RATE OF CONVERGENCE IN THE COMPOUND DECISION PROBLEM FOR TWO COMPLETELY SPECIFIED DISTRIBUTIONS[1]

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0. Summary. Simultaneous consideration of n statistical decision problems having identical generic structure constitutes a compound decision problem. The risk of a compound decision problem is defined as the average risk of the component problems. When the component decisions are between two fully specified distributions P_0 and P_1, $P_0 \neq P_1$, Hannan and Robbins [2] give a decision function whose risk is uniformly close (for n large) to the risk of the best "simple" procedure based on knowing the proportion of component problems in which P_1 is the governing distribution. This result was motivated by heuristic arguments and an example (component decisions between $N(-1, 1)$ and $N(1, 1)$) given by Robbins [4]. In both papers, the decision functions for the component problems depended on data from all n problems.

The present paper considers, as in Hannan and Robbins [2], compound decision problems in which the component decisions are between two distinct completely specified distributions. The decision functions considered are those of [2]. The improvement is in the sense that a convergence order of the bound is obtained in Theorem 1. Higher order bounds are attained in Theorems 2 and 3 under certain continuity assumptions on the induced distribution of a suitably chosen function of the likelihood ratio of the two distributions.

1. Introduction and notation. Consider the following statistical decision problem. Let X be a random variable (of arbitrary dimensionality) known to have one of two distinction distributions P_θ, $\theta \varepsilon \Omega = \{0, 1\}$. Based on observing X, we are required to decide whether the true value of the parameter θ is 0 or 1. We incur zero loss for correct decision and loss $a\theta + b(1 - \theta)$, $a > 0$, $b > 0$, for wrong decision.

If we simultaneously consider n decision problems each having this generic structure, then the n-fold global problem is called a compound decision problem. More precisely, let X_k, $k = 1, \cdots, n$ be n independent observations, X_k distributed according to P_{θ_k} with $\theta_k = 0$ or 1. Based on all n observations, a decision d_k, $d_k = 0$ or 1, is made for each of the n component problems. Note that in the case considered here all n decisions are held in abeyance until all n random variables X_k, $k = 1, \cdots, n$, have been observed. This is the same problem as treated in [2], [4], and [6]. The sequential problem, where the kth decision depends only on X_i, $i \leqq k$, is studied in [1] and [5], and is not dealt with in the present paper.

Received 13 November 1964; revised 5 August 1965.

[1] This research was supported in part by the National Science Foundation, Grant G-18976. The paper was prepared under the auspices of the U. S. Atomic Energy Commission.

Before proceeding, we introduce the following notation. Define $\mathbf{\Omega}$ as the set of all 2^n binary vectors $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$, $\theta_k \,\varepsilon\, \Omega$, $k = 1, \cdots, n$. Note that $\mathbf{\Omega}$ is the parameter space of the n-fold compound decision problem. For any $\boldsymbol{\theta} \,\varepsilon\, \mathbf{\Omega}$, define $\mathbf{P}$ as the product probability measure $\bigtimes_{k=1}^{n} P_{\theta_k}$. Thus under the assumption of independence of the X_k's, the observation $\mathbf{X} = (X_1, \cdots, X_n)$ of the compound problem is distributed as $\mathbf{P}$, $\boldsymbol{\theta} \,\varepsilon\, \mathbf{\Omega}$. Expectation with respect to $\mathbf{P}$, P_1 and P_0 will be denoted by $\mathbf{E}$, E_1 and E_0 respectively.

With X as the generic name of the X_k's, we have the following notation. Let μ be a dominating measure for P_0 and P_1. Then there exist densities, $\theta = 0, 1$,

$$f_\theta(x) = dP_\theta(x)/d\mu. \tag{1}$$

We can (and do) assume throughout the paper that $\max_\theta f_\theta(x) \leqq K'$ a.e. μ for some $K' < \infty$ (e.g., $\mu = P_0 + P_1$, $K' = 1$). Furthermore, we assume without loss of generality that both densities $f_0(x)$ and $f_1(x)$ do not vanish at each x.

2. Decision functions. A randomized decision function for the compound decision problem is any vector of n measurable functions of $\mathbf{x}$, $\mathbf{t} = (t_1, \cdots, t_n)$, where $t_k(\mathbf{x}) = \Pr\{d_k = 1 \mid \mathbf{x}\}$. A decision function $\mathbf{t}$ is called *simple* if $t_k(\mathbf{x}) = t(x_k)$, $k = 1, \cdots, n$ for some function t. A simple decision function will be denoted by t. For any $\boldsymbol{\theta} \,\varepsilon\, \mathbf{\Omega}$ the risk function for the decision $\mathbf{t}$ which is defined to be the average of the component risks is given by

$$\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}) = n^{-1} \textstyle\sum_{k=1}^{n} \mathbf{E}\{a\theta_k(1 - t_k(\mathbf{X})) + b(1 - \theta_k)t_k(\mathbf{X})\}. \tag{2}$$

The risk (2) may be considerably simplified in the case of a simple decision function. For $\boldsymbol{\theta} \,\varepsilon\, \mathbf{\Omega}$, $\bar{\theta} = n^{-1} \sum_{k=1}^{n} \theta_k$ is the relative frequency of problems in which P_1 is the governing distribution. For the simple decision function t, (2) reduces to

$$\begin{aligned} R(\bar{\theta}, t) &= a\bar{\theta}E_1\{1 - t(X)\} + b(1 - \bar{\theta})E_0\{t(X)\} \\ &= a\bar{\theta} + \int \{b(1 - \bar{\theta})f_0(x) - a\bar{\theta}f_1(x)\}t(x)\, d\mu(x) \end{aligned} \tag{3}$$

where the second equality follows from (1). The choice of t which minimizes (3) is any Bayes solution of the component statistical decision problem with $(1 - \bar{\theta}, \bar{\theta})$ considered as an *a priori* distribution on Ω, which is found by minimizing the integrand in (3) for each x. We arbitrarily choose the non-randomized admissible Bayes rule $t_{\bar{\theta}}(x)$, where for $0 \leqq p \leqq 1$

$$\begin{aligned} t_p(x) &= 1 \quad \text{if} \quad apf_1(x) > b(1 - p)f_0(x) \\ &= 1 \quad \text{if} \quad f_0(x) = 0 \quad \text{and} \quad p = 0 \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{4}$$

Defining the measurable transformation $Z(x)$ into $[0, 1]$ by

$$Z(x) = bf_0(x)/[af_1(x) + bf_0(x)], \tag{5}$$

we rewrite (4) conveniently for later use as

$$
(6) \qquad \begin{aligned} t_p(x) &= 1 && \text{if } Z(x) < p \text{ and } Z(x) \,\varepsilon\, (0, 1) \\ &= 0 && \text{if } Z(x) \geqq p \text{ and } Z(x) \,\varepsilon\, (0, 1) \\ &= 1 - Z(x) && \text{if } Z(x) = 0 \text{ or } 1. \end{aligned}
$$

Define $\phi(\bar{\theta})$ as the minimum of $R(\bar{\theta}, t)$ with respect to t. Then

$$
(7) \qquad \phi(\bar{\theta}) = \inf_t R(\bar{\theta}, t) = R(\bar{\theta}, t_{\bar{\theta}}).
$$

Note that from (3), (5), and (7) we have

$$
(8) \qquad R(\bar{\theta}, t_p) - \phi(\bar{\theta}) = \int \{Z(x) - \bar{\theta}\}\{t_p(x) - t_{\bar{\theta}}(x)\}\{af_1(x) + bf_0(x)\}\, d\mu(x).
$$

In [2], the following decision procedure is proposed for the compound problem. Let $h(x)$ be an unbiased estimate of $\theta \,\varepsilon\, \Omega$, i.e.,

$$
(9) \qquad E_\theta\{h(X)\} = \theta \quad \text{for} \quad \theta = 0 \quad \text{or} \quad 1.
$$

(Existence of such h will be discussed later.) Then form as an estimator of $\bar{\theta}$ the average $\bar{h}$ given by

$$
(10) \qquad \bar{h} = n^{-1} \textstyle\sum_{k=1}^{n} h(X_k).
$$

Let $\bar{h}^* = \bar{h}^*(\mathbf{x})$ be the truncation of $\bar{h}$ to the unit interval, i.e.,

$$
(11) \qquad \begin{aligned} \bar{h}^* &= \bar{h} && \text{if } 0 \leqq \bar{h} \leqq 1 \\ &= 0 && \text{if } \bar{h} < 0 \\ &= 1 && \text{if } \bar{h} > 1. \end{aligned}
$$

Now define the non-simple decision procedure $\mathbf{t}^* = (t_1^*, \cdots, t_n^*)$, where the component functions are obtained by substituting $\bar{h}^*$ for $\bar{\theta}$ in the simple rule $t_{\bar{\theta}}$ given by (6). Hence, we have the rule $\mathbf{t}^*$ where

$$
(12) \qquad \begin{aligned} t_k^*(\mathbf{X}) = t_{\bar{h}^*}(X_k) &= 1 && \text{if } Z(X_k) < \bar{h}^* \text{ and } Z(X_k) \,\varepsilon\, (0, 1) \\ &= 0 && \text{if } Z(X_k) \geqq \bar{h}^* \text{ and } Z(X_k) \,\varepsilon\, (0, 1) \\ &= 1 - Z(X_k) && \text{if } Z(X_k) = 0 \text{ or } 1. \end{aligned}
$$

Let $\mathcal{H}$ be the class of all μ-square integrable functions which are unbiased estimators of θ (i.e., satisfy (9)). This is a non-void class since it contains the bounded function $h(x) = (c_{00}c_{11} - c_{01}^2)^{-1}\{c_{00}f_1(x) - c_{01}f_0(x)\}$, where $c_{\theta j} = E_\theta f_j$ for $\theta, j = 0, 1$. Since $P_0 \neq P_1$, Schwarz inequality yields $c_{00}c_{11} - c_{01}^2 > 0$. For a fixed member of $\mathcal{H}$, we also define $\sigma_\theta^2 = E_\theta(h - \theta)^2$ for $\theta = 0, 1$, $\bar{\sigma}^2 = \max_{\theta=0,1} \sigma_\theta^2$ and for any p in the unit interval [0, 1], $\sigma_p^2 = p\sigma_1^2 + (1 - p)\sigma_0^2$. In [2], a constructive procedure is given for obtaining, for fixed p, $0 < p < 1$, a bounded kernel h_p satisfying (9) which minimizes σ_p^2 in the class $\mathcal{H}$.

Finally, the class $\mathcal{H}$ is important because of the following inequality on $\bar{h}$ with h in $\mathcal{H}$. We have, for any $\boldsymbol{\theta} \,\varepsilon\, \boldsymbol{\Omega}$,

$$
(13) \qquad \mathbf{E}(\bar{h} - \bar{\theta})^2 = n^{-1}\sigma_{\bar{\theta}}^2 \leqq n^{-1}\bar{\sigma}^2.
$$

Henceforth in this paper, we shall concern ourselves only with decision procedures $\mathbf{t}^*$ of the form (12), where the estimator $\bar{h}^*$ is defined through (10) and (11) with $h \varepsilon \mathcal{H}$.

3. The regret function. The question immediately arises: How good is the procedure $\mathbf{t}^*$ in (12)? As a partial answer to this question, consider the function

$$\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) - \phi(\bar{\theta}) \tag{14}$$

for the decision function $\mathbf{t}^*$ and $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$. This function will be called the *regret function* of the procedure $\mathbf{t}^*$ against the class of simple procedures. In Theorems 1–3 uniform (in $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$) upper bounds on (14) are given as functions of n.

We now develop a useful inequality (see (15)) for the regret function (14). Let W be the set $W = \{x \mid 0 < Z(x) < 1\}$ and let $\int_W$ denote integration restricted to the set W.

In the remainder of the paper we make extensive use of the characteristic function of a set A, which we denote by A enclosed in square brackets; that is $[A](a) = 1$ or 0 according as $a \varepsilon A$ or $a \not\varepsilon A$.

The regret function for the decision procedure $\mathbf{t}^*$ defined by (12) satisfies the following decomposition lemma.

LEMMA. *Let X be a random variable independent of $\mathbf{X}$ and let h satisfy* (9). *With* $\bar{h}_k = n^{-1}\{\sum_{j\neq k} h(X_j) + h(X)\}$, *then for* $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$,

$$\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) - \phi(\bar{\theta}) \leqq A_n + B_n + C_n, \tag{15}$$

where

$$A_n = \mathbf{E} \int_W (Z(x) - \bar{\theta})\{[\bar{\theta} \leqq Z(x) < \bar{h}] - [\bar{h} \leqq Z(x) < \bar{\theta}]\}\{af_1(x) + bf_0(x)\}\, d\mu(x)$$

$$B_n = n^{-1}a\textstyle\sum_{k\varepsilon I_1}\mathbf{E} \int_W [\bar{h}_k \leqq Z(x) < \bar{h}]\, dP_1(x)$$

$$C_n = n^{-1}b\textstyle\sum_{k\varepsilon I_0}\mathbf{E} \int_W [\bar{h} \leqq Z(x) < \bar{h}_k]\, dP_0(x)$$

with $I_\theta = \{k \mid \theta_k = \theta\}$, $\theta = 0, 1$.

PROOF. If $\theta_k = 0$, we apply the definitions of t_k^* in (12) and Z in (5), a change of variable x_k to x, an added integration on x_k and the fact that $P_0\{Z(x) = 0\} = 0$ as follows:

$$\begin{aligned}
\mathbf{E}\{t_k^*(\mathbf{X})\} &= \int [Z(x_k) < \bar{h}^*(x_1, \cdots, x_k, \cdots, x_n)]\, dP_{\theta_k}(x_k)\, dP_{\theta_1} \\
&\qquad \cdots dP_{\theta_{k-1}}\, dP_{\theta_{k+1}} \cdots dP_{\theta_n} \\
&= \int [Z(x) < \bar{h}^*(x_1, \cdots, x, \cdots, x_n)]\, dP_0(x)\, dP_{\theta_1} \\
&\qquad \cdots dP_{\theta_{k-1}}\, dP_{\theta_{k+1}} \cdots dP_{\theta_n} \\
&= \int_W [Z(x) < \bar{h}^*(x_1, \cdots, x, \cdots, x_n)]\, dP_0(x)\, dP_{\theta_1} \\
&\qquad \cdots dP_{\theta_{k-1}}\, dP_{\theta_k}\, dP_{\theta_{k+1}} \cdots dP_{\theta_n} \\
&= \mathbf{E} \int_W [Z(x) < \bar{h}_k^*]\, dP_0(x).
\end{aligned}$$

Similarly if $\theta_k = 1$, $\mathbf{E}\{1 - t_k^*(\mathbf{X})\} = \mathbf{E} \int_W \{1 - [Z(x) < \bar{h}_k^*]\} \, dP_1(x)$. Hence, for each $k = 1, \cdots, n$, we have

$$
(16) \quad \begin{aligned} a\theta_k \mathbf{E}\{1 - t_k^*(\mathbf{X})\} &+ b(1 - \theta_k)\mathbf{E}\{t_k^*(\mathbf{X})\} \\ &= a\theta_k \mathbf{E} \int_W \{1 - [Z(x) < \bar{h}_k^*]\} \, dP_1(x) \\ &\quad + b(1 - \theta_k)\mathbf{E} \int_W [Z(x) < \bar{h}_k^*] \, dP_0(x). \end{aligned}
$$

Now, add and subtract $a\theta_k \mathbf{E} \int_W \{1 - [Z(x) < \bar{h}^*]\} \, dP(x) + b(1 - \theta_k) \mathbf{E} \cdot \int_W [Z(x) < \bar{h}^*] \, dP_0(x)$ from the right hand side of (16) to obtain

$$
(17) \quad \begin{aligned} \mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) = {} & an^{-1}\textstyle\sum_{k\varepsilon I_1} \mathbf{E} \int_W \{[Z(x) < \bar{h}^*] - [Z(x) < \bar{h}_k^*]\} \, dP_1(x) \\ & + bn^{-1}\textstyle\sum_{k\varepsilon I_0} \mathbf{E} \int_W \{[Z(x) < \bar{h}_k^*] - [Z(x) < \bar{h}^*]\} \, dP_0(x) \\ & + \mathbf{E}\{R(\bar{\theta}, t_{\bar{h}^*})\}. \end{aligned}
$$

Note that by (8) with $\bar{h}^* = p$ we have,

$$
(18) \quad R(\bar{\theta}, t_{\bar{h}^*}) - \phi(\bar{\theta}) = \int \{Z(x) - \bar{\theta}\}\{t_{\bar{h}^*}(x) - t_{\bar{\theta}}(x)\}\{af_1(x) + bf_0(x)\} \, d\mu(x).
$$

From the definitions of $t_{\bar{h}^*}$, $t_{\bar{\theta}}$ and W, the expected value of the right-hand side of (18) with respect to $\mathbf{P}$ reduces to the term A_n with $\bar{h}^*$ replacing $\bar{h}$, which in turn is bounded by A_n.

The term B_n is an upper bound for the first term on the right-hand side of (17) because the pointwise inequality $[Z(x) < \bar{h}^*] - [Z(x) < \bar{h}_k^*] \leqq [\bar{h}_k^* \leqq Z(x) < \bar{h}^*] \leqq [\bar{h}_k \leqq Z(x) < \bar{h}]$ holds for $k \varepsilon I_1$. Similarly, C_n bounds the second term on the right-hand side of (17), and the lemma is proved.

4. A bound for the regret function. Sufficient conditions for a bound $\alpha_1 n^{-\frac{1}{2}}$, where α_1 is independent of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$, on the regret function of the procedure $\mathbf{t}^*$ will be given. Before proceeding to the theorem, we state the following inequality: If y is a positive real number and if $n^{-1} \leqq p \leqq 1$, then

$$
(19) \quad n^{\frac{1}{2}}p \min \{1, (np - 1)^{-\frac{1}{2}}y\} \leqq (1 + y^2)^{\frac{1}{2}}p^{\frac{1}{2}}.
$$

Verification of Inequality (19) is straightforward: If $(np - 1) \geqq y^2$, then $n^{\frac{1}{2}}p(np - 1)^{-\frac{1}{2}}y = p^{\frac{1}{2}}(1 - (np)^{-1})^{-\frac{1}{2}}y \leqq p^{\frac{1}{2}}(1 + y^2)^{\frac{1}{2}}$, and if $(np - 1) \leqq y^2$, then $n^{\frac{1}{2}}p = p^{\frac{1}{2}}(np)^{\frac{1}{2}} \leqq p^{\frac{1}{2}}(1 + y^2)^{\frac{1}{2}}$.

THEOREM 1. *If $h(x)$ is such that $E_\theta\{h(X)\} = \theta$ and $E_\theta|h(X)|^3 < \infty$ for $\theta = 0$ and 1, then there exists a constant $\alpha_1 = \alpha_1(h)$ such that $\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) - \phi(\bar{\theta}) \leqq \alpha_1 n^{-\frac{1}{2}}$.*

PROOF. In Inequality (15) we bound (i) the term $n^{\frac{1}{2}}A_n$ and (ii) the term $n^{\frac{1}{2}}(B_n + C_n)$.

(i) Since $\int_W \{(Z(x) - \bar{\theta})([\bar{\theta} \leqq Z(x) < \bar{h}] - [\bar{h} \leqq Z(x) < \bar{\theta}])\}\{af_1(x) + bf_0(x)\} \, d\mu(x) \leqq |\bar{h} - \bar{\theta}|(a + b)$ a.e. $\mathbf{P}$, Schwarz inequality implies $A_n \leqq (a + b)\mathbf{E}|\bar{h} - \bar{\theta}| \leqq (a + b)\{\mathbf{E}(\bar{h} - \bar{\theta})^2\}^{\frac{1}{2}}$. Inequality (13) yields $n^{\frac{1}{2}}A_n \leqq (a + b)\bar{\sigma}$, where the bound is independent of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$.

(ii) In bounding the term B_n, we can assume without loss of generality that I_1 is non-void and $\sigma_1 > 0$. If $\sigma_1 = 0$, then $\bar{h}_k = \bar{h} + n^{-1}\{h(x) - h(x_k)\} = \bar{h}$ a.e.

$\mathbf{P} \times P_1$ for all $k \varepsilon I_1$, and hence $[\bar{h}_k \leqq Z(x) < \bar{h}] = 0$ a.e. $\mathbf{P} \times P_1$ for all $k \varepsilon I_1$, that is, $B_n = 0$.

Fix $k \varepsilon I_1$ and let $\sigma_1 > 0$. Define $S = \sum_{i\varepsilon I_1, i \neq k} \{h(X_i) - 1\}$, $\sigma^2 = \text{Var}(S)$, $T = n\{Z(X) - \bar{\theta}\} + 1 - \sum_{i \varepsilon I_0} h(X_i)$. Then

$$(20) \qquad [\bar{h}_k \leqq Z(X) < \bar{h}] = [T - h(X_k) < S \leqq T - h(X)].$$

Apply the Berry-Esseen theorem (Loève [3], p. 288) for fixed x, x_k, and x_i, $i \varepsilon I_0$, to the normalized sum $\sigma^{-1}S$ at the endpoints $\sigma^{-1}\{T - h(x_k)\}$ and $\sigma^{-1}\{T - h(x)\}$ and bound the resulting absolute difference of normal df's by $(2\pi)^{-\frac{1}{2}}|h(x) - h(x_k)|\sigma^{-1}$. Noting that $\sigma^2 = (n\bar{\theta} - 1)\sigma_1^2$, this Berry-Esseen bound for the $\mathbf{P} \times P_1$ integral of (20) yields

$$(21) \quad \mathbf{E} \int_W [\bar{h}_k \leqq Z(x) < \bar{h}]\, dP_1(x) \leqq \mathbf{E}E_1[\bar{h}_k \leqq Z(X) < \bar{h}]$$
$$\leqq \min\{1, (n\bar{\theta} - 1)^{-\frac{1}{2}}((2\pi)^{-\frac{1}{2}}\sigma_1^{-1}E_{\theta_k}E_1|h(X) - h(X_k)| + 2\beta a_1)\},$$

where $a_1 = \sigma_1^{-3}E_1|h - 1|^3$ and β is the Berry-Esseen constant.

Weakening the bound in (21) by the Schwarz inequality $E_{\theta_k}E_1|h(X) - h(X_k)| \leqq \{E_{\theta_k}E_1|h(X) - h(X_k)|^2\}^{\frac{1}{2}} = 2^{\frac{1}{2}}\sigma_1$, and summing (21) over all $k \varepsilon I_1$, we have $B_n \leqq a\bar{\theta} \min\{1, (n\bar{\theta} - 1)^{-\frac{1}{2}}b_1\}$, where $b_1 = \pi^{-\frac{1}{2}} + 2\beta a_1$. Inequality (19) yields the desired bound $n^{\frac{1}{2}}B_n \leqq a(1 + b_1^2)^{\frac{1}{2}}(\bar{\theta})^{\frac{1}{2}}$.

A similar argument shows that $n^{\frac{1}{2}}C_n \leqq b(1 + b_0^2)^{\frac{1}{2}}(1 - \bar{\theta})^{\frac{1}{2}}$, where $b_0 = \pi^{-\frac{1}{2}} + 2\beta a_0$ with $a_0 = \sigma_0^{-3}E_0|h|^3$. The Schwarz inequality on the sum of the bounds for $n^{\frac{1}{2}}B_n$ and $n^{\frac{1}{2}}C_n$ implies $n^{\frac{1}{2}}(B_n + C_n) \leqq \{a^2(1 + b_1^2) + b^2(1 + b_0^2)\}^{\frac{1}{2}}$, which is independent of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$.

The theorem now follows from (i) and (ii) and Inequality (15) by defining $\alpha_1 = (a + b)\bar{\sigma} + \{a^2(1 + b_1^2) + b^2(1 + b_0^2)\}^{\frac{1}{2}}$.

5. Higher order bounds. Bounds for the regret function of order higher than that in Theorem 1 are obtainable under successively stronger sufficient conditions. Under P_θ, $\theta = 0$ or 1, let P_θ^* denote the induced probability measure on the unit interval [0, 1] under the measurable transformation Z defined by (5). Let $F_\theta(z)$ denote the corresponding distribution function. The following conditions on the continuity of the induced distributions are pertinent for the theorems to follow.

(I) The function $Z(x)$ in (5) has an induced distribution function $F_\theta(z)$ which is continuous on (0, 1) under P_θ for $\theta = 0$ and 1.

Observe that under (I), P_θ^* may assign positive probability to the values $z = 0$ and $z = 1$.

It is an immediate equivalence of (I) that

$$H(z) = \int_W [Z(x) < z]\{af_1(x) + bf_0(x)\}\, d\mu$$

and $H_\theta(z) = \int_W [Z(x) < z]\, dP_\theta(x)$ for $\theta = 0$ and 1 are continuous (and hence uniformly continuous) on the closed interval [0, 1].

Consider also the following condition:

(I′) Let $L(x) = f_1(x)/f_0(x)$ be the likelihood ratio of the densities in (1) (with the usual interpretation when $f_0(x) = 0$). The function $L(x)$ has an induced distribution function which is continuous over $(0, \infty)$ under P_θ for $\theta = 0$ and 1.

It is an easy matter to show that Conditions (I) and (I′) are equivalent, since the transformation from $(0, \infty)$ to $(0, 1)$ given by $z(l) = b(al + b)^{-1}$ is 1-1 and thus it and its inverse preserve singleton points of Lebesgue measure zero. In application, the Condition (I′) is often easier to check than (I). However, the proof of Theorem 3 takes a simpler form under (I).

(II) The function $Z(x)$ in (5) has an induced probability measure P_θ^* which is absolutely continuous with respect to Lebesgue measure (λ) and there exists a $K < \infty$ such that a.e. λ,

$$p_\theta^*(z) = dP_\theta^*(z)/d\lambda \leqq K \tag{22}$$

for $\theta = 0$ and 1.

Theorem 2. *Let $h(x)$ be such that $E_\theta\{h(X)\} = \theta$ and $|h(x)| \leqq M$* a.e. P_θ *for $\theta = 0$ and 1. If Assumption* (II) *holds, then there exist a constant $\alpha_2 = \alpha_2(h)$ such that $\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) - \phi(\bar{\theta}) \leqq \alpha_2 n^{-1}$.*

Proof. We bound the terms A_n, B_n, and C_n in (15). With $p_\theta^*(z)$ as in (22) express A_n in the integral form below, and use (22) to obtain

$$\begin{aligned} A_n &= \mathbf{E} \int \{(z - \bar{\theta})([\bar{\theta} \leqq z < \bar{h}] - [\bar{h} \leqq z < \bar{\theta}])\}\{ap_1^*(z) + bp_0^*(z)\}\, dz \\ &\leqq (a + b)K\mathbf{E} \int (z - \bar{\theta})\{[\bar{\theta} \leqq z < \bar{h}] - [\bar{h} \leqq z < \bar{\theta}]\}\, dz \\ &= (a + b)K(\mathbf{E}\{\textstyle\int_{\bar{\theta}}^{\bar{h}} (z - \bar{\theta})\, dz\}[\bar{h} \geqq \bar{\theta}] + \mathbf{E}\{\textstyle\int_{\bar{h}}^{\bar{\theta}} (\bar{\theta} - z)\, dz\}[\bar{h} < \bar{\theta}]) \\ &= \tfrac{1}{2}(a + b)K\mathbf{E}(\bar{h} - \bar{\theta})^2 \\ &\leqq n^{-1}\tfrac{1}{2}(a + b)K\bar{\sigma}^2, \end{aligned}$$

where the last inequality follows from (13).

The term B_n can be treated in a similar manner by bounding $\bar{h}_k = \bar{h} + n^{-1}\{h(x) - h(x_k)\}$ from below by $\bar{h} - 2Mn^{-1}$ for each $k \in I_1$ to obtain

$$\begin{aligned} B_n &\leqq a\bar{\theta}\mathbf{E}E_1[\bar{h} - 2Mn^{-1} \leqq Z(X) < \bar{h}] \\ &= a\bar{\theta}\mathbf{E} \int [\bar{h} - 2Mn^{-1} \leqq z < \bar{h}]p_1^*(z)\, dz \\ &\leqq n^{-1}\, 2aKM, \end{aligned}$$

where the last inequality follows from (22). In a similar manner, we have $C_n \leqq n^{-1}\, 2bKM$.

Substituting these three upper bounds for A_n, B_n, and C_n respectively into Inequality (15) yields the theorem with $\alpha_2 = (a + b)K\{\tfrac{1}{2}\bar{\sigma}^2 + 2M\}$.

Assumption (II) is quite stringent as can be seen from examining the examples in Section 6. However, as the following theorem illustrates, a convergence rate of $o(n^{-\frac{1}{2}})$ is still obtainable even without (II) by imposing Condition (I) or (I′).

THEOREM 3. *Let $h(x)$ be such that $E_\theta\{h(X)\} = \theta$ and $E_\theta|h(X)|^3 < \infty$ for $\theta = 0$ and 1. If (I) or (I′) holds, then for every $\epsilon > 0$ there exists an $n_0 = n_0(h)$ not depending on $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$ such that $\mathbf{R}(\boldsymbol{\theta}, \mathbf{t}^*) - \phi(\bar\theta) \leqq \epsilon n^{-\frac{1}{2}}$ for all $n \geqq n_0$.*

PROOF. In Inequality (15), we bound (i) the term $n^{\frac{1}{2}}A_n$ and (ii) the terms $n^{\frac{1}{2}}B_n$ and $n^{\frac{1}{2}}C_n$.

(i) Let $\epsilon > 0$ be given. Under (I), $H(z)$ is uniformly continuous on [0, 1] (and hence on the real line). Therefore, there exists a $\delta > 0$, such that

$$|H(z_2) - H(z_1)| \leqq (32)^{-\frac{1}{2}}\bar\sigma^{-1}\epsilon$$

whenever $|z_2 - z_1| < \delta$. Choose n_1 sufficiently large such that $n_1 \geqq 32(\delta\epsilon)^{-2}(a + b)^2\bar\sigma^4$. Let $E = \{|\bar h - \bar\theta| \geqq \delta\}$ and observe that by Tchebichev's inequality and (13),

$$\int_E d\mathbf{P} \leqq \delta^{-2}\mathbf{E}(\bar h - \bar\theta)^2 \leqq n^{-1}\delta^{-2}\bar\sigma^2. \tag{23}$$

Let $d\nu(x) = \{af_1(x) + bf_0(x)\}\, d\mu(x)$. Consider now the term $A_{1,n}^2 = n\{\mathbf{E} \int_W (Z(x) - \bar\theta)[\bar\theta \leqq Z(x) < \bar h]\, d\nu(x)\}^2$. Using the pointwise inequality $(Z(x) - \bar\theta)[\bar\theta \leqq Z(x) < \bar h] \leqq |\bar h - \bar\theta|[\bar\theta \leqq Z(x) < \bar h]$ in $A_{1,n}^2$, followed by the Schwarz integral inequality yields the bound

$$A_{1,n}^2 \leqq \sigma_{\bar\theta}^2\mathbf{E}\, \{\textstyle\int_W [\bar\theta \leqq Z(x) < \bar h]\, d\nu(x)\}^2.$$

In the second factor of this bound, partition the space under the $\mathbf{P}$ integral into E and its complement E^c, noting that on E^c, $\int_W [\bar\theta \leqq Z(x) < \bar h]\, d\nu(x) \leqq |H(\bar h) - H(\bar\theta)| \leqq (32)^{-\frac{1}{2}}\bar\sigma^{-1}\epsilon$, while on E, $\int_W [\bar\theta \leqq Z(x) < \bar h]\, d\nu(x) \leqq (a + b)$. Hence, $A_{1,n}^2 \leqq \sigma_{\bar\theta}^2\{(32)^{-1}\bar\sigma^{-2}\epsilon^2 + (a + b)^2 \int_E d\mathbf{P}\} \leqq (32)^{-1}\epsilon^2 + (a + b)^2\bar\sigma^2 \int_E d\mathbf{P}$. Inequality (23) and the choice of n_1 yield for $n \geqq n_1$, $A_{1,n} \leqq \frac{1}{4}\epsilon$.

By a similar argument, we obtain for $n \geqq n_1$,

$$A_{2,n} = n^{\frac{1}{2}}\{\mathbf{E} \textstyle\int_W \{\bar\theta - Z(x)\}[\bar h \leqq Z(x) < \bar\theta]\, d\nu(x)\} \leqq \frac{1}{4}\epsilon.$$

Since $n^{\frac{1}{2}}A_n = A_{1,n} + A_{2,n}$ the previous two inequalities yield $n^{\frac{1}{2}}A_n \leqq \frac{1}{2}\epsilon$ for $n \geqq n_1$. Note that n_1 was chosen independently of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$.

(ii) Let $\epsilon > 0$ be given. Choose $\gamma > 0$ such that $a\gamma\{\gamma + \pi^{-\frac{1}{2}} + 2\beta a_1\gamma\} \leqq \frac{1}{8}\epsilon$ where $a_1 = \sigma_1^{-3}E_1|h - 1|^3$ and β is the Berry-Esseen constant as in Theorem 1. By uniform continuity of $H_1(z)$ on the real line, there exists a $\delta = \delta(\gamma) > 0$ such that $|H_1(z_2) - H_1(z_1)| \leqq \frac{1}{2}\gamma^2$ if $|z_2 - z_1| < \delta$. The proof for the term B_n depends on properly bounding the two terms on the right-hand side of the expression

$$\begin{aligned} B_n = {} & n^{-1}a\textstyle\sum_{k\varepsilon I_1} \int_{W\cap F} \{\mathbf{E}[\bar h_k \leqq Z(x) < \bar h]\}dP_1(x) \\ & + n^{-1}a\textstyle\sum_{k\varepsilon I_1} \int_{W\cap F^c} \{\mathbf{E}[\bar h_k \leqq Z(x) < \bar h]\}\, dP_1(x) \end{aligned} \tag{24}$$

where $F = \{|Z(x) - \bar\theta| < \delta\}$. The two terms on the right-hand side of (24) will be denoted B_n' and B_n'' respectively.

We first bound B_n' in (24) by a Berry-Esseen approximation argument. As in the proof of Theorem 1, we assume without loss of generality that $\sigma_1 > 0$ and I_1 is non-void. By a Berry-Esseen approximation for fixed x, x_k, and x_i, $i \varepsilon I_0$ applied to the kth summand in $B_{n'}$, we have by (20) and (21),

$$(25)\quad \int_{W\cap F} \{\mathbf{E}[\bar{h}_k \leqq Z(x) < \bar{h}]\}\, dP_1(x) \leqq \min\{\int_{W\cap F} dP_1, (n\bar{\theta} - 1)^{-\frac{1}{2}} \cdot ((2\pi)^{-\frac{1}{2}}\sigma_1^{-1}E_{\theta_k}\int_{W\cap F} |h(x) - h(X_k)|\, dP_1(x) + 2\beta a_1 \int_{W\cap F} dP_1)\}.$$

Weakening in (25) by $E_{\theta_k}\int_{W\cap F} |h(x) - h(X_k)|\, dP_1(x) \leqq 2^{\frac{1}{2}}\sigma_1\{\int_{W\cap F} dP_1\}^{\frac{1}{2}}$, observing that our choice of δ implies $\int_{W\cap F} dP_1 \leqq H_1(\bar{\theta} + \delta) - H_1(\bar{\theta} - \delta) \leqq \gamma^2$, and summing over all $k \varepsilon I_1$, the definition of B_n' and Inequalities (25) and (19) yield

$$(26)\quad \begin{aligned} n^{\frac{1}{2}}B_n' &\leqq a\gamma^2 n^{\frac{1}{2}}\bar{\theta} \min\{1, (n\bar{\theta} - 1)^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\gamma^{-1} + 2\beta a_1) \\ &\leqq a\gamma(\gamma + \pi^{-\frac{1}{2}} + 2\beta a_1\gamma) \leqq \tfrac{1}{8}\epsilon, \end{aligned}$$

where the last inequality follows from our choice of γ.

We now bound B_n'' in (24). Observe the following set inclusion:

$$\{|Z(x) - \bar{\theta}| \geqq \delta, \bar{h}_k \leqq Z(x) < \bar{h}\} \subset \{\bar{h} - \bar{\theta} \geqq \delta\} \cup \{\bar{h}_k - \bar{\theta} \leqq -\delta\}.$$

Substituting this set inclusion in B_n'' and observing that a simple change of variable implies $\mathbf{E}\int_W [|\bar{h}_k - \bar{\theta}| \leqq -\delta]\, dP_1(x) \leqq \mathbf{E}\int [|\bar{h}_k - \bar{\theta}| \leqq -\delta]\, dP_1(x) = \mathbf{P}\{\bar{h} - \bar{\theta} \leqq -\delta\}$ for all $k \varepsilon I_1$, we obtain $B_n'' \leqq a\bar{\theta}\mathbf{P}\{|\bar{h} - \bar{\theta}| \geqq \delta\}$. Hence, by Tchebichev's inequality and (13) we have,

$$(27)\quad \begin{aligned} B_n'' &\leqq a\bar{\theta}\mathbf{P}\{|\bar{h} - \bar{\theta}| \geqq \delta\} \\ &\leqq a\delta^{-2}\mathbf{E}(\bar{h} - \bar{\theta})^2 \leqq a(\bar{\sigma}\delta^{-1})^2 n^{-1}. \end{aligned}$$

Note that the bound in (27) is independent of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$, and when multiplied by $n^{\frac{1}{2}}$ approaches zero as $n \to \infty$. Hence there exists an n_2 independent of $\boldsymbol{\theta} \varepsilon \boldsymbol{\Omega}$ such that $n^{\frac{1}{2}}B_n'' \leqq \frac{1}{8}\epsilon$ for $n \geqq n_2$. This result together with (24) and (26) implies $n^{\frac{1}{2}}B_n \leqq \frac{1}{4}\epsilon$ for all $n \geqq n_2$.

By a similar argument there exists an n_3 such that $n^{\frac{1}{2}}C_n \leqq \frac{1}{4}\epsilon$ for $n \geqq n_3$, and Part (ii) of the proof is completed.

By choosing $n_0 = \max(n_1, n_2, n_3)$ the results of (i) and (ii) substituted into (15) completes the proof.

6. Examples.

As remarked earlier the estimator

$$h(x) = (c_{00}c_{11} - c_{01}^2)^{-1}\{c_{00}f_1(x) - c_{01}f_0(x)\}$$

where $c_{\theta j} = E_\theta\{f_j(X)\}$ for $\theta, j = 0, 1$ is always a bounded (a.e. μ) member of $\mathcal{H}$. Hence, the examples given below illustrate when Condition (I) or (I′) and (II) are satisfied.

EXAMPLE 1. This example exhibits a whole class of pairs of distribution for which Assumption (I′) and hence Theorem 3 and (I) are verified. Let the generic random variable in the component problem be X. If $\theta = 0$ or 1, assume X has Lebesgue (μ) density $f_\theta(x) = a_\theta\xi(x) \exp\{\omega_\theta T(x)\}$, where $T(x)$ has a non-zero derivative in x, $\omega_1 \neq \omega_0$. Then, by the definition of the likelihood ratio $L(x)$ in (I′), we have $L(x) = f_1(x)/f_0(x) = a_1a_0^{-1}\exp\{(\omega_1 - \omega_0)T(x)\}$. Note that $T = T(x)$ having a non-zero derivative and X having a density (either under P_0 or P_1) implies T has a density. But L as a function of T having non-zero derivative implies L has a density. Thus, in particular, (I′) is satisfied.

EXAMPLE 2. This is an example for which (II) and hence Theorem 2 holds. Let X be the generic random variable of the component problem. Take $a = b$. If $\theta = 0$ or 1, assume X has a Lebesgue (μ) density $f_\theta(x) = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x-\theta)^2\}$. Then, $Z(x)$ defined by (5) is

$$z = Z(x) = \{1 + \exp(x - \tfrac{1}{2})\}^{-1}. \tag{28}$$

Note that $Z(x)$ in (28) is monotone and approaches 0 or 1 as $x \to +\infty$ or $-\infty$. Then, the density $p_\theta^*(z)$ in (22) is

$$\begin{aligned} p_\theta^*(z) &= f_\theta(x)\{|Z'(x)|\}^{-1} \\ &= f_\theta(x) z^{-2} \exp\{\tfrac{1}{2} - x\}. \end{aligned} \tag{29}$$

But (29) clearly approaches 0 as $z \to 0$ or 1 (that is, as $x \to +\infty$ or $-\infty$).

Since the densities $p_\theta^*(z)$ are continuous on the open interval (0, 1), the above convergence to 0 as the endpoints $z = 0$ and 1 establishes continuity on the closed interval [0, 1]. Thus, boundedness on [0, 1] follows and (II) is verified for this example.

EXAMPLE 3. An example where (I) or (I′) holds but (II) fails is the following special case of Example 1. Let $f_\theta(x) = \omega_\theta \exp(-\omega_\theta x)$, $x > 0$, and assume $\omega_1 > 2\omega_0 > 0$. Then, $Z(x)$ defined by (5) is

$$z = Z(x) = \{1 + (a\omega_1/b\omega_0) \exp[(\omega_0 - \omega_1)x]\}^{-1}$$

and

$$p_\theta^*(z) = \omega_0 \{\exp(-\omega_0 x)\}\, z^{-2}\, (b\omega_0/a\omega_1)\, \{\exp[(\omega_1 - \omega_0)x]\}\, (\omega_1 - \omega_0)^{-1}. \tag{30}$$

Observe that the density (30) $\to \infty$ as $z \to 1 (x \to \infty)$, and, hence, is unbounded on (0, 1). Therefore Assumption (II) of Theorem 2 is violated for this example. Whether or not the conclusion of Theorem 2 can still be proved for this example we have not been able to show.

7. Acknowledgments. The authors would like to thank the referees for pointing out certain changes which clarified presentation and corrected errors in an earlier version of the manuscript.

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