A CLASS OF TESTS WITH MONOTONE POWER FUNCTIONS FOR TWO PROBLEMS IN MULTIVARIATE STATISTICAL ANALYSIS[1]

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1. Summary. The problem of testing the general multivariate linear hypothesis, also known as MANOVA, and the problem of testing independence between two sets of multivariate normal random variables are the two problems considered in this paper. In [3] two sufficient conditions for the power function of an invariant test of the general linear hypothesis to be a monotone increasing function of each of the noncentrality parameters have been obtained. In [2] one of these two conditions has been extended to invariant tests of the hypothesis of independence between two sets of variates. These conditions are in terms of, respectively, the convexity and the symmetry of certain sections of the acceptance regions of the tests; and their verification is, in general, nontrivial. In this paper it is shown that the power functions of the members of a class of invariant tests based on statistics "generated" by symmetric gauge functions of convex increasing functions of the maximal invariants are monotone increasing functions of the relevant noncentrality parameters. In this process we have explained an interesting tie-up between the monotonicity properties of the invariant tests for the two problems (Theorem 2) and have obtained extension of some results on the symmetric gauge functions and convexity in the matrix theory (Theorem 3 and Theorem 4). The terms "invariance," "invariant tests," and "maximal invariants" have been used throughout this paper in connection with the relevant groups of transformations mentioned in the Section 2 without their explicit mention each time.

2. Introduction. In this section we shall state the canonical forms and the relevant results on the monotonicity properties of the power functions of the invariant tests; and obtain a tie-up, in the context of the monotonicity properties, between the MANOVA problem and the problem of testing independence between two sets of variates.

In canonical form [3], [5] the MANOVA problem involves a $(p \times n)$ random matrix

$$[\mathbf{X}(p \times s), \mathbf{Y}(p \times (n - r)), \mathbf{Z}(p \times (r - s))], \qquad s \leqq r \leqq n - p,$$

whose column vectors are independently distributed according to p-variate normal distributions with the same covariance matrix $\mathbf{\Sigma}(p \times p)$ and expectations given by

$$E\mathbf{X} = \mathbf{\Delta}(p \times s), \qquad E\mathbf{Y} = \mathbf{0}(p \times (n - r)), \qquad E\mathbf{Z} = \mathbf{\Gamma}(p \times (r - s)).$$

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In this canonical form the MANOVA hypothesis is $\mathcal{H}_0 : \mathbf{\Delta} = \mathbf{0}(p \times s)$. It is well known [5] that the problem is invariant under transformations: $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \rightarrow (\mathbf{BXF}_1, \mathbf{BYF}_2, \mathbf{BZF}_3 + \mathbf{G})$, ($\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ orthogonal, $\mathbf{B}$ nonsingular) and the invariant tests of $\mathcal{H}_0$ depend only on the characteristic roots of $\mathbf{S}_h\mathbf{S}_e^{-1}$, where $\mathbf{S}_h$ and $\mathbf{S}_e$ are the sums of products matrices, respectively, due to the hypothesis and due to error, $\mathbf{S}_h = \mathbf{XX}'$, $\mathbf{S}_e = \mathbf{YY}'$. Moreover, the power functions of the invariant tests depend only on $\theta_1, \theta_2, \cdots, \theta_t$, the $t = \min(p, s)$ noncentrality parameters, which are the characteristic roots of $\mathbf{\Delta\Delta}'\mathbf{\Sigma}^{-1}$.

The canonical form [2], [10] for the problem of testing independence between two sets of variates, referred to in the sequel as the independence problem, involves two random matrices $\mathbf{X}(p \times n) = (x_{ij})$ and $\mathbf{Y}(q \times n) = (y_{ij})$, $q \geqq p$, with the joint probability density

$$(2\pi)^{-n(p+q)/2}\prod_{i=1}^{p} (1 - \rho_i^2)^{-n/2} \exp [-\tfrac{1}{2}[\textstyle\sum_{i=1}^{p} (1 - \rho_i^2)^{-1} \cdot \sum_{\alpha=1}^{n} (x_{i\alpha}^2 + y_{i\alpha}^2 - 2\rho_i x_{i\alpha} y_{i\alpha}) + \sum_{i=p+1} \sum_{\alpha=1} y_{i\alpha}^2)],$$

where $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_p$ are the squares of the population canonical correlation coefficients. The hypothesis of independence is equivalent to $\mathcal{H}_0 : \rho_1 = \rho_2 = \cdots = \rho_p = 0$. It is well known [2], [5] that the problem is invariant under the transformation

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \rightarrow \begin{pmatrix}\mathbf{B}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\mathbf{F},$$

where $\mathbf{B}_1$, $\mathbf{B}_2$ are nonsingular matrices of order p and q, respectively, and $\mathbf{F}$ is orthogonal. An invariant test of $\mathcal{H}_0$ depends only on the sample canonical correlation coefficients, that is, the characteristic roots of $(\mathbf{XX}')^{-1}(\mathbf{XY}')\cdot(\mathbf{YY}')^{-1}(\mathbf{YX}')$, and the power function of such a test depends only on the noncentrality parameters $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_p$.

The following theorem on the monotonicity of the power function of an invariant test for the MANOVA problem has been proved in [3].

THEOREM 1. *If the acceptance region of an invariant test for the* MANOVA *problem is convex in the space of each column vector of* $\mathbf{X}$, *for each set of fixed values of* $\mathbf{Y}$ *and of the other columns of* $\mathbf{X}$, *then the power function of the test is a monotone increasing function of each of the noncentrality parameters* θ_i, $i = 1, 2, \cdots, t$.

The following corollary to the above theorem is used in this paper as a basic result.

COROLLARY 1.1. *If the acceptance region of an invariant test for the* MANOVA *problem is convex in* $\mathbf{X}$, *for each fixed* $\mathbf{Y}$, *then the power of the test increases monotonically in each* θ_i.

In [2] the above results were extended to cover the monotonicity of the power of an invariant test for the independence problem. We shall state this result in a more general form. Toward this end, let us denote the ordered characteristic roots of both, $\mathbf{S}_h\mathbf{S}_e^{-1}$ and $(\mathbf{XX}')^{-1}(\mathbf{XY}')(\mathbf{YY}')^{-1}(\mathbf{YX}')$, by $c_1 \geqq c_2 \geqq \cdots \geqq c_p$. Also let us use $\mathcal{A} = \mathcal{A}(c_1, c_2, \cdots, c_p)$ to denote a region in the space of $c_1, c_2, \cdots, c_p$. Then the proof of the following theorem is analogous to the proof of the Theorem 1 of [2].

THEOREM 2. *Suppose that the power of an invariant test, which accepts the* MANOVA *hypothesis over a region* $\mathfrak{A}(c_1, c_2, \cdots, c_p)$ *in space of the characteristic roots of* $S_hS_e^{-1}$, *increases monotonically in each noncentrality parameter* θ_i. *Then the power function of the invariant test for the independence problem which accepts the hypothesis over the region* $\mathfrak{A}(e_1, e_2, \cdots, e_p)$, *where* e_i *are related to the characteristic roots* c_i of $(\mathbf{XX}')^{-1}(\mathbf{XY}')(\mathbf{YY}')^{-1}(\mathbf{YX}')$ by $e_i = c_i(1 - c_i)^{-1}, i = 1, 2, \cdots, p$, *is a monotone increasing function of each population canonical correlation coefficient* ρ_i, $i = 1, 2, \cdots, p$.

3. Symmetric gauge functions. A real valued function

$$\psi(\mathbf{a}) = \psi(a_1, a_2, \cdots, a_p)$$

on the p-dimensional space of p-tuples of real numbers is said to be a gauge function if

(i) $\psi(a_1, a_2, \cdots, a_p) \geqq 0$, with equality if, and only if $a_1, = a_2 = \cdots = a_p = 0$,

(ii) $\psi(ca_1, ca_2, \cdots, ca_p) = |c| \psi(a_1, a_2, \cdots, a_p)$ for any real number c,

(iii) $\psi(a_1 + b_1, a_2 + b_2, \cdots, a_p + b_p) \leqq \psi(a_1, a_2, \cdots, a_p) + \psi(b_1, b_2, \cdots b_p)$.

$\psi(\mathbf{a})$ is said to be a symmetric gauge function if, in addition to (i), (ii) and (iii), it also satisfies

(iv) $\psi(\epsilon_1 a_{i_1}, \epsilon_2 a_{i_2}, \cdots, \epsilon_p a_{i_p}) = \psi(a_1, a_2, \cdots, a_p)$, where $\epsilon_i = \pm 1$, for all i, and $i_1, i_2, \cdots, i_p$ is a permutation of $1, 2, \cdots, p$.

As a convenience, one may require a symmetric gauge function to satisfy the normalizing condition:

(v) $\psi(1, 1, \cdots, 1) = 1$

Now suppose that $a_{(1)} \geqq a_{(2)} \geqq \cdots \geqq a_{(p)}$ denote the ordered values of p real numbers $a_1, a_2, \cdots, a_p$. Then it is easy to verify that

$$\psi(\mathbf{a}) = \textstyle\sum_{i=1}^{q} |a_{(i)}|, \qquad 1 \leqq q \leqq p,$$

and

$$\psi(\mathbf{a}) = (\textstyle\sum_{i=1}^{p} |a_i|^l)^{1/l}, \qquad 1 \leqq l < \infty,$$

are symmetric gauge functions. In the latter case the condition (iii) is the well known Minkowski inequality. For constructing another class of symmetric gauge functions let us define

$$\mathbf{T}_{(r,k,p)}(a_1, a_2, \cdots, a_p) = \textstyle\sum_{i_1+\cdots+i_p=r} \delta_{i_1}\delta_{i_2} \cdots \delta_{i_p} a_1^{i_1} a_2^{i_2} \cdots a_p^{i_p},$$

where $a_1, a_2, \cdots, a_p$ are non-negative, $i_j \geqq 0, j = 1, 2, \cdots, p, k < 0$, r is a positive integer and $\delta_i = (-1)^i\binom{k}{i}$, $i = 1, 2, \cdots p$. Then $\mathbf{T}_{(r,k,p)}^{1/r}$ is known [6] to be a convex function of $(a_1, a_2, \cdots, a_p)$. Thus

$$\psi(\mathbf{a}) = \mathbf{T}_{(r,k,p)}^{1/r}(|a_1|, |a_2|, \cdots, |a_p|)$$

is a symmetric gauge function. In particular, when $k = -1$,

$$\mathbf{T}_{(r,-1,p)}(a_1, a_2, \cdots, a_p) = h_r(a_1, a_2, \cdots, a_p),$$

where h_r is the rth completely symmetric function of $a_1, a_2, \cdots, a_p$ and is formally given by

$$[\prod_{i=1}^{p}(1 - a_i x)]^{-1} = 1 + h_1 x + h_2 x^2 + \cdots.$$

The following theorem will be used in the sequel.

THEOREM 3. *Let $a_1 \geqq a_2 \geqq \cdots \geqq a_p \geqq 0$ and $b_1 \geqq b_2 \geqq \cdots \geqq b_p \geqq 0$ be* such that

$$\sum_{i=1}^{q} a_i \leqq \sum_{i=1}^{q} b_i \qquad (\text{for } q = 1, 2, \cdots, p),$$

then for any convex increasing function ω on the positive half of the real line and any symmetric gauge function ψ of p variables we have $\psi(\omega(a_1), \cdots, \omega(a_p)) \leqq \psi(\omega(b_1), \cdots, \omega(b_p))$.

PROOF. The theorem is an immediate consequence of the following two results:

LEMMA 3.1 (G. Polya [8]). *Let $a_1 \geqq a_2 \geqq \cdots \geqq a_p$ and $b_1, b_2, \cdots, b_p$ be real numbers such that $\sum_{i=1}^{m} a_i \leqq \sum_{i=1}^{m} b_i$ for $m = 1, 2, \cdots, q$, then the inequality $\sum_{i=1}^{q} \omega(a_i) \leqq \sum_{i=1}^{q} \omega(b_i)$ holds for any convex increasing function ω on the real line.*

LEMMA 3.2 (Ky Fan [4]). *Let $a_1 \geqq a_2 \geqq \cdots \geqq a_p \geqq 0$ and $b_1 \geqq b_2 \geqq \cdots \geqq b_p \geqq 0$ then a necessary and sufficient condition that $\psi(a_1, a_2, \cdots, a_p) \leqq \psi(b_1, b_2, \cdots, b_p)$ for all symmetric gauge functions ψ of p variables is that $\sum_{i=1}^{q} a_i \leqq \sum_{i=1}^{q} b_i$ for all $q = 1, 2, \cdots, p$.*

4. Convex functions of matrices. Let $a_1 \geqq a_2 \geqq \cdots \geqq a_p$ be the ordered characteristic roots of $\mathbf{AA}'$, where $\mathbf{A}$ is a $(p \times n)$, $p \leqq n$, real matrix, and for any convex increasing function ω on the positive half of the real line and any symmetric gauge function ψ of p variables let us have the

DEFINITION. $\|\mathbf{A}\|_{\psi,\omega} = \psi(\omega(a_1^{\frac{1}{2}}), \cdots, \omega(a_p^{\frac{1}{2}}))$.

In this section we shall prove that $\|\mathbf{A}\|_{\psi,\omega}$ is a convex function on the space of $(p \times n)$ real matrices. This result, when $\mathbf{A}$ is a square matrix and $\omega(x) = x$, is due to von Neumann [7]. Ky Fan [4] has given a simpler proof of von Neumann's result in a more general set-up, where $\mathbf{A}$ is a completely continuous operator in a Hilbert space. Our proof, although in the spirit of Ky Fan's proof, will be a direct matrix proof. We shall develop the proof, through a number of lemmas.

LEMMA 4.1. *Let $\alpha_1 \geqq \cdots \geqq \alpha_p$ be the ordered characteristic roots of a positive semidefinite matrix $\mathbf{A}$; and $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_q$ be any set of q orthonormal p-vectors. Then for any $p \times p$ orthogonal matrix $\mathbf{F}$*

$$|\sum_{i=1}^{q} \mathbf{x}_i'\mathbf{AFx}_i| \leqq \sum_{i=1}^{q} \alpha_i.$$

PROOF. Let $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ be a set of orthonormal vectors such that $\mathbf{A}'\mathbf{y}_i = \alpha_i\mathbf{y}_i$, $i = 1, 2, \cdots, p$. Let $\mathbf{x}_i = \sum_{j=1}^{p} c_{ij}\mathbf{y}_j$, $i = 1, 2, \cdots p$. Then $\sum_{j=1}^{p} c_{ij}^2 = 1$ and $\sum_{i=1}^{q} c_{ij}^2 \leqq 1$. Now

$$\begin{aligned} \mathbf{x}_i'\mathbf{A}\mathbf{F}\mathbf{x}_i &= \sum_{j=1}^{p} (\mathbf{y}_j'\mathbf{F}\mathbf{x}_i) c_{ij}\alpha_j \\ &= \sum_{j=1}^{p} c_{ij}\, d_{ij}\alpha_j, \text{ where } d_{ij} = \mathbf{y}_j'\mathbf{F}\mathbf{x}_i \end{aligned}$$

Therefore,

$$\begin{aligned} 2|\mathbf{x}_i'\mathbf{A}\mathbf{F}\mathbf{x}_i| &\leqq \sum_{j=1}^{p} \alpha_j(c_{ij}^2 + d_{ij}^2) \\ &\leqq 2\alpha_q + \sum_{j=1}^{q} (\alpha_j - \alpha_q)(c_{ij}^2 + d_{ij}^2). \end{aligned}$$

Thus,

$$\begin{aligned} 2\left|\sum_{j=1}^{q} \mathbf{x}_i'\, \mathbf{A}\mathbf{F}\mathbf{x}_i\right| &\leqq 2\sum_{i=1}^{q} \alpha_i \\ &+ 2\sum_{j=1}^{q} (\alpha_j - \alpha_q)\left[\left(1 - \sum_{i=1}^{q} c_{ij}^2\right) + \left(1 - \sum_{i=1}^{q} d_{ij}^2\right)\right] \leqq 2\sum_{i=1}^{q} \alpha_i, \end{aligned}$$

which completes the proof of the Lemma 4.1.

Lemma 4.2 (*Polar factorization*). *If* $\mathbf{A}$ *is a* $(p \times p)$ *real matrix then there exist orthogonal matrices* $\mathbf{U}$ *and* $\mathbf{V}$ *such that* $\mathbf{A} = \mathbf{U}\mathbf{H} = \mathbf{K}\mathbf{V}$, *where symmetric, positive semidefinite* $\mathbf{H} = (\mathbf{A}'\mathbf{A})^{\frac{1}{2}}$ *and* $\mathbf{K} = (\mathbf{A}\mathbf{A}')^{\frac{1}{2}}$ *are the nonnegative square roots of, respectively,* $\mathbf{A}'\mathbf{A}$ *and* $\mathbf{A}\mathbf{A}'$.

Proof. See, e.g., Marcus and Minc [6].

Lemma 4.3. *Let* $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_p$ *be the ordered characteristic roots of* $\mathbf{A}\mathbf{A}'$, *where* $\mathbf{A}$ *is a* $(p \times p)$ *real matrix;* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_q$ *be set of* q *orthonormal* p*-vectors and* $\mathbf{F}$ *any* $(p \times p)$ *orthogonal matrix. Then*

$$\left|\sum_{i=1}^{q} \mathbf{x}_i'\mathbf{A}\mathbf{F}\mathbf{x}_i\right| \leqq \sum_{i=1}^{q} \alpha_i^{\frac{1}{2}}.$$

Proof. Let $\mathbf{A} = \mathbf{K}\mathbf{V}$ be the polar factorization of $\mathbf{A}$, where $\mathbf{K} = (\mathbf{A}\mathbf{A}')^{\frac{1}{2}}$ and $\mathbf{V}$ is orthogonal. Then the characteristic roots of the matrix $\mathbf{K}$ are $\alpha_1^{\frac{1}{2}} \geqq \alpha_2^{\frac{1}{2}} \geqq \cdots \geqq \alpha_p^{\frac{1}{2}}$. Therefore

$$\begin{aligned} \left|\sum_{i=1}^{q} \mathbf{x}_i'\mathbf{A}\mathbf{F}\mathbf{x}_i\right| &= \left|\sum_{i=1}^{q} \mathbf{x}_i'\mathbf{K}\mathbf{V}\mathbf{F}\mathbf{x}_i\right| \\ &\leqq \sum_{i=1}^{q} \alpha_i^{\frac{1}{2}}, \end{aligned}$$

which follows from the Lemma 4.1 since $\mathbf{V}\mathbf{F}$ is an orthogonal matrix.

Lemma 4.4. Let $\alpha_1 \geqq \cdots \geqq \alpha_p$, $\beta_1 \geqq \cdots \geqq \beta_p$ and $\gamma_1 \geqq \cdots \geqq \gamma_p$ *be the ordered characteristic roots of, respectively,* $\mathbf{A}\mathbf{A}'$, $\mathbf{B}\mathbf{B}'$ *and* $\mathbf{C}\mathbf{C}'$, *where* $\mathbf{A}$, $\mathbf{B}$ *are real* $(p \times p)$ *matrices and* $\mathbf{C} = \mathbf{A} + \mathbf{B}$. *Then we have*

$$\sum_{i=1}^{q} \gamma_i^{\frac{1}{2}} \leqq \sum_{i=1}^{q} \alpha_i^{\frac{1}{2}} + \sum_{i=1}^{q} \beta_i^{\frac{1}{2}}, \qquad q = 1, 2, \cdots, p.$$

Proof. Let $\mathbf{C} = \mathbf{K}\mathbf{V}$ be the polar factorization of $\mathbf{C}$, where $\mathbf{V}$ is orthogonal and $\mathbf{K} = (\mathbf{C}\mathbf{C}')^{\frac{1}{2}}$. Let $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_p$ be a set of orthonormal p-vectors such that $\mathbf{K}\mathbf{y}_i = \gamma_i^{\frac{1}{2}}\mathbf{y}_i$, $i = 1, 2, \cdots, p$. Thus

$$\begin{aligned} \sum_{i=1}^{q} \gamma_i^{\frac{1}{2}} &= \left|\sum_{i=1}^{q} \mathbf{y}_i'\mathbf{K}\mathbf{y}_i\right| \\ &= \left|\sum_{i=1}^{q} \mathbf{y}_i'\mathbf{A}\mathbf{V}'\mathbf{y}_i\right| + \left|\sum_{i=1}^{q} \mathbf{x}_i'\mathbf{B}\mathbf{V}'\mathbf{y}_i\right| \\ &\leqq \sum_{i=1}^{q} \alpha_i^{\frac{1}{2}} + \sum_{i=1}^{q} \beta_i^{\frac{1}{2}}. \end{aligned}$$

THEOREM 4. *Let* $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_p$ *be the characteristic roots of* $\mathbf{AA}'$, *where* $\mathbf{A}(p \times p)$, $p \leqq n$, *is a real matrix. For a symmetric gauge function* ψ *of* p *variables and a convex increasing function* ω, *defined on the positive half of the real line define*

$$\|\mathbf{A}\|_{\psi,\omega} = \psi(\omega(\alpha_1^{\frac{1}{2}}), \cdots, \omega(\alpha_p^{\frac{1}{2}})).$$

Then $\|\mathbf{A}\|_{\psi,\omega}$ *is a convex function of* $\mathbf{A}$.

PROOF. Assume that $p = n$, and let α_i, β_i and γ_i, $i = 1, 2, \cdots, p$ denote the ordered characteristic roots of $\mathbf{AA}'$, $\mathbf{BB}'$ and $(t\mathbf{A} + (1 - t)\mathbf{B})(t\mathbf{A} + (1 - t)\mathbf{B})'$, where $\mathbf{A}$ and $\mathbf{B}$ are $p = n$ dimensional real matrices and $0 \leqq t \leqq 1$. Since the ordered characteristic roots of $(t\mathbf{A})(t\mathbf{A})'$ and $((1 - t)\mathbf{B})((1 - t)\mathbf{B}')$ are, respectively, $t^2\alpha_i$ and $(1 - t)^2\beta_i$, $i = 1, 2, \cdots, p$, we have by Lemma 3.4

$$\textstyle\sum_{i=1}^{q} \gamma_i^{\frac{1}{2}} \leqq \sum_{i=1}^{q} (t\alpha_i^{\frac{1}{2}} + (1 - t)\beta_i^{\frac{1}{2}}), \qquad q = 1, 2, \cdots, p.$$

Thus by Theorem 4

$$\begin{aligned} \|t\mathbf{A} + (1 - t)\mathbf{B}\|_{\psi,\omega} &= \psi(\omega(\gamma_1^{\frac{1}{2}}), \cdots, \omega(\gamma_p^{\frac{1}{2}})) \\ &\leqq \psi(\omega(t\alpha_1^{\frac{1}{2}} + (1 - t)\beta_1^{\frac{1}{2}}), \cdots, \omega(t\alpha_p^{\frac{1}{2}} + (1 - t)\beta_p^{\frac{1}{2}})) \\ &\leqq \psi(t\omega(\alpha_1^{\frac{1}{2}}) + (1 - t)\omega(\beta_1^{\frac{1}{2}}), \cdots, t\omega(\alpha_p^{\frac{1}{2}}) \\ &\quad + (1 - t)\omega(\beta_p^{\frac{1}{2}})) \\ &\leqq t\|\mathbf{A}\|_{\psi,\omega} + (1 - t)\|\mathbf{B}\|_{\psi,\omega}. \end{aligned}$$

Now to see the result for $p < n$ it is sufficient to notice that the characteristic roots of $\mathbf{AA}'$ are the same as the nonzero characteristic roots of

$$\begin{bmatrix} \mathbf{A}(p \times n) \\ \mathbf{0}((n - p) \times n) \end{bmatrix} [\mathbf{A}'(n \times p)\mathbf{0}(n \times (n - p))],$$

where $\mathbf{0}$ denotes a null matrix.

5. The tests with monotone power functions. Let $c_1 \geqq c_2 \geqq \cdots \geqq c_p$ denote the ordered characteristic roots of the matrix $\mathbf{S}_h\mathbf{S}_e^{-1}$ of the MANOVA problem then we have the following:

THEOREM 5. *The power function of an invariant test, which accepts the general multivariate linear hypothesis over* $\mathfrak{A}(c_1, c_2, \cdots, c_p)$: $\psi(\omega(c_1^{\frac{1}{2}}), \omega(c_2^{\frac{1}{2}}), \cdots, \omega(c_p^{\frac{1}{2}})) \leqq \mu$, *where* ψ, ω *and* μ *are, respectively a symmetric gauge function of* p *variables, a convex increasing function over the positive half of the real line and a constant determined by the significance level of the test, is a monotone increasing function in each* θ_i, $i = 1, 2, \cdots, t$.

PROOF. Let $(\mathbf{YY}')^{-1} = \mathbf{TT}'$, where $\mathbf{T}(p \times p)$ is a nonsingular matrix. Then the characteristic roots of $(\mathbf{XX}')(\mathbf{YY}')^{-1}$ are the same as those of the matrix $(\mathbf{TX})(\mathbf{TX}')$; and $\psi(\omega(c_1^{\frac{1}{2}}), \omega(c_2^{\frac{1}{2}}), \cdots, \omega(c_p^{\frac{1}{2}})) = \|\mathbf{TX}\|_{\psi,\omega}$. Thus by the Corollary 1.1 it is sufficient to show that $\|\mathbf{TX}\|_{\psi,\omega}$ is convex in $\mathbf{X}$ for any fixed value of $\mathbf{T}$. This is an easy deduction from the Theorem 4.

Now let ψ_j and ω_j, $j = 1, 2, \cdots, k$ be, respectively, any k symmetric gauge functions of p variables and any k convex increasing functions defined on the positive half of the real line. Let

$$\psi_j = \psi_j(\omega_j(c_1^{\frac{1}{2}}), \cdots, \omega_j(c_p^{\frac{1}{2}}), \qquad j = 1, 2, \cdots, k.$$

Further let g be any convex increasing function of k variables then we have the following:

THEOREM 6. *The power of an invariant test which accepts the* MANOVA *hypothesis over* $\mathfrak{A}(c_1, c_2, \cdots, c_p)$: $g(\psi_1, \psi_2, \cdots, \psi_k) \leqq \mu$, *where* μ *is a constant determined by the significance level of the test, increases monotonically in each* θ_i, $i = 1, 2, \cdots, t$.

PROOF. Let $\psi_j(\mathbf{X}, \mathbf{Y})$ and $g(\psi_1, \cdots, \psi_k)(\mathbf{X}, \mathbf{Y})$ denote, respectively, the values of ψ_j and $g(\psi_1, \cdots, \psi_k)$ when observations are $(\mathbf{X}, \mathbf{Y})$. Let $(\mathbf{X}_1, \mathbf{Y})$ and $(\mathbf{X}_2, \mathbf{Y})$ be such that $g(\psi_1, \psi_2, \cdots, \psi_k)(\mathbf{X}_1, \mathbf{Y}) \leqq \mu$ and

$$g(\psi_1, \psi_2, \cdots, \psi_k)(\mathbf{X}_2, \mathbf{Y}) \leqq \mu.$$

Then for any t, $0 \leqq t \leqq 1$, we have

$$\psi_j(t\mathbf{X}_1 + (1 - t)\mathbf{X}_2, \mathbf{Y}) \leqq t\psi_j(\mathbf{X}_1, \mathbf{Y}) + (1 - t)\psi_j(\mathbf{X}, \mathbf{Y}).$$

Since g is increasing in each component we have

$$\begin{aligned} g(\psi_1, \psi_2, \cdots, \psi_k)(t\mathbf{X}_1 + (1 - t)\mathbf{X}_2, \mathbf{Y}) & \\ \leqq g(t\psi_1(\mathbf{X}_1, \mathbf{Y}) + (1 - t)\psi_1(\mathbf{X}_2, \mathbf{Y}), \cdots, t\psi_k(\mathbf{X}_1, \mathbf{Y})t + (1 - t)\psi_k(\mathbf{X}_2, \mathbf{Y})) & \\ \leqq tg(\psi_1, \cdots, \psi_k)(\mathbf{X}_1, \mathbf{Y}) + (1 - t)g(\psi_1, \cdots, \psi_k)(\mathbf{X}_2, \mathbf{Y}) & \\ \leqq t\mu + (1 - t)\mu = \mu. & \end{aligned}$$

Now let $c_1 \geqq c_2 \geqq \cdots \geqq c_p$ denote the ordered characteristic roots of the matrix $(\mathbf{XX}')^{-1}(\mathbf{XY}')(\mathbf{YY}')^{-1}(\mathbf{YX}')$ associated with the independence problem, and let $e_i = c_i(1 - c_i)^{-1}$, $i = 1, 2, \cdots, p$. Then we have, in view of the Theorem 2 the following:

THEOREM 7. *The power of an invariant test which accepts the independence hypothesis over* $\mathfrak{A}(e_1, e_2, \cdots, e_p)$, *where* $\mathfrak{A}$ *is as defined in the Theorem* 5 *or the Theorem* 6, *increases monotonically in each population canonical correlation* ρ_i, $i = 1, 2, \cdots, p$.

REMARK 1. Among the three well known [1], [9] tests of the MANOVA problem, namely, the trace criterion due to Hotelling, the largest root test due to Roy and the likelihood ratio test, the former two belong to the class of procedures considered in this paper. It is known that the likelihood ratio test satisfies the hypothesis of the Theorem 1 [3], and, therefore, has monotone power function. However, there are reasons to believe that it does not satisfy the condition of the Corollary 1.1. One of these reasons is that the kth, $k > 1$, root of the kth elementary symmetric function of the characteristic roots of a matrix is a concave function of the matrix [6]. Statistical implications of this possibility should be worth investigation. A similar remark holds for the problem of testing independence between two sets of variates.

REMARK 2. The functions $||\mathbf{A}||_{\psi,\omega}$ used in this paper have certain unitarily invariant norm properties, that is, they satisfy (i) $||\mathbf{A}||_{\psi,\omega} \geqq 0$, with equality if, and only if $\mathbf{A} = 0$, (ii) $||\mathbf{A} + \mathbf{B}||_{\psi,\omega} \leqq ||\mathbf{A}||_{\psi,\omega} + ||\mathbf{B}||_{\psi,\omega}$ and (iii) $||\mathbf{AF}||_{\psi,\omega} = ||\mathbf{A}||_{\psi,\omega}$ for any orthogonal $\mathbf{F}$. Furthermore when $\omega(x) = x$, in addition to (i), (ii) and (iii) we have the homogeneity condition (iv) $||c\mathbf{A}|| = |c| ||\mathbf{A}||_{\psi,\omega}$. Statistical implications of this aspect will be discussed in another paper.

REMARK 3. It may be observed that the results in [2] and [3] deal with invariant tests in general of the two hypotheses. The results of the present paper have some limitations, in that they deal with invariant tests "generated" by symmetric gauge functions of the maximal invariants. Essentially, in this paper, we have constructed a class of procedures which satisfy one of the two conditions of [3], sufficient for the monotonicity of the power functions.

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