.

(2) We use (i)–(iii) to conclude that for any $\epsilon > 0$, there exists $\delta > 0$ such that $-\epsilon \leqq \psi(z) \leqq 0$ and $f(z) > 0$ a.e. for $|z| < \delta$.

(3) Notice that $I(n) \geqq \int_{\leqq} f(z)e^{n\psi(z)}\, dz \geqq \int_{\leqq} f(z)\, dz > 0$, by (2), and that $I(n) \leqq I(0) < \infty$, by (i) of Condition (a) and (1). Thus, $0 < e^{-n\epsilon} \int_{\leqq} f(z)\, dz \leqq I(n) \leqq I(0) < \infty$ and $-\epsilon + n^{-1} \ln \int_{\leqq} f(z)\, dz \leqq n^{-1} \ln I(n) \leqq n^{-1} \ln I(0)$.

(4) Letting $n \to \infty$ first and then letting $\epsilon \to 0$ we obtain $n^{-1} \ln I(n) \to \psi(0)$.

COROLLARY 3.1. *Let the sequence of densities $\{q_{\theta n}(x)\}$ be given by* (3.11) *and* (3.12) *where, for each (θ, x), $I(n; \theta, x)$ satisfies the hypothesis of Theorem* 3.4 *with $R = E^{r_0} \times E_0^{r-r_0}$. If $\partial^2\psi(0; \theta, x)/\partial\theta\partial x$ exists and is > 0 for all $\theta, x \varepsilon \Theta$, then Condition* A_1 *is satisfied with $h(\theta, x) = \psi(0; \theta, x)$.*

PROOF. Condition A_1 (i) follows from Theorem 3.4 and Condition A_1 (ii) follows from the assumption that $\partial^2\psi(0; \theta, x)/\partial\theta\partial x > 0$.

We shall encounter in our applications sequences of densities given in a closed form, rather than in an integral form, as follows:

$$q_{\theta n}(x) = \gamma_n(\theta, x)K(n)C(\theta, x)e^{nh(\theta,x)}, \tag{3.16}$$

where K, C and the γ_n are positive valued.

THEOREM 3.5. *Let the sequence of densities $\{q_{\theta n}(x)\}$ be given by* (3.16). *If $\gamma_n(\theta, x) \to 1$ and if $\partial^2h(\theta, x)/\partial\theta\partial x$ exists and is > 0 for all $\theta, x \varepsilon \Theta$, then Condition* B (*and therefore Condition* A_1) *is satisfied. Let θ_0 be the solution of $h(\theta_2, x) = h(\theta_1, x)$, where $\theta_1 < \theta_2$. If, in addition $C(\theta, x)$ is continuous in x, if $h(\theta, x)$ possesses for each θ a continuous derivative $h'(\theta, x)$ with respect to x in a neighborhood of $x = \theta_0$ and if $\gamma_n(\theta, x) \to 1$ uniformly in x for x in a neighborhood of θ_0, then Condition* A_3 *is satisfied with $\beta = g'(\theta_0)$, where $g(x) \equiv h(\theta_2, x) - h(\theta_1, x)$.*

PROOF. The first conclusion is obvious and similar to Theorem 3.3. To prove the second conclusion, we have for n large enough and $c \neq 0$: $h(\theta, \theta_0 + c/n) = h(\theta, \theta_0) + (c/n)h'(\theta, \theta_0 + (c/n)\, \alpha_n)$, where $0 < \alpha_n < 1$. Thus

$$\begin{aligned} q_{\theta n}(\theta_0 + c/n) \\ = \gamma_n(\theta, \theta_0 + c/n)K(n)C(\theta, \theta_0 + c/n)e^{nh(\theta,\theta_0)} \exp\, [ch'(\theta, \theta_0 + (c/n)\alpha_n]. \end{aligned} \tag{3.17}$$

The hypotheses of the theorem imply that $C(\theta, \theta_0 + c/n) \to C(\theta, \theta_0)$, $h'(\theta, \theta_0 + (c/n)\, \alpha_n) \to h'(\theta, \theta_0)$ and $\gamma_n(\theta, \theta_0 + c/n) \to 1$. Therefore, since $h(\theta_2, \theta_0) = h(\theta_1, \theta_0)$,

$$r_n(\theta_0 + c/n) \to \alpha e^{c\beta} \tag{3.18}$$

where $\beta = g'(\theta_0) > 0$ and $\alpha = \{C(\theta_2, \theta_0)/C(\theta_1, \theta_0)\} > 0$. Thus Condition A_3 is satisfied.

NOTE. In the applications of Section 4 we shall use a theorem on the convergence of random variables in probability law to establish Condition A_4 by showing that $\gamma_n = n^{\frac{1}{2}}(X_n - \theta)$ is asymptotically normal with mean 0 and variance

$\sigma^2(\theta)$ with respect to P_θ. We remark that under certain conditions, in addition to those of either Theorem 3.3 or Theorem 3.5, the following stronger result can be established ($q^*_{\theta n}$ denotes the density of Y_n):

$$q^*_{\theta n}(y) \to [2\pi\sigma^2(\theta)]^{-\frac{1}{2}} \exp[-y^2/2\sigma^2(\theta)], \quad \text{pointwise,} \tag{3.19}$$

where

$$\sigma^2(\theta) = -[\partial^2 h(\theta, \theta)/\partial x^2]^{-1}. \tag{3.20}$$

We shall notice in Section 4 that $\sigma^2(\theta)$ is indeed given by (3.20).

4. Applications. We shall study in this section some special cases of the general problem described in Section 1. In each case it is desired to test the hypothesis $\theta \leqq \theta_1$ against the alternative $\theta \geqq \theta_2$ where $\theta_1 < \theta_2$. G is a group of transformations that leaves the sequential testing problem invariant. Let $\mathcal{A}_n'$ be the Borel field generated by $(Z_1, \cdots, Z_n)$. For $n \geqq n_0$, X_n is obtained through a reduction of the problem by sufficiency and invariance on $\mathcal{A}_n'$ applied in that order. It will be shown below that $\{X_n\}$ satisfies Assumptions (i)–(v) of (1.4). A SPRT based on the sequence $\{q_{\theta_2 n}(x)/q_{\theta_1 n}(x)\}$ has then a monotone power function, as shown by Ghosh [9], and as implied by Lemma 2.4 in [27]. It suffices therefore to control the error probabilities at θ_1 and θ_2.

Assumptions (i)–(v) of (1.4) hold in each case: (i) and (iv) are obvious, (iii) is a well-known property of the family of distributions in question, and (ii) and (iv) will be discussed in the following paragraphs. We recall that X_n was not required to be related to $(Z_1, \cdots, Z_n)$ by a symmetric function, and, in fact, it is not in the last application treated in this section.

Assumption (ii) requires X_n to be sufficient on $\mathcal{A}_n$, where $\mathcal{A}_n$ is as defined in Section 1. Since $(X_j, j \leqq n)$ is an invariant function of $(Z_1, \cdots, Z_n)$ under G, it is enough to show that X_n is invariantly sufficient [10], i.e., sufficient for the family of distributions of any invariant function. Let $\mathcal{A}_n''$ be the Borel field generated by a maximal invariant of $(Z_1, \cdots, Z_n)$. Notice that $\mathcal{A}_n \subset \mathcal{A}_n''$ and that it is enough to establish the sufficiency of X_n on $\mathcal{A}_n''$ or, equivalently, the interchangeability of the order in which sufficiency and invariance are applied. Different conditions guaranteeing this interchangeability are given in [10]. One such condition is what the authors of [10] call Assumption A. In order to avoid the introduction of further notation we describe Assumption A in view of (iii) of (1.4), instead of quoting [10], as follows: (i) each $g \varepsilon G$ induces a corresponding transformation in the space of any sufficient statistic, (ii) any $\mathcal{A}(X_n)$-measurable function ($\mathcal{A}(X_n)$ is the Borel field generated by X_n) that is almost invariant under G is equivalent to an invariant $\mathcal{A}(X_n)$-measurable function. Assumption A(ii) holds under some conditions, stated in Theorem 4 of Chapter 6 in [20], the least immediate of which is the following: there exist a Borel field $\mathcal{B}$ of subsets of G and a σ-finite measure ν over $(G, \mathcal{B})$ such that for any $B \varepsilon \mathcal{B}$ with $\nu(B) = 0$, $\nu(Bg) = 0$ for all $g \varepsilon G$. This property is in particular satisfied by a right Haar measure [22]. It is shown in [22] that a right (left) Haar measure exists for a locally compact metric group. This will be a case in all our applications except Application VII in which case Assumption A(ii) follows from the fact that G is a finite group.

Since in every instance X_n is a continuous function of sample moments (and perhaps sequences which converge a.e. to constants), we conclude that X_n converges a.e. to some constant. The reasoning is as follows: (1) sample moments about the origin converge a.e. to the population moments when the latter exist; (2) if $U_n \to a$ a.e. and $V_n \to b$ a.e. then $(U_n, V_n) \to (a, b)$ a.e. and hence sample moments converge jointly; (3) if $W_n \to c$ a.e. and $g(\cdot)$ is continuous at c then $g(W_n) \to g(c)$ a.e., and hence continuous functions of sample moments converge a.e.; (4) sample central moments are continuous functions of sample moments. The constant to which X_n converges a.e. is shown to be θ (Assumption (v)) by using a theorem of Cramér [3] (28.4 and 27.7.3), according to which the sample moments appearing in the expression of X_n should be replaced by the corresponding population moments to obtain the constant. This procedure is illustrated in Application I.

Condition A_4 follows from the stronger result that $n^{\frac{1}{2}}(X_n - \theta)$ is asymptotically $N(0, \sigma^2(\theta))$ where $\sigma^2(\theta)$ is given by (3.20). We have verified this directly in our application but we shall illustrate it only in Applications I and VII. An alternative procedure is possible by the theorem of Cramér [3] (28.4) which guarantees the asymptotic normality of continuous functions of sample moments with a variance of the form c/n (where c is given by (27.7.3) in [3]) provided that $c > 0$. This alternative procedure will be illustrated in Application I.

NOTATION. A sequence of random variables $\{W_n\}$ is said to converge in law to the random variable W if the sequence of distribution functions $\{F_n(w)\}$ of $\{W_n\}$ converges completely to the distribution function $F(w)$ of W ([16], [21]) i.e., at points of continuity of F. We shall denote this convergence by: $W_n \to W$; and say that W_n is asymptotically distributed as W. Given $\{Z_{ij}, i = 1, \cdots, I, j = 1, \cdots, J\}$ we let $Z_{\cdot j}$ stand for $I^{-1} \sum_{i=1}^{I} Z_{ij}$; $Z_{i\cdot}$ and $Z_{\cdot\cdot}$ have a similar meaning. If a random variable has the t-distribution with n degrees of freedom, it will be referred to as a t_n variable. If the distribution is noncentral with noncentrality parameter ξ, it will be referred to as a $t_n'(\xi)$ variable. The symbols $\chi_n^{2'}(\xi)$ and $F'_{m,n}(\xi)$ denote similarly defined chi-square and F variables. A normal variable with mean ξ and variance σ^2 will be denoted by $N(\xi, \sigma^2)$.

CONCLUSIONS (4.0). In each of the first seven applications the density of X_n is written either in the integral form (3.11)–(3.12) or in the closed form (3.16). In either case $h(\theta, x)$ is obtained. Conditions B, A_1 and A_3 follow from Theorem 3.3 in the first case and Theorem 3.5 in the second together with the positivity of $\partial^2 h/\partial\theta\partial x$. In the last application these conditions are verified directly. The position of θ_0 which is the solution of $h(\theta_2, x) = h(\theta_1, x)$ will be specified and compared with $(\theta_1 + \theta_2)/2$. By Corollary 2.1, $\{R_n\}$ possesses the limiting behavior (1.1)–(1.2) in Applications I–VIII. Theorem 2.3 implies that (1.3) holds also and that Condition A_2 is necessarily satisfied in Applications I–VII and need not be checked. All the relevant conditions are checked in Application II only since the rest are similar. Thus, in each application, the test terminates a.e. P_θ for all $\theta \varepsilon \Theta$.

When the density of X_n is put in the form (3.11)–(3.12), θ and x will be often suppressed in the notation for f, ψ and the k_n, while the vector z will be denoted by v, (v, z) or (v, y, z) in case the dimension is 1, 2 or 3 respectively.

I. *The sequential t-test.* The Z_j are identically and independently distributed $N(\zeta, \sigma^2)$ variables, $\theta = \zeta/\sigma$, $= (-\infty, \infty)$, $n_0 = 2$ and $X_n = \bar{Z}_n/S_n$ where $n\bar{Z}_n = \sum_{i=1}^n Z_i$ and $(n - 1)\, S_n^{\;2} = \sum_{i=1}^n (Z_i - \bar{Z}_n)^2$. For each n, sufficiency and invariance under some group of transformations reduce the data to X_n ([20], pp. 222–223), which has the distribution of a $n^{-\frac{1}{2}}t'_{n-1}(n^{\frac{1}{2}}\theta)$ variable. The family of distributions of X_n is a MLR family ([18], [17], [20]).

Notice that $X_n = U_n/V_n$ where U_n is $N(\theta, n^{-1})$, $V_n^{\;2}$ is $\chi^2_{n-1}/(n-1)$ and U_n is independent of V_n. Making use of the relation between the densities of U_n, V_n and X_n, we obtain for some $K(n)$

$$q_{\theta n}(x) = K(n)\textstyle\int_0^\infty f(v)e^{n\psi(v)}\,dv \quad \text{for} \quad x \,\varepsilon\, (-\infty, \infty), \tag{4.1}$$

where $f(v) = v^{-1}e^{v^2/2}$ and $\psi(v) = -(1 + x^2)v^2/2 + \theta xv - \theta^2/2 + \ln v$.

Let $\xi = x/(1 + x^2)^{\frac{1}{2}}$, $\alpha(z) = [z + (z^2 + 4)^{\frac{1}{2}}]/2$ and $v_0 = \alpha(\xi\theta)/(1 + x^2)^{\frac{1}{2}}$. It is easy to show that ψ has a unique maximum at v_0. After changing the origin to v_0, we see that $q_{\theta n}(x)$ is of the form (3.11)–(3.12). Denoting $\psi(v_0)$ by $h(\theta, x)$ we obtain

$$h(\theta, x) = \tfrac{1}{2}\alpha^2(\xi\theta) + \ln \alpha(\xi\theta) - \tfrac{1}{2}\theta^2 + \tfrac{1}{2}\ln(1 - \xi^2) - 1, \tag{4.2}$$

$$\partial^2 h/\partial\theta\partial x = \{2\alpha^2(\xi\theta)/(\xi^2\theta^2 + 4)^{\frac{1}{2}}\}\, d\xi/dx > 0, \tag{4.3}$$

$$\partial^2 h/\partial x^2|_{x=\theta} = -(1 + \tfrac{1}{2}\theta^2)^{-1}. \tag{4.4}$$

We remark that (4.2) is in agreement with the results obtained in [4] and [27]. It was shown in [27] that θ_0 lies between 0 and $(\theta_1 + \theta_2)/2$. Thus, Conclusions (4.0) hold.

Let us illustrate two methods for showing that $n^{\frac{1}{2}}(X_n - \theta)$ is asymptotically $N(0, \sigma^2(\theta))$ where $\sigma^2(\theta)$ is given by (3.20) and (4.4). Since the distribution of X_n is independent of σ we may assume that the Z_j are $N(\theta, 1)$ variables. Let $W_n = (W_{n1}, W_{n2}, W_{n3}, W_{n4})$, where $W_{n1} = \sum_{i=1}^n (Z_i - \theta)/n^{\frac{1}{2}}$, $W_{n2} = (n/2)^{\frac{1}{2}}(S_n^{\;2} - 1)$, $W_{n3} = S_n$ and $W_{n4} = (S_n + 1)/2$. Also let $W = (W_1, W_2, W_3, W_4)$, where W_1 and W_2 are independently $N(0, 1)$ while W_3 and W_4 are degenerate at 1. Finally, let $Y_n = n^{\frac{1}{2}}(X_n - \theta)$. It follows easily that $Y_n = f(W_n) \equiv (W_{n1}/W_{n3}) - (\theta W_{n2}/2^{\frac{1}{2}}W_{n3}W_{n4})$. $W_{n1} \to W_1$ because W_{n1} is $N(0, 1)$. Since a χ^2_{n-1} variable has mean $(n-1)$ and variance $2(n-1)$, $W_{ni} \to W_i$ for $i = 2, 3, 4$. Thus, $W_n \to W$ since W_{n1} and W_{n2} are independent and hence (e.g. [6], p. 211 where the independence restriction is unnecessary) $f(W_n) \to f(W) \equiv W_1 - (\theta W_2/2^{\frac{1}{2}})$ which is $N(0, 1 + (\theta^2/2))$. Alternatively, since $X_n = a_1/(a_2 - a_1^{\;2})^{\frac{1}{2}}$, where a_k denotes the sample kth moment about the origin, we use the theorem of (28.4) in [3] to conclude that X_n is asymptotically $N(c', c/n)$ where c' and c are given by formulas analogous to (27.7.3) of [3] (but in terms of population moments about the origin) as θ and $1 + (\theta^2/2)$ respectively.

II. *The sequential F-test.* The Z_j are identically and independently distributed k-dimensional vectors with $Z_j = (Z_{j1}, \cdots, Z_{jk})$, where the Z_{ji} are independent and Z_{ji} is $N(\zeta_i, \sigma^2)$. It is known that for some $s < k$, $\zeta_{s+1} = \cdots = \zeta_k = 0$, $\theta = \sum_{i=1}^q \zeta_i^2/k\sigma^2$ for some q, $1 \leqq q \leqq s$, $\Theta = [0, \infty)$, $n_0 = 2$ and $X_n = U_n/V_n$ where $kU_n = \sum_{i=1}^q Z_{\cdot i}^2$ and $knV_n = \sum_{j=1}^n [\sum_{i=1}^s (Z_{ji} - Z_{\cdot i})^2 + \sum_{i=s+1}^k Z_{ji}^2]$. It follows by a procedure similar to that of p. 267 in [20], that for each n sufficiency and invariance under some group of transformations reduce the data to X_n. U_n and V_n are independently $\sigma^2\chi_q^{2\prime}(kn\theta)/kn$ and $\sigma^2\chi_{kn-s}^2/kn$ variables respectively. Thus, X_n itself is a $qF'_{q,kn-s}(kn\theta)/kn$ variable and the family of its distributions is a MLR family ([17], [20]).

The density of U_n is given in Application III while that of V_n has a simple form. As in Application I, for $q = 1$ and some $K_1(n)$, we have

$$q_{\theta n}(x) = K_1(n)\int_0^\infty f(v)k_n(v)e^{kn\psi(v)}\,dv, \qquad x > 0, \tag{4.5}$$

where

$$k_n(v) = 1 + \exp[-2kn(xv)^{\frac{1}{2}}], \qquad f(v) = (vx)^{-\frac{1}{2}},$$
$$\psi(v) = -\tfrac{1}{2}(1 + x)v + (\theta xv)^{\frac{1}{2}}z - \tfrac{1}{2}\theta + \tfrac{1}{2}\ln v,$$

while for $q > 1$ and some $K(n)$,

$$q_{\theta n}(x) = K(n)\int_0^\infty\int_0^1 f(v, z)k_n(v, z)e^{kn\psi(v,z)}\,dz\,dv, \tag{4.6}$$

where

$$k_n(v, z) = 1 + \exp[-2kn(\theta xv)^{\frac{1}{2}}z], \qquad f(v, z) = (vx)^{(q/2)-1}(1 - z^2)^{(q-3)/2},$$
$$\psi(v, z) = -\tfrac{1}{2}(1 + x)v + (\theta xv)^{\frac{1}{2}}z - \tfrac{1}{2}\theta + \tfrac{1}{2}\ln v.$$

Let $\xi = x/(1 + x)$, $\alpha(z) = [(z)^{\frac{1}{2}} + (z + 4)^{\frac{1}{2}}]/2$, $z_0 = 1$ and $v_0 = \alpha^2(\xi\theta)/(1 + x)$. It is easy to show that, for $q = 1$, $\psi(v)$ has a unique maximum at v_0, and that for $q > 1$, $\psi(v, z)$ has a unique maximum at (v_0, z_0). For $q = 1$, we change the origin to v_0 and observe that $q_{\theta n}(x)$ is of the form (3.11)–(3.12). We include here in some detail the corresponding procedure for the case $q > 1$. We change the origin to (v_0, z_0) and denote the new variables by (u, y). Thus, $z = z_0 + y$ and $v = v_0 + u$. We then change the sign of y and call the new variable ζ, i.e., $\zeta = -y$. It follows from (4.6) that

$$q_{\theta n}(x) = K(n)\int_{-v_0}^\infty\int_0^1 g(u, \zeta)l_n(u, \zeta)e^{kn\varphi(n,\zeta)}\,d\zeta\,du, \tag{4.7}$$

where

$$g(u, \zeta) = \{(u + v_0)x\}^{q/2-1}\{\zeta(2 - \zeta)\}^{(q-3)/2},$$
$$l_n(u, \zeta) = 1 + \exp[-2kn[\theta x(v_0 + u)]^{\frac{1}{2}}(1 - \zeta)] \quad \text{and}$$
$$\varphi(u, \zeta) = -\tfrac{1}{2}(1 + x)(v_0 + u) + [\theta x(v_0 + u)]^{\frac{1}{2}}(1 - \zeta) - \tfrac{1}{2}\theta$$
$$+ \tfrac{1}{2}\ln(v_0 + u).$$

We observe that $q_{\theta n}(x)$ is of the form (3.11)–(3.12), by enlarging the domain of integration to $E^1 \times E_0^1$ and setting $g(u, \zeta)$ in (4.7) equal to 0 outside the region indicated in (4.7). The functions g, φ, and l_n satisfy Condition (b) on $E^1 \times E_0^1$: (i) follows from the fact that ψ has a unique maximum at (v_0, z_0), (ii) is implied by the fact that φ decreases as (u, ζ) recedes from the origin, (iii) and (iv) are obvious, (v) follows from the monotonicity of $\{l_n\}$ and the continuity of the l_n and of $\lim_{n\to\infty} l_n$ in a neighborhood of the origin (Dini's theorem), and (vi) is implied by the fact that $q_{\theta n}(x)$ is a density for $n \geqq 2$. Condition (c) is satisfied on $E^1 \times E_0^1$ and at each $x > 0$; (i) and (ii) are obvious while (iii) follows from the immediate extension of Dini's theorem to functions on any finite dimensional space. We also observe $g(u, \zeta)$ is factored in the form required by Theorem 3.2.

It follows from (4.5) and (4.6) that $\psi(v_0) = \psi(v_0, z_0)$. Denote this common value by $h(\theta, x)$. We have for $q \geqq 1$,

$$h(\theta, x) = k[\tfrac{1}{2}\alpha^2(\xi\theta) + \ln \alpha(\xi\theta) - \tfrac{1}{2}\theta + \tfrac{1}{2}\ln(1 - \xi) - 1], \tag{4.8}$$

$$\partial^2 h/\partial\theta\partial x = \tfrac{1}{2}k\alpha^2(\xi\theta)[\xi\theta(\xi\theta + 4)]^{-\frac{1}{2}}\, d\xi/dx > 0, \tag{4.9}$$

$$\partial^2 h/\partial x^2|_{x=\theta} = -[k\{4\theta(1 + \tfrac{1}{2}\theta)\}]^{-1}. \tag{4.10}$$

Let us denote the function given in (4.2) by $h^*(\theta, x)$ to distinguish it from the function given by (4.8) and remember that α and ξ appearing in (4.2) are different from, but related to, α and ξ appearing in (4.8). Algebraic substitution shows that $h(\theta, x) = h^*(\theta^{\frac{1}{2}}, x^{\frac{1}{2}})$. Thus, it follows from Application I that

$$\theta_1 < \theta_0 < (\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}})^2/4 < (\theta_1 + \theta_2)/2. \tag{4.11}$$

Thus, Conclusions (4.0) hold.

III. *The sequential* χ^2*-test.* The Z_j are as in Application II except that σ^2 is assumed to be known and is taken to be unity without loss of generality. $\theta = \sum_{i=1}^{q} \zeta_i^2$ and $\Theta = [0, \infty)$, $n_0 = 2$, and $X_n = \sum_{i=1}^{q} Z_{\cdot i}^2$. For each n sufficiency and invariance under some group of transformations reduce the data to X_n which is a $n^{-1}\chi_q^{2'}(n\theta)$ variable (p. 321 in [20]). The family of distributions of X_n is a MLR family [20]. For $q = 1$, X_n is $n^{-1}\chi_1^{2'}(n\theta)$ and

$$q_{\theta n}(x) = (1 + \exp[-2n(\theta x)^{\frac{1}{2}}])n^{\frac{1}{2}}(2(2\pi x)^{\frac{1}{2}})^{-1} \exp[-(n/2)(x^{\frac{1}{2}} - \theta^{\frac{1}{2}})^2]. \tag{4.12}$$

For $q > 1$, $X_n = U_n + V_n$ where U_n is $n^{-1}\chi_1^{2'}(n\theta)$, V_n is $n^{-1}\chi_{q-1}^2$ and U_n is independent of V_n. Using the relation between the densities of the three variables and changing the variable of integration, we obtain for some $K(n)$,

$$q_{\theta n}(x) = K(n)\int_0^1 f(z)k_n(z)e^{n\psi(z)}\, dz, \tag{4.13}$$

where

$$f(z) = x^{q/2-1}(1 - z^2)^{(q-3)/2}, \qquad k_n(z) = 1 + \exp[-2n(\theta x)^{\frac{1}{2}}z]$$

and

$$\psi(z) = -\tfrac{1}{2}(x + \theta) + (\theta x)^{\frac{1}{2}}z.$$

It is immediate that ψ has a unique maximum at $z = 1$. After changing the origin to 1, we see that $q_{\theta n}(x)$ is of the form (3.11)–(3.12). For $q = 1$, $q_{\theta n}(x)$ is of the form (3.16) with

$$h(\theta, x) = -\tfrac{1}{2}(x + \theta) + (\theta x)^{\frac{1}{2}}, \tag{4.14}$$

$$\partial^2 h/\partial\theta\partial x = (16x\theta)^{-\frac{1}{2}} > 0, \tag{4.15}$$

$$\partial^2 h/\partial x^2|_{x=\theta} = -(4\theta)^{-1}. \tag{4.16}$$

For $q > 1$, let $\psi(1)$ be denoted by $h(\theta, x)$. It follows from (4.13) that $h(\theta, x)$ is given by (4.14) which implies

$$(\theta_0)^{\frac{1}{2}} = [(\theta_1)^{\frac{1}{2}} + (\theta_2)^{\frac{1}{2}}]/2 \tag{4.17}$$

so that $\theta_1 < \theta_0 < (\theta_1 + \theta_2)/2$. Thus, Conclusions (4.0) hold.

IV. *The sequential T^2-test.* The Z_j are identically and independently distributed k-variate normal variables, $Z_j = (Z_{j1}, \cdots, Z_{jk})$, and have mean vector ζ and covariance matrix Σ. Let $\bar{Z}_n$ and S_n be the sample mean vector and the sample covariance matrix (at the nth stage) respectively. $\theta = \zeta'\Sigma^{-1}\zeta$ and $X_n = \bar{Z}_n S_n^{-1} \bar{Z}_n$. The sequential T^2-test is discussed in [14] and the fixed sample size T^2-test is treated in [20] (p. 300). It follows from Sections 9 and 10 of Chapter 7 in [20] that for each n sufficiency and invariance reduce the data to X_n. By an argument similar to that of Application VI, we are able to show that $\bar{Z}_n$ and S_n converge a.e. P_θ. It is shown in [14] and [20] that the distribution of X_n is the same as in Application II.

V. *The sequential ordinary correlation coefficient test.* The Z_j are identically and independently distributed where $Z_j = (U_j, V_j)$ is a bivariate normal variable with means (μ, η), variances (σ^2, τ^2), and correlation coefficient ρ. $\theta = \rho/(1 - \rho^2)^{\frac{1}{2}}$ and $\Theta = (-\infty, \infty)$, $n_0 = 2$ and $X_n = Y_n/(1 - Y_n^2)^{\frac{1}{2}}$ where $n\bar{U}_n = \sum_{i=1}^n U_i$ and $n\bar{V}_n = \sum_{i=1}^n V_i$, $Y_n = \sum_{i=1}^n (U_i - \bar{U}_n)(V_i - \bar{V}_n)/[\sum_{i=1}^n (U_i - \bar{U}_n)^2 \sum_{i=1}^n (V_i - \bar{V}_n)^2]^{\frac{1}{2}}$. It is shown in [20] (p. 251) that for each n sufficiency and invariance under some group of transformations reduce the data to X_n whose family of distributions is a MLR family ([20], [25]).

We use a result of Wijsman [25] according to which X_n is representable as $W_0 + (\theta\chi_n/\chi_{n-1})$, where W_0 is $N(0, 1)$, χ_k denotes $(\chi_k^2)^{\frac{1}{2}}$ and all three variables are independent. It is remarked in [25] that X_n can thus be described as a constant times a t'_{n-1} variable with a random noncentrality parameter which is a $\theta\chi_n$ variable. It follows that X_n is a $(n - 1)^{-\frac{1}{2}} t'_{n-1}((n)^{\frac{1}{2}} Y_n)$ variable where Y_n is $\theta n^{-\frac{1}{2}} \chi_n$. Letting $U_n(y)$ denote a $(n - 1)^{-\frac{1}{2}} t'_{n-1}((n)^{\frac{1}{2}} y)$ variable and p denote the density function of the variable appearing as a superscript, we obtain [25]

$$q_{\theta n}(x) = \int_0^\infty p^{U_n(y)}(x) p_\theta^{Y_n}(y)\, dy. \tag{4.18}$$

The densities of $U_n(y)$ and Y_n are easily obtained from Application I. Thus, for some $K(n)$,

$$q_{\theta n}(x) = K(n) \int_0^\infty \int_0^\infty f(v, y) k_n(v, y) e^{n\psi(v,y)}\, dv\, dy, \tag{4.19}$$

where

$$f(v, y) = (vy)^{-1} \exp [\tfrac{1}{2}(1 + x^2)v^2],$$
$$k_n(v, y) = \exp [([(n - 1)n]^{\frac{1}{2}} - n)yxv],$$
$$\psi(v, y) = -\tfrac{1}{2}(1 + x^2)v^2 + yxv - \tfrac{1}{2}y^2 + \ln v + \ln (y/\theta) - \tfrac{1}{2}y^2/\theta^2.$$

For the purpose of maximizing $\psi(v, y)$ we can use the fact that only the first four terms of $\psi(v, y)$ involve v and are as in (4.1) with θ replaced by y. Let ξ and α be as in Application I. Also let $v_0 = \alpha(\xi\theta)/(1 + x^2)^{\frac{1}{2}}$ and $y_0 = \varphi^2/(1 - \varphi\xi)$ where $\varphi = \theta/(1 + \theta^2)^{\frac{1}{2}}$. It can be shown that ψ has a unique maximum at (v_0, y_0). After changing the origin to (v_0, y_0), we see that $q_{\theta n}(x)$ is of the form (3.11)–(3.12) (see Application II).

Denote $\psi(v_0, y_0)$ by $h(\theta, x)$. We compute

$$h(\theta, x) = -1 - \ln [(1 + x^2)^{\frac{1}{2}}(1 + \theta^2)^{\frac{1}{2}} - x\theta], \tag{4.20}$$

$$\partial^2 h/\partial\theta\partial x = \{(1 + x^2)^{\frac{1}{2}}(1 + \theta^2)^{\frac{1}{2}}[(1 + x^2)^{\frac{1}{2}}(1 + \theta^2)^{\frac{1}{2}} - x\theta]^2\}^{-1} > 0, \tag{4.21}$$

$$\partial^2 h/\partial x^2|_{x=\theta} = -(1 + \theta^2)^{-1}, \tag{4.22}$$

$$\theta_0 = \xi_0/(1 - \xi_0^2)^{\frac{1}{2}}, \quad \text{where} \quad \xi_0 = (\theta_1 + \theta_2)/[(1 + \theta_1^2)^{\frac{1}{2}} + (1 + \theta_2^2)^{\frac{1}{2}}]. \tag{4.23}$$

By studying the sign of $(\theta_1 + \theta_2 - 2\theta_0)(\theta_1 + \theta_2)$, we see that θ_0 does not lie between 0 and $(\theta_1 + \theta_2)/2$. Thus, Conclusions (4.0) hold.

VI. *The sequential multiple correlation coefficient test.* The Z_j are identically and independently distributed where Z_j is a p-variate normal variable $(Z_{j1}, \cdots, Z_{jp})$, $p \geqq 2$, with mean $\zeta = (\zeta_1, \cdots, \zeta_p)$ and covariance matrix $\Sigma = [\sigma_{ij}]$. Let Σ be partitioned as

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where Σ_{11} is 1×1. Let ρ^2 be the square of the population multiple correlation coefficient between Z_{j1} and $(Z_{j2}, \cdots, z_{jp})$ defined by: $\rho^2 = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}/\Sigma_{11}$. $\theta = \rho^2/(1 - \rho^2)$, $\Theta = [0, \infty)$ and $n_0 = p - 1$. Let S_n be the $p \times p$ matrix where the (i, i') element $S_{ii'}^2$ is given by: $(n - 1)S_{ii'}^2 = \sum_{l=1}^{n} (Z_{li} - Z_{\cdot i})(Z_{li'} - Z_{\cdot i'})$. It is shown in [20] (p. 320) that for each n sufficiency and invariance under some group of transformations reduce the data to the square of the sample multiple correlation coefficient given by: $R_n^2 = S_{n,12}S_{n,22}^{-1}S_{n,21}/S_{,\,,11}$, where S_n is partitioned in the same way as Σ. Let $X_n = R_n^2/(1 - R_n^2)$. The family of distributions of X is a MLR family [25].

It is shown in [25] that

$$X_n = [\chi_{p-2}^2 + (W_0 + \theta^{\frac{1}{2}}\chi_n)^2]/\chi_{n-p+1}^2, \tag{4.24}$$

where the variables appearing on the right hand side of (4.24) are independent, W_0 is $N(0, 1)$, and X_n can be interpreted as a constant times an $F'_{p-1,n-p+1}$ variable with a random noncentrality parameter which is a $\theta\chi_n^2$ variable, i.e., $X_n = U_n(Y_n)$ where $U_n(y)$ is $\chi_{p-1}^{2\prime}(ny)/\chi_{n-p+1}^2$ and Y_n is $\theta\chi_n^2/n$. By (4.6) we obtain for $p > 2$ and some $K(n)$,

$$q_{\theta n}(x) = K(n)\int_0^\infty\int_0^\infty\int_0^1 f(v, y, z)k_n(v, y, z)e^{n\psi(v,y,z)}dzdydv, \tag{4.25}$$

where

$$f(v, y, z) = (vx)^{(p-3)/2}y^{-1}(1 - z^2)^{(p-4)/2},$$
$$k_n(v, y, z) = 1 + \exp[-2n(yxv)^{\frac{1}{2}}z],$$
$$\psi(v, y, z) = -\tfrac{1}{2}(1 + x)v + (yxv)^{\frac{1}{2}}z - \tfrac{1}{2}y + \tfrac{1}{2}\ln v + \tfrac{1}{2}\ln(y/\theta) - \tfrac{1}{2}(y/\theta).$$

For $p = 2$, $q_{\theta n}(x)$ can be written as a double integral using (4.5) instead of (4.6). This will be omitted because its treatment is exactly as in the case $p > 2$. However, we mention that $h(\theta, x)$ is the same for the case $p = 2$ as for the case $p > 2$. In order to maximize ψ, z should be assigned the value 1. Let ψ^* denote the function given in (4.19) which we write now as $\psi^*(v, y; \theta, x)$. Let $\psi(v, y, z; \theta, x)$ denote the function given in (4.25). Algebraic substitution shows that $\psi(v, y, 1; \theta, x) = \psi^*(v^{\frac{1}{2}}, y^{\frac{1}{2}}; \theta^{\frac{1}{2}}, x^{\frac{1}{2}})$. Using the results of the preceding application regarding ψ^*, we conclude that ψ has a unique maximum at (v_0, y_0, z_0) where $z_0 = 1$ and v_0, y_0 are different from, but related to, their counterparts in Application V. By changing the origin to (v_0, y_0, z_0) we see that $q_{\theta n}(x)$ is of the form (3.11)–(3.12) (see Application II).

It follows from (4.20) that

$$h(\theta, x) = -1 - \ln[(1 + x)^{\frac{1}{2}}(1 + \theta)^{\frac{1}{2}} - (x\theta)^{\frac{1}{2}}]. \tag{4.26}$$

The positiveness of $\partial^2 h/\partial\theta\partial x$ can be established by using the relation

$$h(\theta, x) = h^*(\theta^{\frac{1}{2}}, x^{\frac{1}{2}}), \tag{4.27}$$

where h^* denotes the function given by (4.20). We conclude from (4.26) that

$$\theta_0 = \xi_0/(1 - \xi_0) \text{ where } \xi_0^{\frac{1}{2}} = (\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}})/[(1 + \theta_1)^{\frac{1}{2}} + (1 + \theta_2)^{\frac{1}{2}}]. \tag{4.28}$$

Using the information obtained about θ_0 of Application V, we find

$$\tfrac{1}{2}(\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}}) < \theta_0^{\frac{1}{2}} < \theta_2^{\frac{1}{2}}. \tag{4.29}$$

Notice that Conclusions (4.0) hold and that

$$\partial^2 h/\partial_x^2|_{x=\theta} = -[4\theta(1 + \theta)]^{-1}. \tag{4.30}$$

VII. *A sequential test of the absolute value of the parameter in an exponential family.* The Z_j are identically and independently distributed real variables having density $c(\vartheta)e^{\vartheta z}$ with respect to some σ-finite measure μ. The parameter ϑ is real and hypotheses are made about $\varphi = |\vartheta|$. We assume further that $c(\vartheta) = c(-\vartheta) = c(\varphi)$, that $\mu(A) = \mu(-A)$ for every Borel set A and that $\int_{-\infty}^{\infty}$

$e^{\vartheta z} d\mu(z) < \infty$ for $0 \leqq \varphi < K < \infty$, where $K > 0$. G consists of the identity and reflection transformations. Reduction by sufficiency and invariance leads to the test statistic $X_n = |\bar{Z}_n|$. It is shown in [20] that $c(\vartheta)$ is analytic. Letting $c'(\varphi)$ denote the derivative of $c(\varphi)$, we take

$$\theta = -c'(\varphi)/c(\varphi). \tag{4.32}$$

The density of X_n, with respect to the n-fold convolution of μ, is given by

$$q_{\theta n}(x) = n[c(\varphi)]^n(e^{n\vartheta x} + e^{-n\vartheta x}), \varphi, x, > 0. \tag{4.33}$$

From this, the MLR property can be easily established. It is well known [20] that

$$E_\vartheta Z_j = -c'(\vartheta)/c(\vartheta). \tag{4.34}$$

Furthermore, from the symmetry assumptions it follows that $E_{-\vartheta} Z_j = -E_\vartheta Z_j$. Thus, $E_0 Z_j = 0$. Moreover, the MLR property implies that $E_\vartheta Z_j$ is a strictly increasing function of ϑ. Thus, $|E_\vartheta Z_j| = E_\varphi Z_j = -[c'(\varphi)/c(\varphi)] = \theta$ (by (4.32)) is a strictly increasing continuous function of φ, which justifies the parametrization by θ.

We mention two examples of the situation described above: (1) The Z_j are $N(\vartheta, 1)$. $p^Z(z) = e^{-\vartheta^2/2} e^{\vartheta z}$ with respect to $(e^{-z^2/2}/(2\pi)^{\frac{1}{2}})$ times the Lebesgue measure. $\theta = \varphi$. (2) The $Z_j + \frac{1}{2}$ is Bernoulli (p), $\vartheta = \ln p/(1-p)$ and, for $z \,\varepsilon\, [-\frac{1}{2}, \frac{1}{2}]$, $p^Z(z) = [p(1-p)]^{\frac{1}{2}} e^{\vartheta z}$ with respect to the counting measure on $\{-\frac{1}{2}, \frac{1}{2}\}$. Here $[c(\vartheta)]^{-1} = e^{\vartheta/2} + e^{\vartheta/2}$ and $2\theta = \tanh(\varphi/2)$.

It follows from (4.33) and (4.32) that $q_{\theta n}(x)$ is of the form (3.16) and that

$$q_{\theta n}(x) = (1 + e^{-2n\varphi x})n \exp[n\{\varphi x + \ln c(\varphi)\}], \tag{4.35}$$

$$h(\theta, x) = \varphi x + \ln c(\varphi), \tag{4.36}$$

$$\partial^2 h/\partial\theta\partial x = d\varphi/d\theta > 0, \tag{4.37}$$

$$\theta_0 = -[\varphi(\theta_2) - \varphi(\theta_1)]^{-1} \ln [c(\varphi(\theta_2))/c(\varphi(\theta_1))]. \tag{4.38}$$

In (4.32) we let φ_i be the solution of $\theta_i = -c'(\varphi)/c(\varphi)$, for $i = 1, 2$. $\theta_0 >, =,$ or $< (\theta_1 + \theta_2)/2$ according as $\ln c(\varphi_1) - \ln c(\varphi_2) - (\varphi_2 - \varphi_1)/2 >, =,$ or < 0. Let $\psi(\varphi) = \ln c(\varphi_1) - \ln c(\varphi_2) - (\varphi - \varphi_1)(\theta_1 + \theta)/2$ for $\varphi > \varphi_1$. Then, $\psi(\varphi_1) = 0$ so that $\psi(\varphi)$ has the same sign as $\psi'(\varphi)$ provided the latter does not change sign. Denoting $d\theta/d\varphi$ by $\theta'(\varphi)$, we compute $2\,\psi'(\varphi) = (\theta - \theta_1) - (\varphi - \varphi_1)\theta'(\varphi)$ which has the same sign as $(\theta - \theta_1)/(\varphi - \varphi_1) - \theta'(\varphi) = \theta'(\varphi^*) - \theta'(\varphi)$ for some φ^*, $\varphi_1 < \varphi^* < \varphi$. If $\theta'(\varphi)$ is monotonic, we conclude that $\theta_0 >, =,$ or $< (\theta_1 + \theta_2)/2$ according as θ is a strictly concave, linear, or strictly convex function of φ. Applied to our two examples, this criterion gives: $\theta_0 = (\theta_1 + \theta_2)/2$ in the first, and $(\theta_1 + \theta_2)/2 < \theta_0 < \theta_2$ in the second.

If $\vartheta > 0$, i.e., $\vartheta = \varphi$, $E_\vartheta Z_j$ given by (4.34) equals $-c'(\varphi)/c(\varphi) = \theta$ (by (4.32)). Furthermore, the variance of Z_j is $\sigma^2(\vartheta) = \sigma^2(\varphi) = -\partial^2 \ln c(\varphi)/\partial\varphi^2$ ([20], p. 58). Thus, the central limit theorem implies that, for $\vartheta > 0$, $n^{\frac{1}{2}}(\bar{Z}_n - \theta)$ is asymptotically $N(0, \sigma^2(\varphi))$. Since $\bar{Z}_n > 0$ with probability tending to unity, $n^{\frac{1}{2}}(X_n - \theta)$ has the same limiting distribution. Thus, Conclusions (4.0) hold.

VIII. *A sequential test of Model II analysis of variance in a one-way classification.* We describe first the usual fixed sample size Model II analysis of variance test. Let $(Z_1, \cdots, Z_n)$ be s-dimensional vectors with $Z_i = (Z_{i1}, \cdots, Z_{is})$ and for $i = 1, \cdots, n, j = 1, \cdots, s$; $Z_{ij} = \mu + a_j + e_{ij}$ where μ is a constant, the a_j and e_{ij} are samples from $N(0, \sigma_a^2)$ and $N(0, \sigma_e^2)$ respectively. Notice that unlike the Model I analysis of variance, where $\sigma_a^2 = 0$, the Z_{ij} are not independent. Hypotheses are made about $\theta = \sigma_a^2/\sigma_e^2$, $\theta \varepsilon [0, \infty)$. It is shown in [20], (p. 287) that some proper orthogonal transformations reduce the problem to the following canonical form: $\{Y_{ij}, i = 1, \cdots, n \text{ and } j = 1, \cdots, s\}$ are independent where Y_{11} is $N((ns)^{\frac{1}{2}}\mu, \sigma_e^2 + n\sigma_a^2)$, $Y_{12}, \cdots, Y_{1s}$ are $N(0, \sigma_e^2 + n\sigma_a^2)$ and the remaining variables are $N(0, \sigma_e^2)$. Reduction by sufficiency and invariance under some group of transformations leads, for each n, to the test statistic [20]:

$$(4.39)\quad W_n = \textstyle\sum_{j=2}^s Y_{ij}^2 / \sum_{i=2}^n \sum_{j=1}^s Y_{ij}^2$$
$$= n \textstyle\sum_{j=1}^s (Z_{.j} - Z_{..})^2 / \sum_{i=1}^n \sum_{j=1}^s (Z_{ij} - Z_{.j})^2.$$

Notice that $\sum_{j=2}^s Y_{ij}^2$ is $(1 + n\theta)\sigma_e^2\chi_{s-1}^2$ and that $\sum_{i=1}^n \sum_{j=1}^s Y_{ij}^2$ is $\sigma_e^2\chi_{(n-1)s}^2$. By making use of the density of an F-variable, the family of distributions of W_n is shown to be a MLR family.

An SPRT based on $\{T_n\}$ defined by (4.39) does not exhibit the desired behavior (1.1). This is not surprising because such a procedure increases our information about one population $(N(0, \sigma_e^2))$ but not the other, by holding s fixed and increasing n sequentially, whereas testing is desired about σ_a^2/σ_e^2. We modify the design so that at the nth stage we have available n observations on each of n "individuals":

$$[Z_{11}], \quad \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \end{bmatrix}, \quad \begin{bmatrix} Z_{11} & Z_{21} & Z_{31} \\ Z_{12} & Z_{22} & Z_{32} \\ Z_{13} & Z_{23} & Z_{33} \end{bmatrix}, \cdots,$$

where Z_{ij} has the same distribution as given above. The analysis of this problem is exactly as before with $s = n$. We thus have from (4.39),

$$(4.40)\qquad X_n = n \textstyle\sum_{j=1}^n (Z_{.j} - Z_{..})^2 / \sum_{i=1}^n \sum_{j=1}^n (Z_{ij} - Z_{.j})^2.$$

It follows that X_n is a $(1 + n\theta)n^{-1}F_{n-1,n(n-1)}$ variable.

We use the density of an F-variable to obtain for some $K(n)$

$$(4.41)\quad q_{\theta n}(x) = K(n)x^{(n-3)/2}(1 + n\theta)^{-(n-1)/2}(1 + x(1 + n\theta)^{-1})^{-(n^2-1)/2},$$

and

$$(4.42)\qquad q_{\theta n}(x) \sim n^{-(n-1)/2}K(n)\theta^{\frac{1}{2}}x^{-\frac{3}{2}} \exp [x^2/4\theta^2 - 1/2\theta]e^{nh(\theta,x)}$$

where

$$(4.43) \qquad h(\theta, x) = \tfrac{1}{2}[-x/\theta + \ln(x/\theta)],$$

$$(4.44) \qquad \partial^2 h/\partial\theta\partial x = (2\theta^2)^{-1} > 0,$$

$$(4.45) \qquad \partial^2 h/\partial x^2|_{x=\theta} = -(2\theta^2)^{-1}.$$

Conditions B and A_1 follow from (4.42) and (4.44). Condition A_2 is checked by noting that $\partial h/\partial\theta = 0$ for $\theta = x$ and that $\partial^2 h/\partial\theta^2 < 0$ for all θ. We obtain

$$(4.46) \qquad \theta_0 = \theta_1\theta_2(\theta_2 - \theta_1)^{-1} \ln(\theta_2/\theta_1).$$

By showing that $2[h(\theta_2, (\theta_1 + \theta_2)/2) - h(\theta_1, (\theta_1 + \theta_2)/2] = f(\theta_2/\theta_1)$, where $(\xi) = -\ln\xi + \frac{1}{2}(\xi - \xi^{-1})$ and that $f(1) = 0$ while $f'(\xi) > 0$ for $\xi \geqq 1$, we conclude

$$(4.47) \qquad \theta_1 < \theta_0 < (\theta_1 + \theta_2)/2.$$

Since $q_{\theta n}(x)$ is not written in either of the forms (3.11)–(3.12) or (3.16), we verify directly, using (4.42), that $r_n(\theta_0 + c/n) \to \alpha \exp[cg'(\theta_0)]$, where $g'(\theta_0) = \frac{1}{2}(\theta_1^{-1} - 0_2^{-1})$, and $\alpha = (\theta_2/\theta_1) \exp[(\theta_0^2/4\theta_0^2 - 1/2\theta_0) - (\theta_0^2/4\theta_0^2 - 1/2\theta_0)] > 0$. Thus, Conclusions (4.0) hold.

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