ASYMPTOTIC OPTIMUM QUANTILES FOR THE ESTIMATION OF THE PARAMETERS OF THE NEGATIVE EXPONENTIAL DISTRIBUTION

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1. Introduction. Assume that we are sampling from the negative exponential distribution

$$F(x) = 1 - e^{-(x-\mu)/\sigma}, \qquad x \geqq \mu;\ \sigma > 0, \tag{1.1}$$

where μ and σ are respectively the location and the scale parameters. Denote the observations of an ordered random sample of size n by $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$. The integer i denotes the rank of the order statistic $x_{(i)}$.

The problem of estimation of μ and σ simultaneously (or one of them when the other is known) based upon a fraction of the sample, say, the observations, $x_{(n_1)} < \cdots < x_{(n_k)}$ arises frequently in practice. The integer k denotes the number of observations for the estimation. In such estimation problem linear estimates, $c_1x_{(n_1)} + c_2x_{(n_2)} + \cdots + c_kx_{(n_k)}$, are useful. The quantities $c_1, c_2, \cdots, c_k$ are the coefficients of the linear function. When both μ and σ are unknown and to be estimated, we obtain two linear functions differing only in coefficients but the sample observations remain the same. The observations that are used to form the estimate will be the relevant sample for the estimation. In such estimation problem, ranks n_i ($i = 1, \cdots, k$) of the observations in the relevant sample form a subset $R_k = \{n_1, n_2, \cdots, n_k\}$ of the set $I_n = \{1, 2, \cdots, n\}$. Therefore, in estimating the parameters based on k ($\leqq n$) observations we have at our disposal $\binom{n}{k}$ subset of the ranks to determine the relevant sample. Some subsets $R_k^0 = \{n_1^0, \cdots, n_k^0\}$ are preferable to others if estimates based on the corresponding order statistics $x_{(n_1^0)}, x_{(n_2^0)}, \cdots, x_{(n_k^0)}$ possess minimum variance property among all other $\binom{n}{k}$ estimates. If such a set R_k^0 exists, the corresponding set of order statistics will be called "optimum set." Therefore, the problem arises as to how the "optimum set" of order statistics be determined for fixed values of n and k such that the linear estimates of μ and σ based on them have optimum variance property.

The problem has been studied numerically by several authors. The small sample situation has been studied by Harter (1961), Kulldorff (1963), Sarhan, Greenberg and Ogawa (1963), Siddiqui (1963). The asymptotic situation ($n \to \infty$) for the estimation of σ when $\mu = 0$ has been considered by Kulldorff (1962), Ogawa (1950) and Sarhan, Greenberg and Ogawa (1963). In this paper, we present the asymptotic situation ($n \to \infty$) and obtain analytically a system of equations which determine the k ($\leqq n$) optimum spacings of the optimum set

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order statistics (quantiles) for the estimation of μ and σ. The solution of the system of equations is shown to be unique.

2. Asymptotically best linear unbiased estimates of the parameters based on k sample quantiles. Consider the two-parameter negative exponential distribution (1.1). Assume that the sample size n is large and k ($\leqq n$) is given. Let the ordered observations in a random sample of size n be $x_{(1)} < \cdots < x_{(n)}$ and consider the k sample quantiles $x_{(n_1)}, \cdots, x_{(n_k)}$ where $n_1, \cdots, n_k$ are the respective ranks which satisfy the inequality $1 \leqq n_1 < \cdots < n_k \leqq n$. The integers $n_1, \cdots, n_k$ are determined by the k fixed spacings $p_1, p_2, \cdots, p_k$ which satisfy the order relation $0 < p_1 < \cdots < p_k < 1$ and $n_i = [np_i] + 1$, $(i = 1, \cdots, k)$. $[np_i]$ is the Euler notation denoting the largest integer not exceeding np_i. Denote $p_0 = 0$ and $p_{k+1} = 1$. It is well known that the joint asymptotic $(n \to \infty)$ distribution of $x_{(n_1)}, x_{(n_2)}, \cdots, x_{(n_k)}$ is a k-variate normal distribution with means $\{\mu + \sigma u_1, \mu + \sigma u_2, \cdots, \mu + \sigma u_k\}$ and dispersion matrix $W = (\sigma^2/n)((W_{ij}))$ with $W_{ii} = W_{ij} = W_{ji} = (e^{u_i} - 1)(i < j); i, j = 1, 2, \cdots, k$. The elements of the inverse W^{-1} of W are

$$W'_{i,i} = (e^{-u_{i-1}} - e^{-u_{i+1}})e^{-2u_i}/(e^{-u_i} - e^{-u_{i+1}})(e^{-u_{i+1}} - e^{-u_i}),$$

$$W'_{i,i-1} = W'_{i-1,i} = 1/(e^{-u_i} - e^{-u_{i-1}}),$$

$$W'_{i,j} = 0 \quad \text{for} \quad |i - j| > 1,$$

where $u_i = \ln(1 - p_i)^{-1}$, $(i = 1, \cdots, k)$ are the quantiles of the standardized exponential distribution corresponding to $p_1, p_2, \cdots, p_k$ [Mosteller (1946)]. Now, by the application of the generalized Gauss-Markoff theorem [Kulldorff (1963)], the asymptotically best linear unbiased estimates of μ and σ based on the k quantiles are given by

$$\hat{\sigma} = \textstyle\sum_{i=1}^{k} b_i x_{(r_i)}, \tag{2.1}$$

$$\hat{\mu} = x_{(n_1)} - \hat{\sigma} u_1, \tag{2.2}$$

where

$$b_1 = -(u_2 - u_1)/(e^{u_2} - e^{u_1})L, \tag{2.3a}$$

$$b_i = (1/L)\{(u_i - u_{i-1})/(e^{u_i} - e^{u_{i-1}}) - (u_{i+1} - u_i)/(e^{u_{i+1}} - e^{u_i})\}, \quad i = 2, \cdots, k-1, \tag{2.3b}$$

$$b_k = (1/L)\{(u_k - u_{k-1})/(e^{u_k} - e^{u_{k-1}})\}, \tag{2.3c}$$

$$L = \textstyle\sum_{i=2}^{k} (u_i - u_{i-1})^2/(e^{u_i} - e^{u_{i-1}}). \tag{2.3d}$$

The variance and the covariance of the estimates are

$$V(\hat{\mu}) = \{u_1^2/L + (e^{u_1} - 1)\}\sigma^2/n, \tag{2.4a}$$

$$V(\hat{\sigma}) = (1/L)\sigma^2/n, \tag{2.4b}$$

$$\text{Cov}(\hat{\mu}, \hat{\sigma}) = (u_1/L)\sigma^2/n \tag{2.4c}$$

and the generalized variance of the estimates is

(2.5) $$\Lambda = V(\hat{\mu})\cdot V(\hat{\sigma}) - \text{Cov}^2(\hat{\mu}, \hat{\sigma}) = [(e^{u_1} - 1)/L]\sigma^4/n^2.$$

3. The optimum quantiles for the estimation of μ and σ. In order to determine the k optimum quantiles for the simultaneous estimation of μ and σ, we minimize the expression in (2.5) with respect to $u_1, u_2, \cdots, u_k$ where $u_i = \ln(1 - p_i)^{-1}$, $(i = 1, \cdots, k)$ over the domain $0 < u_1 < \cdots < u_k < \infty$, or equivalently, maximize $L(e^{u_1} - 1)^{-1}$ where L is given in (2.3d). The partial derivative

(3.1) $$\delta L(e^{u_1} - 1)^{-1}/\delta u_1 = -[e^{-u_1}L/(1 - e^{-u_1})^2 + (e^{u_1} - 1)^{-1}\{2 + e^{u_1}(u_2 - u_1)/(e^{u_2} - e^{u_1})\}(u_2 - u_1)/(e^{u_2} - e^{u_1})] < 0,$$

since $0 < u_1 < u_2$ and $L > 0$ for $0 < u_1 < \cdots < u_k < \infty$. Therefore, $L(e^{u_1} - 1)^{-1}$ is a monotonically decreasing function of u_1 and attains its maximum value when u_1 assumes its smallest value. But according to the theory of quantiles p_1 must be bounded away from zero and we encounter difficulty as to the choice of a proper value of u_1 for which $L(e^{u_1} - 1)^{-1}$ is maximum. To overcome this difficulty, we consider the estimation of μ and σ based on the order statistics $x_{(r)}$, $x_{(n_2)}, \cdots, x_{(n_k)}$ where the integers $n_2, \cdots, n_k$ are determined by $(k - 1)$ fixed spacings $p_2, \cdots, p_k$ satisfying the order relation $0 < p_2 < \cdots < p_k < 1$ and r is an integer such that $1 \leqq r < [np_2] + 1$. It is well known that

(3.2) $$E(x_{(r)}) = \mu + \sigma \textstyle\sum_{j=1}^{r} (n - j + 1)^{-1},$$

(3.3) $$V(x_{(r)}) = \text{Cov}(x_{(r)}, x_{(n_i)}) = \sigma^2 \textstyle\sum_{j=1}^{r} (n - j + 1)^{-2}.$$

The quantities $\sum_{j=1}^{r} (n - j + 1)^{-s}$, $s = 1, 2$ for $1 \leqq r < [np_2] + 1$ can be approximated as

(3.4a) $$\textstyle\sum_{j=1}^{r} (n - j + 1)^{-1} = \ln(1 - r/(n + \frac{1}{2}))^{-1} + O(n^{-2}),$$

(3.4b) $$\textstyle\sum_{j=1}^{r} (n - j + 1)^{-2} = (1/n)[r/(n + \frac{1}{2})]/[1 - r/(n + \frac{1}{2})] + O(n^{-3}),$$

by the use of the following Euler-Maclaurin summation formula:

(3.5) $$\textstyle\sum_{s=0}^{k+1} \phi(s + \frac{1}{2}) = \int_0^k \phi(x)\,dx + \sum_{\nu=1}^{r-1} [B_{2\nu}(\frac{1}{2})/(2\nu)!][\phi_{(x)}^{(2\nu-1)}]_0^k + R,$$

where $B_{2\nu}(\frac{1}{2})$ is the Bernoulli polynomial with argument $\frac{1}{2}$ and the error term R is numerically less than three times the first neglected term if $\phi_{(x)}^{(2\nu)}$ does not change sign [Steffenson (1927)], $\phi_{(x)}^{(2\nu)}$ denoting the (2ν)th order derivative of $\phi_{(x)}$. Now, we set

(3.6) $$p_1 = r/(n + \tfrac{1}{2}) = 1 - e^{-u_1} \quad \text{(say)}.$$

With this correspondence of u_1 in all the expressions in (2.1) through (2.5), we observe that the maximum of $L(e^{u_1} - 1)^{-1}$ is attained when $u_1 = \ln(1 - 1/(n + \frac{1}{2}))^{-1}$ since u_1 is now limited in the interval $\ln(1 - 1/(n + \frac{1}{2}))^{-1} \leqq u_1 < \ln(1 - p_2)^{-1}$. Thus we have

(3.7) $$p_1^0 = 1/(n + \tfrac{1}{2}) \quad \text{and} \quad n_1^0 = 1.$$

To determine the remaining $k - 1$ quantiles we maximize $L(e^{u_1} - 1)^{-1}$ with respect to $u_2, \cdots, u_k$ keeping u_1 fixed to be $\ln(1 - 1/(n + \frac{1}{2}))^{-1}$. For this we use the following transformation:

$$t_{i-1} = u_i - u_1^0, \qquad (i = 1, \cdots, k), \tag{3.8}$$

where $u_1^0 = \ln(1 - 1/(n + \frac{1}{2}))^{-1}$. Then $L(e^{u_1} - 1)^{-1}$ reduces to $[(n - \frac{1}{2})/(n + \frac{1}{2})]Q_{k-1}(t_1, \cdots, t_{k-1})$ where

$$Q_{k-1} = \sum_{i=1}^{k-1} [(t_i - t_{i-1})^2/(e^{t_i} - e^{t_{i-1}})] \quad \text{with} \quad t_0 = 0. \tag{3.9}$$

The functional form (3.9) may be put in the alternative form

$$Q_{k-1} = \sum_{i=1}^{k} [(t_i e^{-t_i} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_i})] \tag{3.10}$$

with $t_0 = 0$ and $t_k e^{-t_k} = 0$, which has been studied numerically by Ogawa (1960) and Sarhan, Greenberg and Ogawa (1963) in determining the optimum spacings when σ is the unknown parameter to be estimated based on $k - 1$ quantiles. Thus maximization of $Q_{k-1}(t_1, \cdots, t_{k-1})$ with respect to $t_1, \cdots, t_{k-1}$ will determine the optimum spacings $\lambda_1^0, \cdots, \lambda_{k-1}^0$ for the $k - 1$ quantiles by the relations

$$\lambda_i^0 = 1 - e^{-t_i^0}, \qquad i = 1. \cdots, k - 1, \tag{3.11}$$

where $(t_1^0, t_2^0, \cdots, t_{k-1}^0)$ denote the point at which Q_{k-1} attains its maximum. The asymptotic relative efficiency (ARE) of the estimate of σ (when μ is known) based on $k - 1$ quantiles compared to the maximum likelihood estimate using all the observations in the sample is Q_{k-1}^0 which is the maximum of Q_{k-1} at $(t_1^0, \cdots, t_{k-1}^0)$. We refer the readers to Table 3 of Sarhan, Greenberg and Ogawa (1963) for the numerical results for $k = 2(1)16$. This table will be referred to in this paper as "Table 3 (1963)." In Section 4 we prove that maximum of Q_{k-1} for each k is unique.

The optimum spacing of the $k - 1$ quantiles for the simultaneous estimation of μ and σ are determined by the relations

$$p_{i+1}^0 = (2 + (2n - 1)\lambda_i^0)/(2n + 1), \qquad (i = 1, \cdots, k - 1)$$

and the relevant sample for the estimation of μ and σ consists of $x_{(1)}$, $x_{(n_2^0)}, \cdots, x_{(n_k^0)}$ where the ranks n_i^0, $(i = 2, \cdots, k)$, are given by $n_i^0 = [np_i^0] + 1$, $(i = 2, \cdots, k)$. The asymptotically best linear unbiased estimates of μ and σ are

$$\hat{\mu} = x_{(1)} - \hat{\sigma} \ln [(2n - 1)/(2n + 1)],$$
$$\hat{\sigma} = b_0^0 x_{(1)} + \sum_{i=1}^{k-1} b_i^0 x_{(n_{i+1}^0)},$$

where the coefficients $b_1^0, \cdots, b_{k-1}^0$ are available in "Table 3 (1963)" and $b_0^0 = -\sum_{i=1}^{k-1} b_i^0$. The joint asymptotic efficiency (JAE) of μ and σ and asymptotic relative efficiency (ARE) compared to the maximum likelihood estimates using all the observations in the sample are

$$\text{JAE}(\hat{\mu}, \hat{\sigma}) = [(2n - 1)^2/2(n^2 - 1)]Q_{k-1}^0,$$
$$\text{ARE}(\hat{\sigma}) = [(2n - 1)/(2n + 1)]Q_{k-1}^0,$$
$$\text{ARE}(\hat{\mu}) = (2n - 1)Q_{k-1}^0/n[(2n - 1) \ln [(2n + 1)/(2n - 1)] + 2Q_{k-1}^0],$$

where Q_{k-1}^0 is the asymptotic relative efficiency of the estimate of σ based on $k-1$ quantiles (when μ is known).

For $k = 5$ and $n = 70$, it is easily verified that the relevant sample consists of the set $x_{(1)}, x_{(33)}, x_{(53)}, x_{(64)}, x_{(69)}$. The estimates are $\mu = x_{(1)} - \hat{\sigma} \ln (141/139)$, $\hat{\sigma} = -.7871x_{(1)} + .3907x_{(33)} + .2361x_{(53)} + .1195x_{(64)} + .0409x_{(69)}$. The coefficients are taken from "Table 3 (1963)" for $k = 4$. The efficiencies are JAE $(\hat{\mu}, \hat{\sigma}) = 90.04\%$, ARE $(\hat{\sigma}) = 91.37\%$ and ARE $(\hat{\mu}) = 95.22\%$.

4. On the equations for optimum spacings. In the previous section, we have observed that for a given k, the determination of the unique set of optimum quantiles depends on the uniqueness of the maximum of the function Q_{k-1}. In order to obtain the maximum of Q_{k-1} we have to solve the following system of equations

$$\delta Q_{k-1}/\delta t_i = e^{-t_i}(\tau_{i+1} - \tau_i)(\tau_{i+1} + \tau_i - 2t_i) = 0, \qquad i = 1, 2, \cdots, k-1,$$

where

$$1 - \tau_i = (t_i e^{-t_i} - t_{i-1}e^{-t_{i-1}})/(e^{-t_{i-1}} - e^{-t_i}). \tag{4.1}$$

It will be shown later that the solution of the system of equations $\tau_{i+1} + \tau_i - 2t_i = 0$ $(i = 1, \cdots, k-1)$ corresponds to maximum of Q_{k-1}. To prove the same we require the following lemmas:

LEMMA 4.1. *The function* $Q_{k-1}(t_1, \cdots, t_{k-1})$ *defined over the domain* $0 < t_1 < \cdots < t_{k-1} < \infty$ *is bounded by* 0 *and* 1, i.e., $0 \leqq Q_{k-1}(t_1, \cdots, t_{k-1}) \leqq 1$.

PROOF. It is easily checked that $Q_{k-1}(t_1, \cdots, t_{k-1}) \geqq 0$. By Schwarz's inequality we have

$$\left(\int_{t_{i-1}}^{t_i} (t-1)e^{-t}\,dt\right)^2 \leqq \left(\int_{t_{i-1}}^{t_i} e^{-t}\,dt\right)\left(\int_{t_{i-1}}^{t_i} (t-1)^2 e^{-t}\,dt\right)$$

or

$$(t_i e^{-t_i} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_i}) \leqq \int_{t_{i-1}}^{t_i} (t-1)^2 e^{-t}\,dt$$

or

$$\sum_{i=1}^{k} [(t_i e^{-t_i} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_i})] \leqq \int_0^\infty (t-1)^2 e^{-t}\,dt = 1.$$

Hence the lemma is proved.

LEMMA 4.2. $t_{i-1} < \tau_i < t_i$ *for* $i = 1, \cdots, k-1$ *where* τ_i *is given in* (4.1).

PROOF. By the mean value theorem, there exists a τ_i such that $\int_{t_{i-1}}^{t_i} te^{-t}\,dt = \tau_i \int_{t_{i-1}}^{t_i} e^{-t}\,dt = \tau_i(e^{-t_{i-1}} - e^{-t_i})$ where $t_{i-1} < \tau_i < t_i$.

Hence $0 < \tau_1 < t_1 < \tau_2 < \cdots < t_{k-1} < \tau_k < \infty$. Further, we note that if $t_i = t_{i-1}$, then $t_i = t_{i-1} = \tau_i$.

LEMMA 4.3. *The function* $Q_{k-1}(t_1, \cdots, t_{k-1})$ *defined over the domain* $0 < t_1 < \cdots < t_{k-1} < \infty$ *satisfy the inequalities*

$$Q_{k-1}(t_1, \cdots, t_i, t_{i+1}, \cdots, t_{k-1}) > \lim_{t_i \to t_{i+1}} Q_{k-1}(t_1, t_2, \cdots, t_i, t_{i+1}, \cdots, t_k),$$

$$Q_{k-1}(t_1, \cdots, t_{k-1}) > \lim_{t_1 \to 0} Q_{k-1}(t_1, \cdots, t_{k-1}),$$

and

$$Q_{k-1}(t_1, \cdots, t_{k-1}) > \lim_{t_{k-1}\to\infty} Q_{k-1}(t_1, \cdots, t_{k-1}).$$

PROOF.

$$Q_{k-1}(t_1, \cdots, t_{k-1}) = \sum_{i=1}^{k} (t_i e^{-t_i} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_i})$$

and

$$\lim_{t_i\to t_{i+1}} (t_{i+1}e^{-t_{i+1}} - t_i e^{-t_i})^2/(e^{-t_i} - e^{-t_{i+1}}) = 0.$$

Therefore,

$$\lim_{t_i\to t_{i+1}} Q_{k-1}(t_1, \cdots, t_{i-1}, t_i, t_{i+1}, \cdots, t_{k-1})$$
$$= Q_{k-2}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{k-1}).$$

But

$$Q_{k-1}(t_1, \cdots, t_{i-1}, t_i', t_{i+1}, \cdots, t_{k-1}) > Q_{k-2}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{k-1})$$

for $t_{i-1} < t_i' < t_{i+1}$ since

$$Q_{k-1}(t_1, \cdots, t_{i-1}, t_i', t_{i+1}, \cdots, t_{k-1}) - Q_{k-1}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{k-1})$$
$$= (t_i e^{-t_1'} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_i'}) + (t_{i+1}e^{-t_{i+1}} - t_i e^{-t_i'})^2/(e^{-t_i'} - e^{-t_{i+1}})$$
$$- (t_{i+1}e^{-t_{i+1}} - t_{i-1}e^{-t_{i-1}})^2/(e^{-t_{i-1}} - e^{-t_{i+1}})$$
$$= (e^{-t_{i-1}} - e^{-t_i'})(e^{-t_i'} - e^{-t_{i+1}})(\tau_{i+1} - \tau_i') > 0.$$

But $Q_{k-1}(t_1, \cdots, t_{i-1}, t_i', t_{i+1}, \cdots, t_{k-1}) = Q_{k-2}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{k-1})$ if $t_i' \to t_{i+1}$ or $t_i' \to t_{i-1}$. It is easily checked that if $t_1 \to 0$ or $t_{k-1} \to \infty$, the proposition remains true. Hence the lemma is proved.

LEMMA 4.4. *The function* $Q_{k-1}(t_1, \cdots, t_{k-1})$ *defined over the domain* $0 \leqq t_1 \leqq \cdots \leqq t_{k-1} \leqq \infty$ *does not have a maximum on the boundary; it has at least one maximum inside the domain* $0 \leqq t_1 \leqq \cdots \leqq t_{k-1} \leqq \infty$.

PROOF. The proof is a direct consequence of Lemmas 4.1 and 4.3.

LEMMA 4.5. *A sufficient condition that the* $(k - 1) \times (k - 1)$ *symmetric matrix of the form*

$$J = \begin{pmatrix} -a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & -a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & -a_3 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & b_{k-3} & -a_{k-2} & b_{k-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_{k-2} & -a_{k-1} \end{pmatrix}$$

for $a_i > 0, b_i > 0$, *be negative definite is that the row-totals* (i) $(b_i - a_{i+1} + b_{i+1}) \leqq 0$ *for* $i = 1, 2, \cdots, k - 3$, *and the row-totals* (ii) $(b_1 - a_1) < 0$; (iii) $(b_{k-2} - a_{k-1}) < 0$.

PROOF. Consider the quadratic form

$$\begin{aligned} X'JX &= \textstyle\sum_{i=1}^{k-1} a_iX_i^2 + 2\sum_{i=1}^{k-1} b_iX_iX_{i-1}, \quad \text{where} \quad X_k = 0, \\ &= (b_1 - a_1)X_1^2 + (b_{k-2} - a_{k-1})X_{k-1}^2 + \textstyle\sum_{i=1}^{k-3}(b_i - a_{i+1} + b_{i+1})X_{i+1}^2 \\ &\quad - \textstyle\sum_{i=1}^{k-1} b_i(X_i - X_{i-1})^2. \end{aligned}$$

It follows immediately that if the stated conditions of the lemma are satisfied, then J is negative definite.

LEMMA 4.6. *The function $Q_{k-1}(t_1, \cdots, t_{k-1})$ defined over the domain $0 \leqq t_1 \leqq \cdots \leqq t_{k-1} \leqq \infty$ has at least one maximum which corresponds to the solutions of the equations*

$$\tau_{i+1} + \tau_i - 2t_i = 0 \qquad (i = 1, \cdots, k-1)$$

where τ_i is defined in (4.1).

PROOF. To obtain the maximum of $Q_{k-1}(t_1, \cdots, t_{k-1})$ we solve the following system of equations, for $i = 1, \cdots, k-1$:

$$(\delta/\delta t_i)Q_{k-1}(t_1, \cdots, t_{k-1}) = e^{-t_i}(\tau_{i+1} - \tau_i)(\tau_{i+1} + \tau_i - 2t_i) = 0.$$

But $e^{-t_i} > 0$ for $0 < t_i < \infty$ and $\tau_{i+1} > \tau_i$ by Lemmas 4.1 and 4.4. Hence the extremes of $Q_{k-1}(t_1, \cdots, t_{k-1})$ correspond to the solutions of the following system of equations $\tau_{i+1} + \tau_i - 2t_i = 0$ $(i = 1, \cdots, k-1)$. To check whether maximum or minimum occurs, we have to show that the matrix of the second partial derivatives

$$(((\delta^2/\delta t_i\delta t_j)Q_{k-1}))$$

at those points where $\tau_{i+1} + \tau_i - 2t_i = 0$ is negative definite or positive definite. In the present case, we show that it is negative definite.

The second partial derivatives of Q_{k-1} at those points which satisfy $\tau_{i+1} + \tau_i - 2t_i = 0$ are given by

$$\begin{aligned} \delta^2 Q_{k-1}/\delta t_i^2 &= \phi_{ii} = e^{-2t_i}(\tau_{i+1} - \tau_i)[(t_i - \tau_i)/(e^{-t_{i-1}} - e^{-t_i}) \\ &\qquad + (\tau_{i+1} - t_i)/(e^{-t_i} - e^{-t_{i-1}}) - 2e^{t_i}], \\ \delta^2 Q_{k-1}/\delta t_i\delta t_{i+1} &= \phi_{i,i+1} = e^{-t_i}e^{-t_{i+1}}(\tau_{i+1} - \tau_i)(t_{i+1} - \tau_{i+1})/(e^{-t_i} - e^{-t_{i+1}}), \\ \delta^2 Q_{k-1}/\delta t_i\delta t_{i-1} &= \phi_{i,i-1} = e^{-t_i}e^{-t_{i-1}}(\tau_{i+1} - \tau_i)(\tau_i - t_{i-1})/(e^{-t_{i-1}} - e^{-t_i}), \\ \delta^2 Q_{k-1}/\delta t_i\delta t_j &= \phi_{i,j} = 0 \qquad \text{for } |i - j| > 1. \end{aligned}$$

Now, for $0 < t_1 < \cdots < t_{k-1} < \infty$, $\phi_{i,i+1} > 0$ and $\phi_{i,i-1} > 0$, since $\tau_{i+1} - \tau_i > 0$, $t_{i+1} > \tau_{i+1}$ and $\tau_i > t_{i-1}$. Further, $\phi_{ii} < 0$ for $i = 2, \cdots, k-2$,

$$\begin{aligned} &\phi_{ii} + \phi_{i,i+1} + \phi_{i,i-1} \\ &\quad = e^{-t_i}(\tau_{i+1} - \tau_i)[(t_i - \tau_i)e^{-t_i}/(e^{-t_{i-1}} - e^{-t_i})] + (\tau_{i+1} - t_i)e^{-t_i}/(e^{-t_i} - e^{-t_{i+1}}) \\ &\qquad - 2 + (t_{i+1} - \tau_{i+1})e^{-t_{i+1}}/(e^{-t_i} - e^{-t_{i+1}}) + (\tau_i - t_{i-1})e^{-t_{i-1}}/(e^{-t_{i-1}} - e^{-t_i})] \\ &\quad = e^{-t_i}(\tau_{i+1} - \tau_i)(1 - \tau_i + \tau_i + 1 - \tau_{i+1} + \tau_{i+1} - 2) = 0. \end{aligned}$$

Also, it is easy to check that

$$\phi_{11} + \phi_{12} = -e^{-t_i}(\tau_2 - \tau_1)\{\tau_2 + \tau_1(e^{t_i} - 1)^{-1}\} < 0$$

and

$$\phi_{k-1,k-2} + \phi_{k-1,k-1} = -e^{-t_{k-1}}(\tau_k - \tau_{k-1})(1 + t_{k-1} + \tau_{k-1}) < 0.$$

Therefore, the elements of the matrix $((\delta^2 Q_{k-1}/\delta t_i \delta t_j))$ satisfy the condition of the Lemma 4.5. Hence, the matrix is negative definite and the lemma is proved.

THEOREM 4.1. *The system of equations*

$$\tau_{i+1} + \tau_i - 2t_i = 0 \qquad (i = 1, \cdots, k-1) \tag{4.2}$$

has one and only one solution and this solution corresponds to the maximum of Q_{k-1}.

PROOF. Applying the following transformation $s_i = t_i - t_{i-1}$ $(i = 1, \cdots, k-1)$ with $t_0 = 0$ to the function $Q_{k-1}(t_1, \cdots, t_{k-1})$ given in (3.9), we obtain

$$\psi(s_1, \cdots, s_{k-1}) = f(s_1) + e^{-s_1}f(s_2) + e^{-s_1-s_2}f(s_3) + \cdots + \exp\left(-\textstyle\sum_{j=1}^{k-2} s_j\right)f(s_{k-1}), \tag{4.3}$$

where $f(s_i) = s_i^2/e^{s_i} - 1$ $(i = 1, \cdots, k-1)$ and $0 < s_1 < s_1 + s_2 < \cdots < \sum_{j=1}^{k-2} s_j < \infty$. Clearly, the Jacobian of transformation is unity. Moreover, by the chain rule, $\delta Q_{k-1}/\delta t_i = (\delta\psi/\delta s_1)(\delta s_1/\delta t_1) + \cdots + (\delta\psi/\delta s_{k-1})(\delta s_{k-1}/\delta t_i)$; $(i = 1, \cdots, k-1)$. It follows that $\delta\psi/\delta s_i = 0$ if and only if $\delta Q_{k-1}/\delta t_i = 0$, $i = 1, \cdots, k-1$. Now, we have to show that the system of equations $\delta\psi/\delta s_i = 0$ $(i = 1, \cdots, k-1)$ has one and only one solution. The solution corresponds to maximum follow from lemma (4.6).

$$\delta\psi/\delta s_i = \exp\left(-\textstyle\sum_{j=1}^{i-1} s_j\right)\{df(s_i)/ds_i - e^{-s_i}G(s_{i+1})\} = 0 \qquad (i = 1, \cdots, k-1) \tag{4.4}$$

where $G(s_k) = 0$, and

$$G(s_{i+1}) = f(s_{i+1}) + e^{-s_{i+1}}f(s_{i+2}) + \cdots + \exp\left(-\textstyle\sum_{j=i+1}^{k-2} s_j\right)f(s_{k-1}).$$

Clearly, $G(s_{i+1})$ is a function of the form $\psi(s_1, \cdots, s_{k-1})$ and $0 \leqq \psi(s_1, \cdots, s_{k-1}) \leqq 1$.

We consider first the $(k-1)$th equation

$$\delta\psi/\delta s_{k-1} = \exp\left(-\textstyle\sum_{j=1}^{k-1} s_j\right)[s_{k-1}/(e^{s_{k-1}} - 1)][2 - s_{k-1}/(1 - e^{-s_{k-1}})] = 0.$$

Since $\sum_{j=1}^{k-1} s_j < \infty$ and $0 < s_{k-1} < \infty$, we consider the factor in the parenthesis. This factor is a decreasing function of s_{k-1} since $s_{k-1}/1 - e^{-s_{k-1}}$ is increasing. Therefore, by monotonicity, the solution is unique, and is $s_{k-1}^0 = 1.5936$.

Next, we consider the $(k-2)$th equation

$$\delta\psi/\delta s_{k-2} = \exp\left(-\textstyle\sum_{j=1}^{k-3} s_j\right)\{df(s_{k-2})/ds_{k-2} - e^{-s_{k-2}}f(s_{k-1})\} = 0.$$

Since $0 < \sum_{j=1}^{k-3} s_j < \infty$, the above equation reduces to

$$1 - f(s_{k-1}) - (s_{k-2}/(1 - e^{-s_{k-2}}) - 1)^2 = 0.$$

Substituting the root of the $(k-1)$th equation, we obtain $s_{k-2}/(1-e^{-s_{k-2}}) = 1 + (1 - f(s_{k-1}^0))^{\frac{1}{2}}$ since $s_{k-2}/(1-e^{-s_{k-2}}) > 1$, where $f(s_{k-1}^0)$ is the maximum value of $f(s_{k-1})$ and is less than unity. Since $s_{k-2}/(1-e^{-s_{k-2}})$ is an increasing function of s_{k-2}, by monotonicity the solution is unique and is $s_{k-2}^0 = 1.0177$.

Now, consider the ith equation given in (4.4). Assuming that $k - i - 1$ equations have been solved and the maximum value of $G(s_{i+1})$ is $G(s_{i+1}^0)$ for $s_{i+1} = s_{i+1}^0, \cdots, s_{k-1} = s_{k-1}^0$, Equation (4.4) reduces to $s_i/(1-e^{-s_i}) = 1 + (1 - G(s_{i+1}^0))^{\frac{1}{2}}$, since $s_i/(1-e^{-s_i}) > 1$ and $G(s_{i+1}^0) < 1$. By previous argument, the solution is unique. Hence, we have proved that the system of equations $\delta\psi/\delta s_i = 0$ $(i = 1, \cdots, k-1)$ has unique solution. The proof of the theorem is complete.

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