ON THE EXTRAPOLATION OF A SPECIAL CLASS OF STATIONARY TIME SERIES

By William F. Trench

Drexel Institute of Technology

In this paper, we consider a problem previously solved by P. A. Kozuljaev [1]. Let $y(t)\{t = 0, \pm 1, \pm 2, \cdots\}$ be a discrete stationary (in the wide sense) time series, with zero mean, unit variance, and independent samples. Thus,

$$E\{y(t)\} = 0, \quad E\{[y(t)]^2\} = 1, \quad E\{y(t)y(t+\tau)\} = 0, \quad \tau \neq 0.$$

For an arbitrary but fixed $m > 1$, form the new random variable

$$x(t, m) = m^{-\frac{1}{2}} \textstyle\sum_{i=1}^{m} y(t + \mathrm{i}). \tag{1}$$

Then $E\{x(t, m)\} = 0$, and $E\{x(t, m)x(t + \tau, m)\} = R(\tau, m)$, where

$$\begin{aligned} R(\tau, m) &= 1 - |\tau|/m, \qquad && \tau = 0, \pm 1, \pm 2, \cdots, \pm(m-1) \\ &= 0, && \tau \geqq m. \end{aligned} \tag{2}$$

The problem considered by Kozuljaev is that of extrapolating the stationary sequence (1). That is, for each fixed pair of positive integers p and n, he has solved the problem of determining coefficients $a_1, a_2, \cdots, a_n$ such that the variance

$$\mu(a_1, a_2, \cdots, a_n; m, p) = E\{(x(t + p, m) - \textstyle\sum_{i=1}^{n} a_i x(t + 1 - i, m))^2\} \tag{3}$$

is minimized. If $a_1, a_2, \cdots, a_n$ are chosen to minimize (3), then

$$\tilde{x}(t + p, m) = \textstyle\sum_{i=1}^{n} a_i x(t + 1 - i, m)$$

is the minimum variance (linear) estimate of $x(t + p, m)$, based on $x(t, m)$, $x(t - 1, m), \cdots, x(t - n + 1, m)$, and (3) is the variance of the estimate. For each m, n, and p Kozuljaev has determined the unique set of coefficients which minimizes (3), and has computed the minimum variance. He shows that the coefficients satisfy a certain linear algebraic system, given below, and solves this system by Cramer's rule. However, the evaluation of the determinants which arise in this method involves a large amount of labor, with the result that Kozuljaev's paper contains a series of some forty-four theorems, all of which are directed at solving this linear system, and calculating the resulting minimum variance.

In this paper, we give a different method of deriving Kozuljaev's results, based on considering the coefficients to be part of a sequence which is the solution of a certain difference equation. This viewpoint leads to a method of solution of the extrapolation problem, for this special case, which is considerably less involved than Kozuljaev's method.

Received 11 January 1965; revised 24 May 1965.

Throughout this paper, we consider m, n, and p to be fixed positive integers. It is understood that $a_1, a_2, \cdots, a_n$, and μ depend on these integers. However, we will not make this dependence explicit by the introduction of appropriate subscripts and superscripts, since the resulting complexity of notation would be quite burdensome. In the interests of simplicity, we replace (2) by

$$\begin{aligned} V_j &= 1 - |j|/m, \qquad & |j| \leqq m - 1, \\ &= 0, & |j| \geqq m. \end{aligned} \tag{4}$$

The following lemma is proved in [1].

LEMMA 1. *The coefficients* $a_1, a_2, \cdots, a_n$ *which minimize* (3) *are the unique solution of*

$$\textstyle\sum_{j=1}^{n} V_{i-j} a_j = V_{i+p-1}, \qquad 1 \leqq i \leqq n. \tag{5}$$

This lemma holds in general for any non-deterministic process. The process studied here is non-deterministic, since it is a finite moving average of a stationary process with independent samples.

LEMMA 2. *If* $p \geqq m$, *then* $a_1 = a_2 = \cdots = a_n = 0$.

PROOF. If $p \geqq m$, it follows from (4) that the system (5) is homogeneous, and the conclusion follows from Lemma 1.

It is straightforward to verify the following lemma.

LEMMA 3. *Let* $1 \leqq p \leqq m - 1$, *and* $a_1, a_2, \cdots, a_n$ *be the solution of* (5). *Define*

$$\begin{aligned} &\text{(a)} \qquad a_j = -\delta_{-p+1,j}, \qquad & -m + 2 \leqq j \leqq 0, \\ &\text{(b)} \qquad a_j = 0, & n + 1 \leqq j \leqq n + m - 1. \end{aligned} \tag{6}$$

Then $\{a_r\}$, $(-m + 2 \leqq r \leqq n + m - 1)$ *is the unique solution of*

$$\textstyle\sum_{j=-(m-1)}^{m-1} V_j a_{j+i} = 0, \qquad 1 \leqq i \leqq n, \tag{7}$$

which satisfies (6).

This lemma reduces the extrapolation problem to that of solving the difference equation (7), subject to Conditions (6). It is easy to see that it holds for any process which is a finite moving average of a stationary process with independent samples:

$$x(t) = \textstyle\sum_{i=1}^{m} K_i y(t + i),$$

$K_1, K_2, \cdots, K_m$ being arbitrary complex numbers, (with $K_1 \neq 0$ and $K_m \neq 0$), and

$$\begin{aligned} V_j &= \textstyle\sum_{i=1}^{m-|j|} \bar{K}_i K_{i+|j|}, \qquad & |j| \leqq m - 1, \\ &= 0, & |j| \geqq m. \end{aligned}$$

For $\{V_j\}$ as given by (4), we have the following theorem. (As was noted by the referee, one can also handle (7) by means of a summation by parts; this would exploit the fact that V_j is piecewise linear.)

THEOREM 1. *The solution of* (6) *and* (7) *can be written in the form*

$$a_j = b_j + jc_j, \quad -m + 2 \leqq r \leqq n + m - 1, \tag{8}$$

where

$$b_j = b_i, \; c_j = c_i, \quad \textit{if } j \equiv i (\operatorname{mod} m), \tag{9}$$

$$b_j - (m - j)c_j = -\delta_{p-1,m-j}, \qquad 2 \leqq j \leqq m, \tag{10}$$

$$b_{n+j} + (j + n)c_{n+j} = 0, \qquad 1 \leqq j \leqq m - 1, \tag{11}$$

$$b_1 + b_2 + \cdots + b_m = 0, \tag{12}$$

$$c_1 + c_2 + \cdots + c_m = 0. \tag{13}$$

PROOF. The characteristic equation of (7) is

$$\textstyle\sum_{j=-(m-1)}^{m-1} V_j z^j = m^{-1} z^{-m+1}(1 - z)^{-2}(1 - z^m)^2 = 0,$$

which has each of the mth roots of unity, except $z = 1$, as a double root. Hence, the general solution of (7) is of the form

$$a_j = \textstyle\sum_{r=1}^{m-1} (B_r + jC_r) \exp(2\pi i r j/m),$$

where B_r and C_r are suitable constants. Setting $b_j = \sum_{r=1}^{m-1} B_r \exp(2\pi i r j/m)$, and $c_j = \sum_{r=1}^{m-1} C_r \exp(2\pi i r j/m)$, we obtain (8), (9), (12), and (13). From (6a) and (8), $b_j + jc_j = -\delta_{-p+1,j}$, $-m + 2 \leqq j \leqq 0$. Replace j by $-j$ to obtain $b_{-j} - jc_{-j} = -\delta_{p-1,j}$, $0 \leqq j \leqq m - 2$. From (9), we can write this as $b_{m-j} - jc_{m-j} = -\delta_{p-1,j}$, $0 \leqq j \leqq m - 2$, and replace j by $m - j$ to obtain (10). Equation (11) follows from (6b) and (8).

Equations (10) through (13) constitute a set of $2m$ linear equations in the $2m$ unknowns $b_1, b_2, \cdots, b_m$ and $c_1, c_2, \cdots, c_m$. The form of the solution depends on the residue class of n modulo m, as will be seen in the following theorems.

THEOREM 2. *Let* $b_1, b_2, \cdots, b_m, c_1, c_2, \cdots, c_m$ *be the solution of* (10) *through* (13), *where* $1 \leqq p \leqq m - 1$. *Then* $b_r = c_r = 0$ *unless* r *meets at least one of the following conditions*:

(i) $r = 1$,
(ii) $r = m - p + 1$,
(iii) $r \equiv n (\operatorname{mod} m)$.

PROOF. Suppose r does not satisfy any of these conditions. Then, from (10),

$$b_r - (m - r)c_r = 0,$$

and, from (11)

$$b_r + (j + n)c_r = 0,$$

where j is chosen so that $n + j \equiv r (\operatorname{mod} m)$. (Since $r \not\equiv n (\operatorname{mod} m)$, $1 \leqq j \leqq m - 1$, and there is an equation of the form (11) which involves b_j.) These two equations have only the trivial solution, which proves the theorem.

THEOREM 3. *If* $1 \leqq p \leqq m - 1$, *and* $n \equiv 1 (\text{mod } m)$, *then*

$$b_1 = -b_{m-p+1} = (n + m - p)/(n + m - 1), \tag{14}$$

and

$$c_1 = -c_{m-p+1} = -1/(n + m - 1). \tag{15}$$

PROOF. From (10)

$$b_{m-p+1} - (p - 1)c_{m-p+1} = -1.$$

Since $n \equiv 1(\text{mod } m)$, setting $j = m - p$ in (11) and using (9) yields

$$b_{m-p+1} + (n + m - p)c_{m-p+1} = 0.$$

These equations have the unique solution given by (14) and (15). From Theorem 2, and from (12) and (13), the first equalities of (14) and (15) follow.

THEOREM 4. *If* $1 \leqq p \leqq m - 1$, *and* $n \equiv m - p + 1(\text{mod } m)$, *then*

$$b_1 = -b_{m-p+1} = (n + p)/(n + 2p - 1), \tag{16}$$

$$c_1 = -c_{m-p+1} = -1/(n + 2p - 1). \tag{17}$$

PROOF. $n + p \equiv 1(\text{mod } m)$. In (11) take $j = p$, and use (9) to obtain

$$b_1 + (n + p)c_1 = 0.$$

From (10)

$$b_{m-p+1} - (p - 1)c_{m-p+1} = -1,$$

and from (12), (13), and Theorem 2,

$$b_1 + b_{m-p+1} = 0,$$

$$c_1 + c_{m-p+1} = 0.$$

These four equations have the unique solution given by (16) and (17).

THEOREM 5. *If* $1 \leqq p \leqq m - 2$, *and* $n \equiv k(\text{mod } m)$, *where* $2 \leqq k \leqq m - p$, *then*

$$\begin{aligned}
c_1 &= -(n + 2m - 2k - p + 1)/[(n + m - k)(2m - 2k + n + 1)] \\
b_1 &= -(n + m - k + 1)c_1, \\
c_k &= -p/[(2m - 2k + n + 1)(n + m - k)], \\
b_k &= (m - k)c_k, \\
c_{m-p+1} &= 1/(n + m - k), \\
b_{m-p+1} &= -(n + m - p - k + 1)c_{m-p+1}.
\end{aligned}$$

PROOF. From (10)

$$b_k - (m - k)c_k = 0, \tag{18}$$

$$b_{m-p+1} - (p-1)c_{m-p+1} = -1. \tag{19}$$

From (12), (13), and Theorem 2

$$b_1 + b_k + b_{m-p+1} = 0, \tag{20}$$

$$c_1 + c_k + c_{m-p+1} = 0, \tag{21}$$

and from (9), (11), and the conditions on n and k,

$$b_1 + (n+m-k+1)c_1 = 0, \tag{22}$$

$$b_{m-p+1} + (n+m-p-k+1)c_{m-p+1} = 0.$$

This system has the unique solution given in the theorem.

THEOREM 6. *If $2 \leqq p \leqq m-1$, and $n \equiv k(\text{mod } m)$, where $m-p+2 \leqq k \leqq m$, then*

$$\begin{aligned}
c_1 &= -(n+3m-2k-p+1)/[(n+2m-2k+1)(n+2m-k)] \\
b_1 &= -(n+m-k+1)c_1, \\
c_k &= (m-p)/[(n+2m-2k+1)(n+2m-k)], \\
b_k &= (m-k)c_k, \\
c_{m-p+1} &= 1/(n+2m-k), \\
b_{m-p+1} &= -(n+2m-k-p+1)c_{m-p+1}.
\end{aligned}$$

PROOF. It is easy to verify that the six quantities in question must satisfy (18) through (22). The sixth equation is obtained from (9) and (11), by noting that, from the conditions on n and k, $j = 2m-k-p+1$ is the number which is congruent to $m-p+1(\text{mod } m)$, and at the same time satisfies $1 \leqq j \leqq m-1$. Hence

$$b_{m-p+1} + (n+2m-k-p+1)c_{m-p+1} = 0.$$

The system consisting of (18) through (22), together with this equation, has the unique solution given in the statement of the theorem.

The following theorem can be established in a straight-forward manner from theorems and lemmas which precede it. It contains the same results as those given by Kozuljaev, except for some minor discrepancies, which are presumably caused by typographical errors in his paper.

THEOREM 7. *For each triple of positive integers m, n, and p, there is a unique set of coefficients $a_1, a_2, \cdots, a_n$ which minimizes the variance (3). Every possibility is contained in the following five cases:*

Case 1. *If $p \geqq m$, $a_j = 0$ for all j.*

Case 2. *If $1 \leqq p \leqq m-1$ and $n \equiv 1(\text{mod } m)$, then*

$$a_j = (n+m-p-j)/(n+m-1), \qquad \text{if } j \equiv 1(\text{mod } m),$$

$$a_j = -(n+m-p-j)/(n+m-1), \qquad \text{if } j \equiv m-p+1(\text{mod } m),$$

$a_j = 0,$ *otherwise.*

Case 3. *If* $1 \leqq p \leqq m - 1$ *and* $n \equiv m - p + 1(\mathrm{mod}\, m)$, *then*

$$a_j = (n + p - j)/(n + 2p - 1), \qquad \text{if } j \equiv 1(\mathrm{mod}\, m),$$

$$a_j = -(n + p - j)/(n + 2p - 1), \qquad \text{if } j \equiv m - p + 1(\mathrm{mod}\, m),$$

$$a_j = 0, \qquad \text{otherwise}$$

Case 4. *If* $1 \leqq p \leqq m - 2$ *and* $n \equiv k(\mathrm{mod}\, m)$, *where* $2 \leqq k \leqq m - p$, *then*

$$a_j = [(n + 2m - 2k - p + 1)(n + m - k - j + 1)]/ [(n + m - k)(2m - 2k + n + 1)], \qquad \text{if } j \equiv 1(\mathrm{mod}\, m),$$

$$a_j = -[p(m - k + j)]/[(n + m - k)(2m - 2k + n + 1)], \qquad \text{if } j \equiv k(\mathrm{mod}\, m),$$

$$a_j = -(n + m - p - k - j + 1)/(n + m - k), \qquad \text{if } j \equiv m - p + 1(\mathrm{mod}\, m),$$

$$a_j = 0, \qquad \text{otherwise.}$$

Case 5. *If* $2 \leqq p \leqq m - 1$ *and* $n \equiv k(\mathrm{mod}\, m)$, *where* $m - p + 2 \leqq k \leqq m$, *then*

$$a_j = [(n + 3m - 2k - p + 1)(n + m - k - j + 1)]/ [(n + 2m - 2k + 1)(n + 2m - k)], \qquad \text{if } j \equiv 1(\mathrm{mod}\, m),$$

$$a_j = [(m - p)(m - k + j)]/[(n + 2m - 2k + 1)(n + 2m - k)], \qquad \text{if } j \equiv k\ (\mathrm{mod}\, m),$$

$$a_j = -(n + 2m - k - p - j + 1)/(n + 2m - k), \qquad \text{if } j \equiv m - p + 1(\mathrm{mod}\, m),$$

$$a_j = 0, \qquad \text{otherwise.}$$

It remains only to calculate the resulting mean square error (3). We give a method which is considerably shorter than Kozuljaev's. From (3) and the definition of V_r

$$\mu = 1 - 2\sum_{j=1}^{n} a_j V_{p+j-1} + \sum_{i,j=1}^{n} a_i a_j V_{i-j}. \tag{23}$$

By substituting into this expression from the results of Theorem 7, Kozuljaev was able to evaluate μ. However, the manipulations involved in this procedure are exceedingly tedious, and, fortunately, unnecessary. From (5) and (23) it follows that

$$\mu = 1 - \sum_{j=1}^{n} a_j V_{p+j-1}. \tag{24}$$

$V_{p+j-1} = 0$ if $j > m - p$. Furthermore, in Cases 1, 2, 3, and 5, $a_j = 0$ if $2 \leqq j \leqq m - p$. Hence, in these cases (24) can be written

$$\mu = 1 - a_1 V_p .$$

In Case 4, (24) can be written

$$\mu = 1 - a_1 V_p - a_k V_{p+k-1} .$$

The computation of μ is now reduced to a triviality. The results, which agree with Kozuljaev's are listed for reference.

CASE 1. $\mu = 1$.

CASE 2. $\mu = p(n + 2m - p - 1)/[m(n + m - 1)]$.

CASE 3. $\mu = p(n + p + m - 1)/[m(n + 2p - 1)]$.

CASE 4. $\mu = [p(n + 2m - 2k - p + 1)(n + 2m - k)]/[m(n + m - k)(n + 2m - 2k + 1)]$.

CASE 5. $\mu = [p(n + m - k)(n + 4m - 2k - p + 1) + m^2(m - k + 1)]/[m(n + 2m - 2k + 1)(n + 2m - k)]$.

Acknowledgment. The author thanks the referee for his suggestions on the presentation of Lemmas 1 and 2 and Theorem 1.

REFERENCE

[1] KOZULJAEV, P. A. (1962). On the theory of extrapolation of stationary sequences. *Select. Transl. Math. Statist. Prob.* **3** 91–127.