OPTIMAL STOPPING AND EXPERIMENTAL DESIGN[1]

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1. Introduction and summary. The problem of finding Bayes solutions for sequential experimental design problems motivates the study of the following type of one-person sequential game. If the game is stopped at any stage a, the loss to the player is the value of a random variable (rv), say Z_a. If the player chooses to continue the game, he can select the next rv to be observed from a class of rv's available at that stage, thus bringing the game to one of the stages succeeding stage a. (The class of all stages can be pictured as a "tree".) At this stage the player can again choose to stop, accepting the value of the chosen rv as his loss, or he may continue by selecting one of the class of rv's now available for the next observation. The player is required to stop sometime, and his decisions at any stage must depend only on information available at that stage.

A model for this situation is given in Section 4. *Control variables*, which correspond to stopping variables in the usual formulation of sequential games, are defined which can be used by the player to decide whether to stop or not at any stage and, if he continues, which rv to observe next. A general characterization of control variables that minimize expected loss is given, and existence of such optimal control variables is proved under conditions applicable to statistical problems. The application to finding Bayes solutions to sequential experimental design problems is given in Section 5.

As a preliminary to the discussion on control variables, Section 3 provides a study of the theory of optimal stopping variables. Let $\{Z_n, F_n, n \geqq 1\}$ be a stochastic process on a probability space (Ω, F, P) with points ω. A stopping variable (sv) is a rv t with values in $\{1, 2, \cdots, \infty\}$ such that $t < \infty$ a.e. and $\{t = n\} \varepsilon F_n$ for each n. For any such sv t, a rv Z_t is defined by

$$Z_t(\omega) = Z_n(\omega), \quad \text{if} \quad t(\omega) = n,$$
$$= \infty, \quad \text{if} \quad t(\omega) = \infty.$$

It is convenient to think of Z_n as the loss after n plays in a one-person sequential game and to consider the σ-field F_n as representing the knowledge of the past after n plays. The problem of finding a strategy for stopping the game to minimize the expected loss corresponds to finding a minimizing sv, i.e., one which minimizes EZ_t among the class of all sv's t.

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The main results in Section 3 are new characterizations of Snell's solution in [12] to the problem of optimal stopping which generalized the well-known Arrow-Blackwell-Girshick theory in [1]. Under the assumption that there is an integrable rv U such that $Z_n \geqq U$ a.e. for each n, Snell showed that when a minimizing sv t^* exists, it can be defined as the first integer j such that $X_j = Z_j$ (or ∞ if no such integer exists) where $\{X_n, F_n, n \geqq 1\}$ is the maximal regular generalized submartingale relative to $\{Z_n, F_n, n \geqq 1\}$. Under the additional assumption that $\{Z_n, n \geqq 1\}$ is an integrable sequence, we now show that the rv's X_n can be further identified as

$$X_n = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_n) \text{ a.e.},$$

where T_n is the class of sv's t such that $t \geqq n$. It follows that if there is a set A_n in F_n for each n such that

(i) $Z_n \leqq E(Z_t \mid F_n)$ a.e. on A_n for each t in T_{n+1}, and

(ii) $Z_n > \text{ess inf}_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)$ a.e. on the complement A_n',

then the minimizing sv t^* defined above is such that, for almost all points ω, $t^*(\omega) = n$ if and only if $\omega \varepsilon A_n - \bigcup_{i=1}^{n-1} A_i$.

This comes very close to stating that the player of the sequential game above should stop playing after n plays if and only if there is no continuation (i.e., sv in T_{n+1}) having conditional expected loss given the past which is less than the present loss $Z_n(\omega)$. Unfortunately, such an interpretation is not valid in general since it is equivalent to stopping the game after n plays if and only if

$$Z_n(\omega) \leqq \inf_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)(\omega).$$

The function $\inf_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)$ not only may not be measurable, but also in many cases it may be changed almost at will by choosing different versions of the conditional expectations involved. On the other hand, the interpretation above is valid, as is shown in Section 3, for the case where each σ-field F_n is generated by finitely many discrete rv's.

Section 3 also contains a reasonably general development of the theory of minimizing sv's, including a more direct proof of Snell's result using the definition $X_n = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_n)$. Illustrations are given and comparisons are made with the approaches used by Arrow, Blackwell, and Girshick in [1] and Chow and Robbins in [4] and [5].

2. Preliminaries. The following definitions are relative to a probability space (Ω, F, P) with points ω. A collection $\{Z_a, F_a, a \varepsilon A\}$ where $(A, \leqq)$ is a partially ordered set is said to be a *stochastic process* if F_a is a σ-field of F-sets for each a in A, Z_a is a rv (with values in the extended real number system) which is F_a-measurable, and $F_a \subset F_b$ whenever $a \leqq b$. A *submartingale* is an integrable stochastic process $\{X_a, F_a, a \varepsilon A\}$ such that $X_a \leqq E(X_b \mid F_a)$ a.e. whenever $a \leqq b$; it is called a *martingale* if $X_a = E(X_b \mid F_a)$ a.e. whenever $a \leqq b$. If the integrability requirement in this definition is relaxed to the condition that EX_a exist for each a in A, such processes are referred to as *generalized submartingales* and *generalized martingales*.

For the case where A is the set of positive integers, stochastic processes will be denoted, for example, by $\{Z_n, F_n, n \geqq 1\}$. For this case a submartingale or generalized submartingale $\{Y_n, F_n, n \geqq 1\}$ is said to be *regular* if for any sv t, EY_t exists and $E(Y_t \mid F_n) \geqq Y_n$ a.e. on $\{t \geqq n\}$ for each $n \geqq 1$. A *maximal submartingale* (*maximal regular submartingale*) relative to a stochastic process $\{Z_n, F_n, n \geqq 1\}$ is a submartingale (regular submartingale) $\{Y_n, F_n, n \geqq 1\}$ such that $Y_n \leqq Z_n$ a.e. for each n and, if $\{V_n, F_n, n \geqq 1\}$ also satisfies these properties, then $V_n \leqq Y_n$ a.e. for each n. Clearly, any two maximal (maximal regular) submartingales relative to the same stochastic process must coincide a.e., and we shall refer to "the" maximal (maximal regular) submartingale relative to a process. *Maximal* (*maximal regular*) *generalized submartingales* are defined similarly by substituting "generalized submartingale" for "submartingale" in the preceding definition. Snell proved in [12] that if there is an integrable rv U such that $Z_n \geqq U$ a.e. for each n, both the maximal generalized submartingale and the maximal regular generalized submartingale relative to $\{Z_n, F_n, n \geqq 1\}$ exist.

The *essential infimum* of a family of rv's $\{Y_t, t \varepsilon T\}$, denoted by $\operatorname{ess\,inf}_{t \varepsilon T} Y_t$, is defined as any rv X such that

(a) $X \leqq Y_t$ a.e. for each t in T, and

(b) if Z is any other rv satisfying (a), $Z \leqq X$ a.e.

It is well-known that such a rv X always exists and can be taken as the infimum of some countable subset of $\{Y_t, t \varepsilon T\}$. Clearly, any two essential infimums must coincide a.e. Also, if $S \subset T$, then $\operatorname{ess\,inf}_{t\varepsilon S} Y_t \geqq \operatorname{ess\,inf}_{t\varepsilon T} Y_t$ a.e.

LEMMA 2.1. *For any family of* rv's $\{Y_t, t \varepsilon T\}$,

(a) *if U is a nonnegative finite-valued* rv,

$$\operatorname{ess\,inf}_{t\varepsilon T} UY_t = U \operatorname{ess\,inf}_{t\varepsilon T} Y_t \text{ a.e.};$$

(b) *if Z is a* rv *such that* $Z \leqq Y_t$ a.e. *on a set A in F for each t in T, then* $Z \leqq \operatorname{ess\,inf}_{t\varepsilon T} Y_t$ a.e. *on A;*

(c) *if* $T = \bigcup_{j=1}^{m} T_j$, *then*

$$\operatorname{ess\,inf}_{t\varepsilon T} Y_t = \min_j (\operatorname{ess\,inf}_{t\varepsilon T_j} Y_t) \text{ a.e.}$$

PROOF. (a) Since $UY_t \geqq U \operatorname{ess\,inf}_{t\varepsilon T} Y_t$ a.e. for each t in T, also $\operatorname{ess\,inf}_{t\varepsilon T} UY_t \geqq U \operatorname{ess\,inf}_{t\varepsilon T} Y_t$ a.e. The opposite inequality also holds because, from the representation $\operatorname{ess\,inf}_{t\varepsilon T} Y_t = \inf_k Y_{t_k}$ for some sequence $\{t_k, k \geqq 1\}$ from T,

$$U \operatorname{ess\,inf}_{t\varepsilon T} Y_t = \inf_k UY_{t_k} \geqq \operatorname{ess\,inf}_{t\varepsilon T} UY_t \text{ a.e.}$$

(b) By hypothesis, $I_A Z \leqq I_A Y_t$ a.e. for each t in T, where I_A denotes the indicator function of the set A. Therefore, by part (a),

$$I_A Z \leqq \operatorname{ess\,inf}_{t\varepsilon T} I_A Y_t = I_A \operatorname{ess\,inf}_{t\varepsilon T} Y_t \text{ a.e.}$$

which implies the result.

(c) This follows easily from the representation of the essential infimum used in part (a).

3. Minimizing stopping variables. Given an integrable stochastic process $\{Z_n, F_n, n \geqq 1\}$ on a probability space (Ω, F, P), the problem is to find a sv t^*, if it exists, which minimizes EZ_t among the class of all sv's t. It will be assumed throughout that $Z_n \geqq 0$ a.e. for each n, but the case where there is an integrable rv U such that $Z_n \geqq U$ a.e. for each n can also be handled, and all the theorems below hold under this assumption.

We shall let $\{X_n, n \geqq 1\}$ be the sequence of rv's defined by

$$X_n = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_n),$$

where T_n denotes the class of sv's t such that $t \geqq n$. By the properties of the essential infimum, we can assume that $X_n = \inf_k E(Z_{t_k} \mid F_n)$ for some sequence $\{t_k, k \geqq 1\}$ from T_n; therefore, we can also assume that X_n is F_n-measurable. With this assumption we shall show that $\{X_n, F_n, n \geqq 1\}$ is a nonnegative submartingale. Clearly, $X_n \geqq 0$ a.e. since $E(Z_t \mid F_n) \geqq 0$ a.e. for each t in T_n. Also, X_n is integrable for each n since $X_n \leqq Z_n$ a.e., as is seen by considering the sv in T_n having constant value n. The submartingale inequality is implied by part (c) of the following lemma.

LEMMA 3.1. *For each positive integer* n,

(a) $X_n = \min [Z_n, \text{ess inf}_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)]$ a.e.;

(b) $E(X_{n+1} \mid F_n) = \text{ess inf}_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)$ a.e.;

(c) $X_n = \min [Z_n, E(X_{n+1} \mid F_n)]$ a.e.

PROOF. (a) Denoting $\text{ess inf}_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)$ by V, we have that $X_n \leqq V$ a.e. since $T_n \supset T_{n+1}$. Since we already have $X_n \leqq Z_n$ a.e., this gives $X_n \leqq \min (Z_n, V)$ a.e. To prove inequality in the opposite sense, let s be any sv in T_n and define $s' = \max (s, n + 1)$. Then $s' \varepsilon T_{n+1}$ and, setting $A = \{s = n\}$, we have

$$E(Z_s \mid F_n) = E(I_A Z_n + I_{A'} Z_{s'} \mid F_n) = I_A Z_n + I_{A'} E(Z_{s'} \mid F_n)$$
$$\geqq I_A Z_n + I_{A'} V \geqq \min (Z_n, V) \text{ a.e.}$$

Hence, $X_n \geqq \min (Z_n, V)$ a.e.

(b) For any sv t in T_{n+1}, $E(X_{n+1} \mid F_n) \leqq E(E(Z_t \mid F_{n+1}) \mid F_n) = E(Z_t \mid F_n)$ a.e. It follows that $E(X_{n+1} \mid F_n) \leqq V$ a.e. To prove the opposite inequality, we first note that $X_{n+1} = \inf_k E(Z_{t_k} \mid F_{n+1})$ a.e. for some sequence $\{t_k, k \geqq 1\}$ from T_{n+1} where t_1 is the sv having value $n + 1$ for all points ω. Next, we define a new sequence $\{s_k, k \geqq 1\}$ from T_{n+1} as follows:

$$\begin{aligned} & s_1(\omega) = t_1(\omega) && \text{for all } \omega; \\ \text{for } k > 1, \quad & s_k(\omega) = s_{k-1}(\omega) && \text{on } A_k \\ & = t_k(\omega) && \text{on } A_k', \end{aligned}$$

where $A_k = \{E(Z_{s_{k-1}} \mid F_{n+1}) \leqq E(Z_{t_k} \mid F_{n+1})\}$. Then

$$\begin{aligned} E(Z_{s_k} \mid F_{n+1}) &= E(I_{A_k} Z_{s_{k-1}} + I_{A_k'} Z_{t_k} \mid F_{n+1}) \\ &= I_{A_k} E(Z_{s_{k-1}} \mid F_{n+1}) + I_{A_k'} E(Z_{t_k} \mid F_{n+1}) \\ &= \min [E(Z_{s_{k-1}} \mid F_{n+1}), E(Z_{t_k} \mid F_{n+1})] \text{ a.e.} \end{aligned}$$

This implies that $\{E(Z_{s_k} \mid F_{n+1}), k \geqq 1\}$ is a nonincreasing sequence a.e. and that $X_{n+1} = \inf_k E(Z_{s_k} \mid F_{n+1}) = \lim_k E(Z_{s_k} \mid F_{n+1})$ a.e. Also, this sequence is dominated above a.e. by an integrable rv; namely, $E(Z_{t_1} \mid F_{n+1}) = Z_{n+1}$. Therefore,

$$\begin{aligned} E(X_{n+1} \mid F_n) &= E(\lim_k E(Z_{s_k} \mid F_{n+1}) \mid F_n) \\ &= \lim_k E(Z_{s_k} \mid F_n) \\ &\geqq \inf_k E(Z_{s_k} \mid F_n) \geqq V \text{ a.e.} \end{aligned}$$

(c) This follows immediately from parts (a) and (b).

THEOREM 3.1. *Let t^* be the* rv *defined by*

$$\begin{aligned} t^*(\omega) &= k \quad \text{if} \quad X_j(\omega) < Z_j(\omega) \quad \text{for} \quad j < k, X_k(\omega) = Z_k(\omega) \\ &= \infty \quad \text{if} \quad X_j(\omega) < Z_j(\omega) \quad \text{for all} \quad j. \end{aligned}$$

(a) *If $t^* < \infty$* a.e., *then t^* is a minimizing* sv; *also,* $X_1 = E(Z_{t^*} \mid F_1)$ a.e.

(b) *If a minimizing* sv *t' exists, then t^* is also a minimizing* sv, *and* $t^* \leqq t'$ a.e.

PROOF. (a) Since $E(Z_t \mid F_1) \geqq X_1$ a.e. for each sv t, it suffices to show that $X_1 \geqq E(Z_{t^*} \mid F_1)$ a.e.; this would imply that $X_1 = E(Z_{t^*} \mid F_1) \leqq E(Z_t \mid F_1)$ a.e. for each t, and taking expectations would then give us that t^* is a minimizing sv. We shall first prove by induction that for each integer $n \geqq 1$,

$$X_1 = E(I_{\{t^*<n\}} Z_{t^*} + I_{\{t^* \geqq n\}} X_n \mid F_1) \text{ a.e.} \tag{1}$$

This clearly holds for $n = 1$. Also, if it holds for any integer n, it follows for $n + 1$ because, by the definition of t^* and Lemma 3.1 (c),

$$\begin{aligned} E(I_{\{t^* \geqq n\}} X_n \mid F_1) &= E(I_{\{t^*=n\}} Z_n + I_{\{t^*>n\}} E(X_{n+1} \mid F_n) \mid F_1) \\ &= E(I_{\{t^*=n\}} Z_n + I_{\{t^*>n\}} X_{n+1} \mid F_1) \text{ a.e.} \end{aligned}$$

By the nonnegativeness of the sequence $\{X_n, n \geqq 1\}$, it follows from (1) that for each n, $X_1 \geqq E(I_{\{t^*<n\}} Z_{t^*} \mid F_1)$ a.e. Since $Z_{t^*} \geqq 0$ a.e., we can take the limit as n becomes infinite and use the monotone convergence theorem for conditional expectations to obtain that $X_1 \geqq E(Z_{t^*} \mid F_1)$ a.e.

(b) It suffices to show that $X_n = Z_n$ a.e. on the set $A_n = \{t' = n\}$ for each $n \geqq 1$, because this would imply that $t^* \leqq n$ a.e. on A_n, thus guaranteeing that $t^* \leqq t' < \infty$ a.e. We first claim that if $t \varepsilon T_n$, then $Z_n \leqq E(Z_t \mid F_n)$ a.e. on A_n. If this were not the case, the set $B = A_n \cap \{Z_n > E(Z_t \mid F_n)\}$ in F_n would have positive probability and the sv s defined by setting $s = I_B t + I_{B'} t'$ would satisfy

$$\begin{aligned} EZ_s = E(I_B Z_t + I_{B'} Z_{t'}) &= E(I_B E(Z_t \mid F_n) + I_{B'} Z_{t'}) \\ &< E(I_B Z_{t'} + I_{B'} Z_{t'}) = EZ_{t'}, \end{aligned}$$

contradicting the assumption that t' is a minimizing sv. Hence, we now have $Z_n \leqq E(Z_t \mid F_n)$ a.e. on A_n for each t in T_n. By Lemma 2.1(b) this implies that $Z_n \leqq X_n$ a.e. on A_n. Since $Z_n \geqq X_n$ a.e., the result follows.

COROLLARY 3.1. *If* $\lim_n Z_n = \infty$ a.e., *t^* is a minimizing* sv.

PROOF. Snell proved in [12] that a minimizing sv exists if $\lim_n Z_n = \infty$ a.e. By part (b) of the theorem, this implies that t^* is a minimizing sv.

THEOREM 3.2. *If there is a set* A_n *in* F_n *for each* $n \geqq 1$ *such that*

(i) $Z_n \leqq E(Z_t \mid F_n)$ a.e. *on* A_n *for each* t *in* T_{n+1}, *and*

(ii) $Z_n >$ ess $\inf_{t \varepsilon T_{n+1}} E(Z_t \mid F_n)$ a.e. *on* A_n',

then the rv t^* *defined in Theorem* 3.1 *is such that, for almost all points* ω, $t^*(\omega) = n$ *if and only if* $\omega \varepsilon A_n - \bigcup_{i=1}^{n-1} A_i$.

PROOF. Condition (i) and Lemma 2.1(b) give us that

$$Z_n \leqq \text{ess inf}_{t\varepsilon T_{n+1}} E(Z_t \mid F_n) \text{ a.e. on } A_n .$$

Therefore, by (ii) and Lemma 3.1(a), $X_n = Z_n$ a.e. on A_n and $X_n < Z_n$ a. e. on A_n'. The conclusion follows by the definition of t^*.

This theorem provides a technique for finding minimizing sv's without reference to the sequence $\{X_n, n \geqq 1\}$. Illustrations will be given at the end of this section.

In typical applications there is a sequence of rv's $Y_1, Y_2, \cdots$ defined on Ω, and F_n is the σ-field generated by $Y_1, Y_2, \cdots, Y_n$. Also $Z_n = g_n(Y_1, Y_2, \cdots, Y_n)$ where g_n is a Borel-measurable function on Euclidean n-space. If the rv's $Y_1, Y_2, \cdots$ are discrete, inverse images of points in n-space under the random vector $Y^n = (Y_1, Y_2, \cdots, Y_n)$ provide a countable partition of the space Ω such that F_n is the class of all unions of sets in this partition. Since both Z_n and $E(Z_t \mid F_n)$ are F_n-measurable, these functions must be constant-valued on each element of the partition. The values of Z_n and $E(Z_t \mid F_n)$ for any sv t in T_{n+1} can be compared for each element in the partition (or, equivalently, for each value of Y^n), and the set A_n in Theorem 3.2 can be taken as the union of sets Q in the partition for which $Z_n \leqq E(Z_t \mid F_n)$ on Q for each t in T_{n+1}. Therefore, by Theorem 3.2 the rv t^* can be characterized as follows: For any point ω, $t^*(\omega)$ is the first integer n such that $Z_n(\omega) \leqq E(Z_t \mid F_n)(\omega)$ for every sv t in T_{n+1}. This justifies, at least for this case, the statement that stopping should occur at the nth stage of a sequential game if and only if no continuation exists having conditional expected loss given the past which is less than the present loss, the value of Z_n.

THEOREM 3.3. *Let* $\{t_n^*, n \geqq 1\}$ *be the sequence of* rv's *defined by*

$$\begin{aligned} t_n^*(\omega) &= k, \quad \text{if} \quad X_k(\omega) = Z_k(\omega) \quad \text{where} \quad k \geqq n \\ &\qquad \text{and} \quad X_j(\omega) < Z_j(\omega) \quad \text{for} \quad n \leqq j < k, \\ &= \infty \quad \text{if} \quad X_j(\omega) < Z_j(\omega) \quad \text{for all} \quad j \geqq n. \end{aligned}$$

(a) *If* $t_n^* < \infty$ a.e., *then* t_n^* *is a minimizing* s.v. *for the class* T_n; *also,* $X_n = E(Z_{t_n^*} \mid F_n)$ a.e.

(b) *If* $t_n^* < \infty$ a.e. *for each* $n \geqq 1$, *the sets* $\{A_n, n \geqq 1\}$ *defined by* $A_n = \{Z_n \leqq E(Z_{t_{n+1}^*} \mid F_n)\}$ *satisfy the conditions of Theorem* 3.2.

PROOF. (a) This becomes a special case of Theorem 3.1(a) when the original stochastic process $\{Z_n, F_n, n \geqq 1\}$ is replaced by $\{Z_j, F_j, j \geqq n\}$.

(b) From part (a),

$$E(X_{n+1} \mid F_n) = E(E(Z_{t_{n+1}^*} \mid F_{n+1}) \mid F_n) = E(Z_{t_{n+1}^*} \mid F_n) \quad \text{a.e.}$$

Therefore, by the definition of A_n and Lemma 3.1(b),

$$Z_n \leqq \text{ess inf}_{t \in T_{n+1}} E(Z_t \mid F_n) \text{ a.e. on } A_n, \text{ and}$$
$$Z_n > \text{ess inf}_{t \in T_{n+1}} E(Z_t \mid F_n) \text{ a.e. on } A_n'.$$

These conditions imply those of Theorem 3.2.

Suppose for the moment that $\lim_n Z_n = \infty$ a.e. Just as this implies that $t^* < \infty$ a.e. (as in Corollary 3.1), it also implies that $t_n^* < \infty$ a.e. for each n. Therefore, under this assumption, part (b) of the above theorem justifies the following characterization of the minimizing sv t^*: Stop at the nth stage if and only if the present loss, the value of Z_n, is less than or equal to the conditional expected loss for the procedure which takes at least one more observation and makes the best use of all future observations.

The next theorem provides necessary and sufficient conditions for a sv to be a minimizing sv. These conditions are similar to those given by Chow and Robbins in [4].

THEOREM 3.4. *A* sv t' *is a minimizing* sv *if and only if the following conditions hold for each* $n \geqq 1$:

(i) $Z_n \leqq E(Z_t \mid F_n)$ a.e. *on* $\{t' = n\}$ *for each* t *in* T_{n+1};

(ii) $Z_n \geqq \text{ess inf}_{t \in T_{n+1}} E(Z_t \mid F_n)$ a.e. *on* $\{t' > n\}$.

PROOF. Suppose that t' is a sv satisfying (i) and (ii). Then, by (i) and Lemma 2.1(b), $Z_n \leqq \text{ess inf}_{t \in T_{n+1}} E(Z_t \mid F_n)$ a.e. on $\{t' = n\}$. It follows by (ii) and Lemma 3.1 that $X_n = Z_n$ a.e. on $\{t' = n\}$ and $X_n = E(X_{n+1} \mid F_n)$ a.e. on $\{t' > n\}$. From these properties of t' the proof that t' is a minimizing sv proceeds exactly as the corresponding proof for t^* in Theorem 3.1(a).

Conversely, if t' is a minimizing sv, then (i) must hold, as was shown in the proof of Theorem 3.1(b). To show that (ii) holds, it suffices to show that $Z_n \geqq E(Z_s \mid F_n)$ a.e. on $\{t' > n\}$ where $s = \max(t', n + 1)$. Let $A = \{Z_n < E(Z_s \mid F_n), t' > n\}$ and assume, contrary to the assertion, that $P(A) > 0$. If we set $s' = nI_A + I_{A'}t'$, then s' is a sv and

$$\begin{aligned} EZ_{s'} &= E(I_A Z_n + I_{A'} Z_{t'}) < E(I_A E(Z_s \mid F_n) + I_{A'} Z_{t'}) \\ &= E(I_A Z_s + I_{A'} Z_{t'}) = E(I_A Z_{t'} + I_{A'} Z_{t'}) = EZ_{t'}. \end{aligned}$$

However, this contradicts the assumption that t' is a minimizing sv, thus completing the proof.

Next we shall show the relationship between the theory above and that given by Snell in [12]. Where we have used the stochastic process $\{X_n, F_n, n \geqq 1\}$ above in characterizing minimizing sv's, Snell uses the maximal regular generalized submartingale relative to $\{Z_n, F_n, n \geqq 1\}$. The next theorem will show that these processes coincide under our assumption that each Z_n is integrable. (With this assumption the word "generalized" above becomes unnecessary.)

THEOREM 3.5. *The stochastic process* $\{X_n, F_n, n \geqq 1\}$ *is the maximal regular submartingale relative to* $\{Z_n, F_n, n \geqq 1\}$.

(This theorem and Lemma 3.1(c) were proved independently by Y. S. Chow. He has also shown that our assumption in these theorems that $\inf_n Z_n$ be integrable can be replaced by the condition that $\sup_n Z_n$ be integrable.)

PROOF. Let $\{Y_n, F_n, n \geqq 1\}$ be the maximal regular submartingale relative to $\{Z_n, F_n, n \geqq 1\}$. Then, for any t in T_n, $Y_n \leqq E(Y_t \mid F_n) \leqq E(Z_t \mid F_n)$ a.e. Hence, $Y_n \leqq X_n$ a.e. Since X_n is integrable, to show that $X_n = Y_n$ a.e., it suffices to show that $EX_n \leqq EY_n$. We already have that $X_n \leqq E(Z_t \mid F_n)$ a.e. for each t in T_n, so that $EX_n \leqq \inf_{t\epsilon T_n} EZ_t$. Since $EY_n = \inf_{t \epsilon T_n} EZ_t$ (Theorem 3.6 of [12]), this completes the proof.

In [12] the assumption that each rv Z_n is integrable is not used. Our assumption could have been relaxed slightly to require only that for each n there exist a sv t in T_n such that $EZ_t < \infty$. It is not known whether Theorem 3.5 holds without some such assumption.

To relate the results above to those given by Arrow, Blackwell, and Girshick in [1], consider the case where the stochastic process $\{Z_n, F_n, n \geqq 1\}$ is "truncated" to the finite process $\{Z_n, F_n, n = 1, 2, \cdots, N\}$, which we shall now denote more concisely by $\{Z_n, n \leqq N\}$. The theory above clearly applies to truncated processes with only minor notational changes. Letting $\{X_n^N, n \leqq N\}$ be the process defined by $X_n^N = \text{ess inf}_{t\epsilon T_n} E(Z_t \mid F_n)$ where T_n now denotes the class of sv's t such that $n \leqq t \leqq N$, we clearly have that $X_N^N = Z_N$ a.e. Also, by Lemma 3.1 (c), $X_n^N = \min [Z_n, E(X_{n+1}^N \mid F_n)]$ a.e. for $n < N$. Hence, the sequence $\{X_n^N, n \leqq N\}$ can be computed by recursion backwards. The sv t^* defined by stopping the first time that $X_n^N = Z_n$ is a minimizing sv by Theorem 3.1.

The above result was proved by Arrow, Blackwell, and Girshick in [1]. Their method of attack for the nontruncated case was to take the limit of the truncated procedures. First the sequences $\{X_n^N, n \leqq N\}$ are defined for each $N \geqq 1$ as follows (compare above):

$$X_N^N = Z_N, \qquad X_n^N = \min [Z_n, E(X_{n+1}^N \mid F_n)] \qquad \text{for } n < N.$$

It is easily seen that $\{X_n^N, n \leqq N\}$ is the maximal submartingale relative to $\{Z_n, n \leqq N\}$. Since $\{X_n^{N+1}, n \leqq N\}$ is another submartingale dominated above by $\{Z_n, n \leqq N\}$, this implies that $X_n^N \geqq X_n^{N+1}$ a.e. for $n \leqq N$. Following [1], we now let $X_n = \lim_N X_n^N$ for each n. Then $\{X_n, n \geqq 1\}$ is again a submartingale since

$$X_n = \lim_N \min [Z_n, E(X_{n+1}^N \mid F_n)] = \min [Z_n, E(X_{n+1} \mid F_n)] \text{ a.e.}$$

To see that it is actually the maximal submartingale relative to $\{Z_n, n \geqq 1\}$, let $\{V_n, n \geqq 1\}$ be any other submartingale such that $V_n \leqq Z_n$ a.e. for each $n \geqq 1$. Since $\{V_n, n \leqq N\}$ is a submartingale such that $V_n \leqq Z_n$ a.e. for $n \leqq N$, and $\{X_n^N, n \leqq N\}$ is the maximal submartingale relative to $\{Z_n, n \leqq N\}$, we have that $V_n \leqq X_n^N$ a.e. With this holding for every $N \geqq n$, also $V_n \leqq \lim_N X_n^N = X_n$ a.e.

It was shown in [1] that if $Z_n = r_n + c_n$ where r_n and c_n are nonnegative F_n-

measurable rv's such that $r_n \leqq K$ for all n and $\{c_n, n \geqq 1\}$ is a nondecreasing sequence such that $\lim_n c_n = \infty$, then stopping the first time that $X_n = Z_n$ gives a minimizing sv. Note that Corollary 3.1 proves the existence of a minimizing sv in this case. Also, since $\{c_n, n \geqq 1\}$ is a submartingale, it follows that $c_n \leqq X_n \leqq c_n + K$ so that $X_m \geqq X_j - K$ whenever $m \geqq j$. Therefore, the maximal submartingale is regular under these conditions by Corollary A.1(b) of [12].

To see that the minimizing sv cannot always be characterized in terms of the maximal submartingale, consider the case where $Y_1, Y_2, \cdots$ are independent rv's on (Ω, F, P), each with distribution given by $P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}$. If we define $Z_n = 2^n Y_1 Y_2 \cdots Y_n$ for each n and let F_n be the σ-field generated by $Y_1, Y_2, \cdots, Y_n$, then it is easily checked that $\{Z_n, F_n, n \geqq 1\}$ is a martingale. Therefore, the maximal submartingale relative to $\{Z_n, n \geqq 1\}$ in this case is again $\{Z_n, n \geqq 1\}$. If stopping occurs the first time that the given process and the maximal submartingale coincide, this would always be at the first stage and the expected loss would be $EZ_1 = 1$. However, the maximal regular submartingale relative to $\{Z_n, n \geqq 1\}$ is obtained by setting $X_n = 0$ for each n. To see this, use the representation given by $X_n = \text{ess inf}_{t \varepsilon T_n} E(Z_t \mid F_n)$ and consider the sv in T_n which stops the first time after stage n that Z_k has value 0 for some integer k. The minimizing sv t^* as given by Theorem 3.1 (stop the first time that $Z_n = 0$) has expected loss $EZ_{t^*} = 0$.

To illustrate the technique of finding a minimizing sv by applying Theorem 3.2, we shall now consider two one-person games which have been treated in [4], [5], [6], [9], and [11]. In both games the player can make observations at a positive cost c per observation on a sequence of independent, identically distributed rv's $Y_1, Y_2, \cdots$. In the first game if the player stops at stage n, he receives the value of the rv $M_n = \max(Y_1, Y_2, \cdots, Y_n)$ so that his net loss is the value of $Z_n = cn - M_n$. In the second game if the player stops at stage n, he receives the value of Y_n and his net loss is the value of $W_n = cn - Y_n$. The problem of finding stopping procedures for these games to minimize the expected loss amounts to finding minimizing sv's for the processes $\{Z_n, F_n, n \geqq 1\}$ and $\{W_n, F_n, n \geqq 1\}$ where F_n is the σ-field generated by $Y_1, Y_2, \cdots, Y_n$. In the solutions given below it is assumed that there exist real numbers a, b such that $a \leqq Y_n \leqq b$ for all n; more general cases are treated in the references cited above.

First note that if G denotes the common distribution function of the rv's $Y_1, Y_2, \cdots$,

$$\begin{aligned} Z_n - E(Z_{n+1} \mid F_n) &= E(-Z_{n+1} + Z_n \mid F_n) = E(M_{n+1} - M_n \mid F_n) - c \\ &= \int (y - M_n)^+ \, dG(y) - c \text{ a.e.} \end{aligned}$$

If α is now defined by $\int (y - \alpha)^+ \, dG(y) = c$, then

(a) $Z_n \leqq E(Z_{n+1} \mid F_n)$ a.e. on $A_n = \{M_n \geqq \alpha\}$,

(b) $Z_n > E(Z_{n+1} \mid F_n)$ a.e. on A_n'.

We shall show that this sequence $\{A_n, n \geqq 1\}$ satisfies the conditions of Theorem 3.2. Condition (ii) clearly holds by (b). To see that condition (i) also holds, observe that since $A_n \subset A_{n+j}$ for all $j \geqq 0$ the collection $\{I_{A_n}Z_{n+j}, F_{n+j}, j \geqq 0\}$ is a submartingale by (a). Since $Z_n \geqq Z_m - (b - a)$ whenever $n \geqq m$, this submartingale is regular by Corollary A.1(b) of [12] so that for any t in T_n,

$$I_{A_n}Z_n \leqq I_{A_n}E(Z_t \mid F_n) \text{ a.e.,}$$

and this implies condition (i). By Theorem 3.2 t^* can now be described as the first integer j such that $M_j \geqq \alpha$ or, equivalently, as the first integer j such that $Y_j \geqq \alpha$. Since $P(Y_n > \alpha) > 0$ for each n by the definition of α, it is clear that $t^* < \infty$ a.e. Therefore, t^* is a minimizing sv by Theorem 3.1.

For the second game as determined by $\{W_n, F_n, n \geqq 1\}$, since $W_n \geqq Z_n$ for each n, we have that for t^* defined as above and for any other sv t,

$$EW_t \geqq EZ_t \geqq EZ_{t^*} = EW_{t^*}.$$

Hence, t^* is also a minimizing sv for $\{W_n, F_n, n \geqq 1\}$; i.e., an optimal procedure again is to stop the first time that $Y_j \geqq \alpha$ for some j.

As a second illustration, consider the problem of testing the hypotheses $H_0:p_0$, $H_1:p_1$ where $Y_1, Y_2, \cdots$ are independent, identically distributed rv's, each having density p_0 or p_1 with respect to a σ-finite measure on the line. If there is an *a priori* probability that H_0 is true and if the cost per observation is constant, the sets A_n of Theorem 3.2 can be taken as complements of sets of the form

$$\{a < \textstyle\prod_{i=1}^{n} [p_1(Y_i)/p_0(Y_i)] < b\}$$

for suitable choices of constants a, b. This will be proved in a more general context in Section 5. The resulting sv is clearly of the type used in the sequential probability ratio test.

4. Extension to the design case. In the above "nondesign case" we were given a stochastic process $\{Z_a, F_a, a \varepsilon A\}$ where A is the set of integers. The following type of problem, which occurs in sequential experimental design, motivates the extension of the above theory to the "design case" where A is a partially ordered set of a particular type, sometimes called a *tree*. The problem can be posed as a one-person sequential game in which, if stopping occurs at stage a, the loss is the value of a rv Z_a, as in the nondesign case. However, one has the option of continuing the game by observing one of finitely many rv's which are the "successors" of Z_a, say $Z_{a1}, Z_{a2}, \cdots, Z_{am}$, where the choice may depend on all information available at that stage. If Z_{ai} is chosen (bringing the game to stage ai) and the observed value is z_{ai}, the player may stop the game and accept the loss z_{ai} or observe one of the successors of Z_{ai}. The over-all goal is to minimize the expected loss. A model to cover this situation will now be given; however, for notational convenience it is assumed that there are m successors available at each stage.

The following notation will be used:

M : set of integers $\{1, 2, \cdots, m\}$,

A : class of finite sequences $(a_1, a_2, \cdots, a_j)$ of elements from M including the "sequence" having no components, which will be denoted by $\varnothing$,

A^* : class of infinite sequences $(a_1, a_2, \cdots)$ of elements from M.

A partial order $\leqq$ is defined on A as follows: If $a = (a_1, a_2, \cdots, a_j)$ and $b = (b_1, b_2, \cdots, b_k)$ are elements of A, then $a \leqq b$ if $j \leqq k$ and $a_i = b_i$ for all $i \leqq j$. This is then extended in the obvious way to a partial order $\leqq$ on $A \cup A^*$. If $a = (a_1, a_2, \cdots, a_j) \varepsilon A$ and $k \varepsilon M$, we shall let ak denote the element $(a_1, a_2, \cdots, a_j, k)$; similarly, if $b = (b_1, b_2, \cdots) \varepsilon A \cup A^*$, ab will denote $(a_1, \cdots, a_j, b_1, b_2, \cdots)$.

It is assumed throughout that $\{Z_a, F_a, a \varepsilon A\}$ is an integrable stochastic process on a probability space (Ω, F, P) where $(A, \leqq)$ is defined as above. We also assume that $Z_a \geqq 0$ a.e. for each a in A, although the methods of proof below also apply with minor changes if there is an integrable rv U such that $Z_a \geqq U$ a.e. for each a.

DEFINITION 4.1. Let t be a function defined on Ω with values in $A \cup A^*$. (The meanings of $t(\omega) \geqq a$ and $t(\omega) > a$ are then clear from the partial order on $A \cup A^*$.) Then t is a control variable (cv) if the following conditions hold:

(i) $t \varepsilon A$ a.e.

(ii) for each a in A, the sets $\{t = a\}$ and $\{t \geqq ak\}$ are elements of F_a for each k in M.

We shall let $T_a(T_{a+})$ denote the class of cv's t such that $t \geqq a (t > a)$.

The interpretation of a cv t in the sequential game above is that if $t(\omega) = a$ where $a = (a_1, a_2, \cdots, a_j)$, then the rv's $Z_\varnothing, Z_{(a_1)}, Z_{(a_1, a_2)}, \cdots, Z_{(a_1, c_2, \cdots, a_j)} = Z_a$ are observed in that order, the game stops after the observation of Z_a, and the loss to the player is the value of Z_a. Condition (i) above is a requirement that the game stop sometime. An interpretation of (ii) is that at any stage of the game both the decision to stop and the choice of the next experiment to perform must depend only on information available at that stage. The problem is to find a cv t to minimize EZ_t where Z_t is defined below.

DEFINITION 4.2. For any cv t, Z_t is the rv defined by

$$\begin{aligned} Z_t(\omega) &= Z_a(\omega), \quad \text{if} \quad t(\omega) = a \quad \text{where} \quad a \varepsilon A, \\ &= \infty, \qquad \text{if} \quad t(\omega) \varepsilon A^*. \end{aligned}$$

We shall let $\{X_a, a \varepsilon A\}$ be the collection defined by

$$X_a = \text{ess inf}_{t \varepsilon T_a} E(Z_t \mid F_a).$$

As in the corresponding situation for the nondesign case (see the second paragraph of Section 3), we can assume that X_a is F_a-measurable for each a in A. Also, it is easily verified that $0 \leqq X_a \leqq Z_a$ a.e., so that $\{X_a, F_a, a \varepsilon A\}$ is a nonnegative integrable stochastic process. That it is, in fact, a submartingale will follow from Lemma 4.1(c) below.

LEMMA 4.1. *For any a in A,*

(a) $X_a = \min [Z_a, \text{ess inf}_{t \varepsilon T_{a+}} E(Z_t \mid F_a)]$ a.e.,

(b) $\text{ess inf}_{t \varepsilon T_{aj}} E(Z_t \mid F_a) = E(X_{aj} \mid F_a)$ a.e.,

(c) $X_a = \min [Z_a, \min_{j \varepsilon M} E(X_{aj} \mid F_a)]$ a.e.

PROOF. (a) Let $V = \text{ess inf}_{t \varepsilon T_{a+}} E(Z_t \mid F_a)$. The proof that $X_a \leqq \min (Z_a, V)$ a.e. parallels the proof for the nondesign case (Lemma 3.1). To show that also $X_a \geqq \min (Z_a, V)$ a.e., let s be any cv in T_a and define s' in T_{a+} as follows:

$$
\begin{aligned}
s'(\omega) &= s(\omega), \quad \text{if} \quad s(\omega) > a, \\
&= a1, \qquad \text{if} \quad s(\omega) = a.
\end{aligned}
$$

Then proceed as in the earlier case.

(b), (c) Since $T_{a+} = \bigcup_{j=1}^{m} T_{aj}$, we have from Lemma 2.1(c) that $V = \min_{j \varepsilon M} V_j$ a.e. where $V_j = \text{ess inf}_{t \varepsilon T_{aj}} E(Z_t \mid F_a)$. Hence, (c) follows immediately from (a) and (b). The proof of (b) parallels the proof of Lemma 3.1(b) and is omitted.

In the theorem below a cv t^* will be defined corresponding to the following strategy in the game described above. At any stage a beginning with $a = \varnothing$, if $X_a(\omega) = Z_a(\omega)$, the game is stopped (i.e., $t^*(\omega) = a$) and the loss to the player is $Z_a(\omega)$. If $X_a(\omega) < Z_a(\omega)$, the player continues the game by observing the value of Z_{ak} provided that $\omega \varepsilon B(ak)$ where

$$
\begin{aligned}
B(ak) = \{E(X_{ak} \mid F_a) &= \min_{j \varepsilon M} E(X_{aj} \mid F_a), \\
&E(X_{ai} \mid F_a) > \min_{j \varepsilon M} E(X_{aj} \mid F_a) \text{ for } 1 \leqq i < k\};
\end{aligned}
$$

by Lemma 4.1(c) this is equivalent to observing Z_{ak} next if k is the first integer such that $X_a(\omega) = E(X_{ak} \mid F_a)(\omega)$. This brings the game to stage ak and the procedure begins anew, i.e., stopping occurs if $X_{ak}(\omega) = Z_{ak}(\omega)$, etc. In proving the optimality of this strategy (assuming that $t^* \ \varepsilon\ A$ a.e.), the following notation will be convenient: If $a = (a_1, a_2, \cdots, a_j) \varepsilon A$ and $i \leqq j$, or if $a = (a_1, a_2, \cdots) \varepsilon A^*$, a^i will denote the element $(a_1, a_2, \cdots, a_i)$.

THEOREM 4.1. *Let $\{B(a), a \varepsilon A, a \neq \varnothing\}$ be the class of sets defined above, and let t^* be defined as follows:*

$$
\begin{aligned}
t^*(\omega) = a = (a_1, a_2, \cdots, a_j), \quad &\textit{if} \quad \omega \varepsilon B(a^{i+1}) \quad \textit{and} \quad X_{a^i}(\omega) < Z_{a^i}(\omega) \\
&\textit{for} \quad 0 \leqq i < j, \quad \textit{and} \quad X_a(\omega) = Z_a(\omega); \\
= (a_1, a_2, \cdots), \quad &\textit{if} \quad \omega \varepsilon B(a^{i+1}) \quad \textit{and} \quad X_{a^i}(\omega) < Z_{a^i}(\omega) \quad \textit{for all } i.
\end{aligned}
$$

If $t^* \varepsilon A$ a.e., *then t^* is a minimizing* cv, i.e., $EZ_{t^*} = \inf_{t \varepsilon T_\varnothing} EZ_t$. *Also,* $EZ_{t^*} = EX_\varnothing$.

PROOF. As in Theorem 3.1(a) in the nondesign case it suffices to show that $EZ_{t^*} \leqq EX_\varnothing$. Let A_n denote the set of those elements in A having exactly n components, e.g., $A_0 = \{\varnothing\}$. We shall write $t^*(\omega) < A_n$ below if $t^*(\omega) < a$ for some element a in A_n. First, we shall prove by induction that for each $n \geqq 0$,

$$X_\varnothing = E(I_{\{t^* < A_n\}} Z_{t^*} + \textstyle\sum_{a \varepsilon A_n} I_{\{t^* \geqq a\}} X_a \mid F_\varnothing) \text{ a.e.}$$

This clearly holds for $n = 0$. If it holds for some integer n, it also holds for $n + 1$ because, by the definition of t^* and Lemma 4.1(c),

$$\begin{aligned} E(\textstyle\sum_{a\varepsilon A_n} I_{\{t^* \geqq a\}} X_a \mid F_\varnothing) &\\ &= E(\textstyle\sum_{a \varepsilon A_n} I_{\{t^* = a\}} Z_a + \sum_{a \varepsilon A_n} \sum_{j \varepsilon M} I_{\{t^* \geqq aj\}} E(X_{aj} \mid F_a) \mid F_\varnothing) \\ &= E(I_{\{t^* \varepsilon A_n\}} Z_{t^*} + \textstyle\sum_{aj \varepsilon A_{n+1}} E(I_{\{t^* \geqq aj\}} X_{aj} \mid F_a) \mid F_\varnothing) \\ &= E(I_{\{t^* \varepsilon A_n\}} Z_{t^*} + \textstyle\sum_{b \varepsilon A_{n+1}} I_{\{t^* \geqq b\}} X_b \mid F_\varnothing) \text{ a.e.} \end{aligned}$$

The rest of the proof proceeds as in Theorem 3.1(a).

Note that the counterpart of Theorem 3.1(b) has not been stated as part of the preceding theorem. The difficulty here is that t^* may "choose the wrong fork in the road" as the following example illustrates. Let $F_a = F = \{\varnothing, \Omega\}$ for all a in A where A is such that $m \geqq 2$. Define $Z_{(m)}$ to be the constant function 0 on Ω; otherwise, define Z_a to be identically equal to $1/(n + 1)$ if $a \varepsilon A_n$. Then t^* always has the value $(1, 1, 1, \cdots)$ and is not a cv, but the cv t' identically equal to (m) is a minimizing cv.

THEOREM 4.2. *Suppose that for each a in A, there is a set D_a in F_a such that*

(i) $Z_a \leqq E(Z_t \mid F_a)$ a.e. *on D_a for each t in T_{a+}, and*

(ii) $Z_a >$ ess $\inf_{t \varepsilon T_{a+}} E(Z_t \mid F_a)$ a.e. *on D_a'.*

Then the cv t^* *defined in Theorem* 4.1 *satisfies the following condition for almost all points ω: If $t^*(\omega) \geqq a$, then*

$$\begin{aligned} t^*(\omega) &= a, \quad \textit{if} \quad \omega \varepsilon D_a, \\ &> a, \quad \textit{if} \quad \omega \varepsilon D_a'. \end{aligned}$$

PROOF. The proof follows from Lemma 4.1 in the same way as the corresponding proof in the nondesign case (Theorem 3.2).

Suppose that the function t^* defined in Theorem 4.1 does not satisfy the condition that $t^* \varepsilon A$ a.e. and therefore is not a cv. Just as the process $\{X_n, F_n, n \geqq 1\}$ can be used to find "ϵ-good" sv's in the nondesign case (see [12]), we would like to state that the process $\{X_a, F_a, a \varepsilon A\}$ can be used to obtain ϵ-good cv's by choosing experiments at each stage as in Theorem 4.1 but stopping the first time that $X_a(\omega) \geqq Z_a(\omega) - \epsilon$. The difficulty is that our procedure for breaking ties in choosing the next experiment to perform may lead to observing a sequence of Z_a's which never get close to the corresponding X_a's. For example, suppose that

(i) $F_a = \{\varnothing, \Omega\}$ for all a in A,

(ii) $Z_a = 1$ if $a = \varnothing$ or if a is of the form $(1, 1, \cdots, 1)$,

(iii) $Z_a = 0$ for all other elements a in A.

Then $X_a = 0$ for all a in A, but $t^* = (1, 1, 1, \cdots)$. Therefore, along the path of experimentation dictated by using t^*, $X_a = 0$ but $Z_a = 1$.

The next theorem will show that if $\{Z_a, F_a, a \varepsilon A\}$ satisfies a further hypothesis which is reasonable in statistical applications (and will be illustrated in the next section), then t^* satisfies the condition that $t^* \varepsilon A$ a.e. and therefore is a minizing cv.

THEOREM 4.3. *Suppose that the stochastic process* $\{Z_a, F_a, a \varepsilon A\}$ *satisfies the condition that* $Z_a \geqq r_n$ a.e. *for each* a *in* A_n *where* $\{r_n, n \geqq 0\}$ *is a sequence of real numbers such that* $\lim_n r_n = \infty$. *Then the function* t^* *defined in Theorem* 4.1 *is a* cv, i.e., $t^* \varepsilon A$ a.e.

PROOF. For any nonnegative integer n consider the set of elements a in A such that $a \leqq A_n$, i.e., the set of those elements in A which have n or fewer components. Note that a partition of Ω is determined by sets of the form $\{t^* = a\}$ for $a < A_n$ and $\{t^* \geqq a\}$ for $a \varepsilon A_n$. Let F_n denote the class of all sets of the form $\bigcup_{a \leqq A_n} H_a$ where $H_a \varepsilon F_a$, $H_a \subset \{t^* = a\}$ if $a < A_n$, and $H_a \subset \{t^* \geqq a\}$ if $a \varepsilon A_n$. Then it is easily checked that F_n is a σ-field for each $n \geqq 0$ and that $\{F_n, n \geqq 0\}$ is an increasing sequence of σ-fields.

Next, let $\{V_n, n \geqq 0\}$ be the sequence of rv's defined by

$$V_n = I_{\{t^* < A_n\}} X_{t^*} + \textstyle\sum_{a \varepsilon A_n} I_{\{t^* \geqq a\}} X_a .$$

We claim that $\{V_n, F_n, n \geqq 0\}$ is a martingale. To see that V_n is F_n-measurable, note that V_n can be written as a finite sum of terms of the form $I_{\{t^*=a\}}X_a$ and $I_{\{t^* \geqq a\}}X_a$ where $a \leqq A_n$. Since each such term is F_n-measurable and integrable, so is V_n. By Lemma 4.1(c) and the definition of t^*, $X_a = E(X_{aj} \mid F_a)$ a.e. on the set $\{t^* \geqq aj\}$. It follows that if $a \varepsilon A_n$, so that the class of F_n-subsets of $\{t^* \geqq aj\}$ are also F_a-subsets,

$$I_{\{t^* \geqq aj\}} X_a = I_{\{t^* \geqq aj\}} E(X_{aj} \mid F_a) = I_{\{t^* \geqq aj\}} E(X_{aj} \mid F_n) \text{ a.e.}$$

The martingale equality can now be verified as follows:

$$\begin{aligned} E(V_{n+1} \mid F_n) &= E(I_{\{t^* < A_{n+1}\}} X_{t^*} + \textstyle\sum_{a \varepsilon A_n} \sum_{j \varepsilon M} I_{\{t^* \geqq aj\}} X_{aj} \mid F_n) \\ &= I_{\{t^* < A_{n+1}\}} X_{t^*} + \textstyle\sum_{a \varepsilon A_n} \sum_{j \varepsilon M} I_{\{t^* \geqq aj\}} E(X_{aj} \mid F_n) \\ &= I_{\{t^* < A_n\}} X_{t^*} + \textstyle\sum_{a \varepsilon A_n} I_{\{t^* = a\}} X_a + \sum_{a \varepsilon A_n} \sum_{j \varepsilon M} I_{\{t^* \geqq aj\}} X_a \\ &= I_{\{t^* < A_n\}} X_{t^*} + \textstyle\sum_{a \varepsilon A_n} I_{\{t^* \geqq a\}} X_a \\ &= V_n \text{ a.e.} \end{aligned}$$

Contrary to the statement of the theorem, suppose that $t^* \not\varepsilon A$ on a set Q of positive probability. For each a in A_n we have that

$$X_a = \text{ess inf}_{t \varepsilon T_a} E(Z_t \mid F_a) \geqq \text{ess inf}_{t \varepsilon T_a} E(\inf_{j \geqq n} r_j \mid F_a) = \inf_{j \geqq n} r_j \text{ a.e.}$$

This implies that $V_n \geqq \inf_{j \geqq n} r_j$ a.e. on Q so that $\lim_n V_n = \infty$ a.e. on Q. However, since $\{V_n, F_n, n \geqq 0\}$ is a nonnegative martingale, $\lim_n V_n$ exists and is finite a.e. (see [7], p. 319). This leads us to a contradiction, thus completing the proof.

To obtain a generalization of the Arrow-Blackwell-Girshick theory to include the design case, suppose that the problem is truncated to N stages, i.e., consider the finite collection $\{Z_a, F_a, a \leqq A_n\}$ where $\{Z_a, F_a, a \varepsilon A\}$ is the given stochastic process and $a \leqq A_N$ again means that $a \varepsilon A_k$ for some $k \leqq N$. The theory above clearly applies to this case after a few notational changes. If $\{X_a^N, a \leqq A_N\}$ is the collection defined by $X_a^N = \text{ess inf}_{t\varepsilon T_a} E(Z_t \mid F_a)$ where T_a is the class of cv's t such that $a \leqq t \leqq A_N$, we clearly have that

$$(1) \qquad X_a^N = Z_a \text{ a.e. if } a \varepsilon A_N,$$

and by Lemma 4.1(c),

$$(2) \qquad X_a^N = \min [Z_a, \min_{j\varepsilon M} E(X_{aj}^N \mid F_a)] \text{ a.e. if } a < A_N.$$

Hence, the collection $\{X_a^N, a \leqq A_N\}$ can be computed directly from the process $\{Z_a, a \leqq A_N\}$ by recursion backwards, first using (1), then using (2) to determine X_a^N for a in A_{N-1}, etc. A minimizing cv for this case is supplied by Theorem 4.1.

Let the collections $\{W_a^N, a \leqq A_N\}$ be defined for each $N \geqq 0$ as follows (cf. (1) and (2) above):

$$W_a^N = Z_a, \quad \text{if} \quad a \varepsilon A_N,$$

$$W_a^{N'} = \min [Z_a, \min_{j\varepsilon M} E(W_{aj}^N \mid F_a)] \quad \text{if} \quad a < A_N.$$

If we extend the definition of maximal submartingale (in Section 2) in the obvious way, it is easy to see that $\{W_a^N, F_a, a \leqq A_N\}$ is the maximal submartingale relative to $\{Z_a, F_a, a \leqq A_N\}$. By following the corresponding discussion for the nondesign case, we can show that $W_a^N, W_a^{N+1}, \cdots$ is a nonincreasing sequence for a in A_N, and if we set $W_a = \lim_N W_a^N$, the resulting process $\{W_a, F_a, a \varepsilon A\}$ is the maximal submartingale relative to $\{Z_a, F_a, a \varepsilon A\}$.

One might wonder when the maximal submartingale $\{W_a, F_a, a \varepsilon A\}$ can be substituted for the process $\{X_a, F_a, a \varepsilon A\}$ in Theorem 4.1 to characterize minimizing cv's. In a sense, by the characterization of minimizing cv's above for the truncated case, this asks the question, "When is the limit of the best truncated procedures an optimal procedure for the nontruncated case?" The example given for the corresponding situation in the nondesign case can be extended to show that the maximal submartingale cannot always be used and that some additional condition is needed. The next theorem provides a partial answer to the question by showing that the stochastic processes $\{W_a, F_a, a \varepsilon A\}$ and $\{X_a, F_a, a \varepsilon A\}$ coincide if a condition is met which holds in statistical applications and will be illustrated in the next section.

THEOREM 4.4. *If* $\{W_a, F_a, a \varepsilon A\}$ *is the maximal submartingale relative to* $\{Z_a, F_a, a \varepsilon A\}$ *and if there is a nonnegative integrable* rv V *such that* $W_b \geqq W_a - V$ a.e. *whenever* $a \leqq b$, *then* $W_a = X_a$ a.e. *for each* a *in* A *where*

$$X_a = \text{ess inf}_{t\varepsilon T_a} E(Z_t \mid F_a).$$

PROOF. Since $\{W_a, F_a, a \varepsilon A\}$ is the maximal submartingale relative to

$\{Z_a, F_a, a \varepsilon A\}$ and $\{X_a, F_a, a \varepsilon A\}$ is another submartingale such that $X_a \leqq Z_a$ a.e. for each a, it follows that $X_a \leqq W_a$ a.e. for each a. To prove that the inequality $W_a \leqq X_a$ also holds a.e., we shall show that $W_a \leqq E(W_t \mid F_a)$ a.e. for any cv t in T_a. Since $W_t \leqq Z_t$ a.e., this will give us that $W_a \leqq E(Z_t \mid F_a)$ a.e. for any t in T_a, which implies that $W_a \leqq X_a$ a.e. This will be proved below for the case $a = \varnothing$, but the proof for any other a in A follows by a change in notation.

Let $\{F_n, n \geqq 0\}$ be the sequence of σ-fields defined as in the proof of Theorem 4.3 by replacing each occurrence of t^* by t; also, let $\{U_n, n \geqq 0\}$ be the sequence of rv's defined by

$$U_n = I_{\{t<A_n\}}W_t + \textstyle\sum_{a\varepsilon A_n} I_{\{t\geqq a\}}W_a .$$

We claim that $\{U_n, F_n, n \geqq 0\}$ is a submartingale. The proof parallels the proof in Theorem 4.3 showing that $\{V_n, F_n, n \geqq 0\}$ is a martingale except that instead of having $X_a = E(X_{aj} \mid F_a)$ a.e. on $\{t^* \geqq aj\}$, we now have $W_a \leqq E(W_{aj} \mid F_a)$ a.e. on $\{t \geqq aj\}$. Another property of the sequence $\{U_n, n \geqq 0\}$ is that $U_{n+k} \geqq U_n - V$ a.e. for each $k \geqq 0$; this follows easily from the condition that $W_b \geqq W_a - V$ a.e. whenever $a \leqq b$. Note that the cv t determines a sv t' for $\{U_n, F_n, n \geqq 0\}$ if we set $t'(\omega) = n$ if $t(\omega) = a$ where $a \varepsilon A_n$. Applying Corollary A.1(b) of [12] to $\{U_n, F_n, n \geqq 0\}$, we obtain that $U_0 \leqq E(U_{t'} \mid F_0)$ a.e. Since $U_0 = W_\varnothing$, $F_0 = F_\varnothing$, and $U_{t'} = W_t$, this implies that $W_\varnothing \leqq E(W_t \mid F_\varnothing)$ a.e., thus completing the proof.

5. Application to sequential experimental design. A formulation of the general sequential decision problem to include the design of experimentation was given by Wald in [14]. After developing a general theory for sequential experimental design, he considered the hypothesis testing case and gave a partial characterization of the Bayes solution. Later, Magwire gave a slightly different formulation for the general case in [10] and extended some of Wald's results on Bayes solutions for the sequential (nondesign) case to the general (design) case. Unfortunately, these results provided little help in characterizing good procedures for a specific problem, and an asymptotic theory introduced by Chernoff in [3] has now arisen which provides workable solutions for some cases. We shall now reconsider the problem of finding Bayes solutions for sequential design problems using the techniques of the previous section.

First, consider the problem of testing the hypotheses $H_0 : \theta = 0$ versus H_1: $\theta = 1$, given that the loss if H_i is true and rejected is w_i where $w_i > 0$ for $i = 0, 1$. Observations can be made sequentially on the rv's in the following matrix:

$$\begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1k} & \cdots \\ Y_{21} & Y_{22} & \cdots & Y_{2k} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Y_{m1} & Y_{m2} & \cdots & Y_{mk} & \cdots \end{bmatrix}$$

The situation we have in mind is that there are m random experiments E_1, E_2,

$\cdots$, E_m available at each stage, and the rv's in the jth row correspond to independent replications of the experiment E_j. In particular, if experiment E_j is performed at the kth stage, this will correspond in our model to observing the rv Y_{jk}, so that rv's in the kth column correspond to possible observations at the kth stage. Accordingly, we assume that the column vectors in the matrix are independent under each of the hypotheses and that rv's in the same row are identically distributed. Also, we assume that there is a σ-finite measure μ on the line such that under H_i each rv Y_{jk} has a Borel measurable density with respect to μ which we shall denote by $p_i(\cdot, j)$. Let the cost of performing experiment E_j be c_j where $c_j > 0$ for $j = 1, 2, \cdots, m$.

If we now suppose that there is an *a priori* probability λ that H_0 is true, we can imbed the problem into a larger probability space (Ω, F, P) and let θ be a rv on this space which takes on the values 0 and 1 with probabilities λ and $1 - \lambda$ respectively. In this framework our assumptions on the rv's Y_{jk} become conditional statements given θ, e.g., the rv's in the jth row are now assumed to be conditionally independent given θ, each having conditional density $p_i(\cdot, j)$ given $\theta = i$ with respect to μ.

If $a = (a_1, a_2, \cdots, a_j)$ is an element of A where $(A, \leqq)$ is the partially ordered set defined in the previous section, let Y^a denote the random vector $(Y_{1a_1}, Y_{2a_2}, \cdots, Y_{ja_j})$. If $a \neq \varnothing$, let F_a be the σ-field generated by Y^a; if $a = \varnothing$, define $F_\varnothing = \{\varnothing, \Omega\}$. Then $\{F_a, a \varepsilon A\}$ is a collection of σ-fields such that $F_a \subset F_b$ whenever $a \leqq b$.

To obtain the Bayes terminal procedure, suppose that the sequence Y^a has been observed where $a = (a_1, a_2, \cdots, a_j)$. Let $p_{\lambda a}$ denote the joint density of Y^a with respect to $\mu^j = \mu \times \mu \times \cdots \times \mu$ on j-space defined by

$$p_{\lambda a}(y^a) = \lambda p_{0a}(y^a) + (1 - \lambda)p_{1a}(y^a), \tag{1}$$

where

$$p_{ia}(y^a) = p_i(y_{1a_1}, a_1)p_i(y_{2a_2}, a_2) \cdots p_i(y_{ja_j}, a_j) \quad \text{for} \quad i = 0, 1.$$

Also, let λ_a denote the version of $P(\theta = 0 \mid F_a)$ given by

$$\begin{aligned} \lambda_a &= \lambda p_{0a}(Y^a)/p_{\lambda a}(Y^a), \quad \text{if} \quad p_{\lambda a}(Y^a) \neq 0, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{2}$$

Since the conditional risks given Y^a of rejecting H_0 and of rejecting H_1 are $w_0\lambda_a + c_a$ and $w_1(1 - \lambda_a) + c_a$ respectively where $c_a = \sum_{i=1}^{j} c_{a_i}$, a Bayes terminal decision rule is to reject H_0 if stopping occurs at this stage on the subset of Ω for which $w_0\lambda_a \leqq w_1(1 - \lambda_a)$. If $h(\lambda_a)$ is the rv defined by $h(\lambda_a) = \min[w_0\lambda_a, w_1(1 - \lambda_a)]$, this is equivalent to rejecting H_0 on the set $\{h(\lambda_a) = w_0\lambda_a\}$. This rule also holds for the initial stage $a = \varnothing$ if we interpret $\lambda_\varnothing$ as the rv having constant value λ.

Let $\{Z_a, a \varepsilon A\}$ be the collection of rv's defined by setting $Z_a = h(\lambda_a) + c_a$. Then Z_a is the conditional risk given Y^a when the Bayes terminal procedure is used. Since Z_a is F_a-measurable, integrable, and nonnegative, the stochastic process $\{Z_a, F_a, a \varepsilon A\}$ satisfies the assumptions of the previous section. Also, the problem of finding a Bayes stopping rule and plan of experimentation coincides with the problem of finding a minimizing cv for $\{Z_a, F_a, a \varepsilon A\}$.

More general Bayesian decision problems can clearly be treated in the same way whenever Bayes terminal procedures exist. Assuming a bounded loss function, we would again have $Z_a = R_a + c_a$ where $0 \leqq R_a \leqq K$ and c_a is defined as above or, more generally, $\{c_a, a \varepsilon A\}$ is a collection of rv's such that c_a is F_a-measurable, integrable, and $c_{ak} \geqq c_a$ a.e. whenever $a \varepsilon A$ and $k \varepsilon M$. At any rate Theorem 4.4 applies since $\{c_a, F_a, a \varepsilon A\}$ is a submartingale such that $c_a \leqq Z_a$ a.e., and it follows that the maximal submartingale $\{W_a, F_a, a \varepsilon A\}$ relative to $\{Z_a, F_a, a \varepsilon A\}$ satisfies the condition that whenever $a \leqq b$, $W_b \geqq c_b \geqq c_a \geqq W_a - K$ a.e. Therefore, the Bayes procedure (if it exists) can be characterized as a limit of the Bayes truncated procedures. If the "experimentation costs" c_a are such that $c_a \geqq r_n$ a.e. for each a in A_n where $\{r_n, n \geqq 0\}$ is a sequence of real numbers such that $\lim_n r_n = \infty$, a Bayes procedure exists by Theorem 4.3 and can be characterized by the cv t^* in Theorem 4.1. In particular, this holds for the hypothesis testing problem above if we set $r_n = n \min_{k \varepsilon M} c_k$.

To characterize the Bayes solution further for the hypothesis testing problem we shall show that the conditions of Theorem 4.2 hold for the family of sets $\{D_a, a \varepsilon A\}$ defined by

$$D_a = \{\lambda_a \leqq \pi_1 \quad \text{or} \quad \lambda_a \geqq \pi_2\}$$

for a choice of constants π_1, π_2 in [0, 1] to be specified below. This in turn will yield a stopping rule of the probability ratio test type since the condition that $\pi_1 < \lambda_a(\omega) < \pi_2$ at a point ω for which $Y^a(\omega) = y^a$ is equivalent to the condition that

$$[\lambda/(1 - \lambda)]\cdot(1 - \pi_2)/\pi_2 < p_{1a}(y^a)/p_{0a}(y^a) < [\lambda/(1 - \lambda)]\cdot(1 - \pi_1)/\pi_1.$$

As in Section 4, $T_a(T_{a+})$ will denote the class of cv's t such that $t \geqq a$ $(t > a)$; for $a = \varnothing$ the symbols T and T_+ will be used. These classes, as well as the process $\{Z_a, a \varepsilon A\}$ itself, depend on the value of λ (assumed fixed up to this point). To exhibit this dependence, the notation $T(\lambda)$, for example, will also be used below. Then when we refer to a cv t in $T(\lambda_0)$, it is to be understood as relative to the process $\{Z_a, a \varepsilon A\}$ corresponding to the *a priori* probability λ_0.

Suppose $t \varepsilon T(\lambda_0)$ for some λ_0. Then t corresponds to a decision procedure which has a risk function, considered as a function of λ, of the form

$$r_t(\lambda) = \lambda[\alpha_0 w_0 + C_0] + (1 - \lambda)[\alpha_1 w_1 + C_1] \tag{3}$$

where $\alpha_i = P(\text{reject } H_i \mid \theta = i)$ and C_i is the expected cost of experimentation under t when $\theta = i$. Let $\rho = \rho(\lambda)$ be defined by

$$\rho(\lambda) = \inf r_t(\lambda),$$

the infimum being over all the classes $T(\lambda')$ for all λ'. Since each r_t is nonnegative and linear on [0, 1] by (3), ρ is a nonnegative concave function on [0, 1]. Moreover, for any fixed λ, a cv not in $T(\lambda)$ has risk at λ at least as great as the cv in $T(\lambda)$ which makes the same choice of observations at each stage and stops at the same stage [recall that cv's in $T(\lambda)$ correspond to decision procedures having Bayes (λ) terminal decision rules]. Therefore,

$$\rho(\lambda) = \inf_{t \varepsilon T(\lambda)} r_t(\lambda) = \text{ess inf}_{t \varepsilon T(\lambda)} E(Z_t \mid F_\varnothing).$$

Similarly, if ρ_+ is defined by $\rho_+(\lambda) = \inf r_t(\lambda)$ where now the infimum is over all classes $T_+(\lambda')$ for all λ', then ρ_+ is a positive concave function on [0, 1] and

$$\rho_+(\lambda) = \inf_{t \varepsilon T_+(\lambda)} r_t(\lambda) = \text{ess inf}_{t \varepsilon T_+(\lambda)} E(Z_t \mid F_\varnothing).$$

For a given λ the minimizing cv $t^* = \varnothing$ if and only if

$$Z_\varnothing \leqq \text{ess inf}_{t \varepsilon T_+} E(Z_t \mid F_\varnothing) = \rho_+(\lambda).$$

By considering the graph of $Z_\varnothing = h(\lambda) = \min [w_0\lambda, w_1(1 - \lambda)]$ as a function of λ on [0, 1] and that of the concave function $\rho_+(\lambda)$, we see that there exist real numbers π_1, π_2 in [0, 1] such that $t^* = \varnothing$ if and only if $\lambda \leqq \pi_1$ or $\lambda \geqq \pi_2$. We shall show below that for these choices of π_1 and π_2 the sets $\{D_a, a \varepsilon A\}$ defined above satisfy conditions (i) and (ii) of Theorem 4.2.

Now suppose $t \varepsilon T_{a+}$ for $a \neq \varnothing$. By an appropriate choice of the original sample space, the set $\{Y^a = y^a\}$ looks exactly like Ω itself (under the correspondence $y^{aa'} \leftrightarrow y^{a'}$ for $a' \varepsilon A^*$). Similarly, the restriction of t in $T_{a+}(\lambda)$ to $\{Y^a = y^a\}$ looks like a cv t' in $T_+(\lambda_a)$ if we define t' on $\{Y^a = y^a\}$ as follows: If $t(\omega) = aa'$ where $a' \varepsilon A \cup A^*$, then $t'(\omega) = a'$.

For any $b = (b_1, \cdots, b_n)$, set

$$(U_{b_1}, U_{b_1b_2}, \cdots, U_b) = (Y_{ab_1}, Y_{ab_1b_2}, \cdots, Y_{ab}).$$

Choose that version of $E(Z_t \mid F_a)$ for $t \varepsilon T_{a+}$ which is determined by the conditional density of U^b given $Y^a(\omega) = y^a$ defined by

$$\begin{aligned} (4) \qquad p_{\lambda_a,b}(u^b) &= [\lambda p_{0a}(y^a)p_{0b}(u^b) + (1 - \lambda)p_{1a}(y^a)p_{1b}(u^b)]/p_{\lambda a}(y^a) \\ &= \lambda_a p_{0b}(u^b) + (1 - \lambda_a)p_{1b}(u^b). \end{aligned}$$

For each b in A_n, let Q_b be the Borel set of points u^b in n-space such that for any ω in $\{Y^a = y^a\}$, $t(\omega) = ab$ if and only if $u^b \varepsilon Q_b$. Then

$$(5) \quad E(Z_t \mid Y^a = y^a) = \textstyle\sum_{n=1}^{\infty} \sum_{b \varepsilon A_n} \int_{Q_b} [h(\lambda_{ab}) + c_{ab}]p_{\lambda_a, b}(u^b)\, d\mu^n(u^b).$$

If R_{ib} is the subset of Q_b corresponding to the subset of $\{t = ab\}$ for which H_i is rejected, we have

$$\begin{aligned} &\textstyle\int_{Q_b} h(\lambda_{ab})p_{\lambda_a,b}(u^b)\, d\mu^n(u^b) \\ &\qquad = \textstyle\int_{R_{0b}} w_0\lambda_{ab}p_{\lambda_a,b}(u^b)\, d\mu^n(u^b) + \int_{R_{1b}} w_1(1 - \lambda_{ab})p_{\lambda_a,b}(u^b)\, d\mu^n(u^b), \end{aligned}$$

and, since

$$\lambda_{ab} = \lambda_a p_{0b}(u^b)/p_{\lambda_a,b}(u^b),$$

the right-hand member reduces to

$$w_0\lambda_a \int_{R_{0b}} p_{0b}(u^b)\, d\mu^n(u^b) + w_1(1-\lambda_a) \int_{R_{1b}} p_{1b}(u^b)\, d\mu^n(u^b).$$

Substituting this and (4) in (5) and using $c_{ab} = c_a + c_b$ gives

$$(6) \qquad E(Z_t \mid Y^a = y^a) = \lambda_a[\alpha_0 w_0 + C_0] + (1-\lambda_a)[\alpha_1 w_1 + C_1] + c_a,$$

where

$$\alpha_i = \sum_{n=1}^{\infty} \sum_{b \varepsilon A_n} \int_{R_{ib}} p_{ib}(u^b)\, d\mu^n(u^b),$$
$$C_i = \sum_{n=1}^{\infty} \sum_{b \varepsilon A_n} c_b \int_{Q_b} p_{ib}(u^b)\, d\mu^n(u^b).$$

By (3) and the correspondence between the cv t in $T_{a+}(\lambda)$ and a cv t' in $T_+(\lambda_a)$, (6) can be written $E(Z_t \mid Y^a = y^a) = r_{t'}(\lambda_a) + c_a$, which implies that $\inf_{t \varepsilon T_{a+}(\lambda)} E(Z_t \mid Y^a = y^a) = \rho_+(\lambda_a) + c_a$. Since ρ_+ is a concave function on [0, 1], it is Borel measurable on [0, 1], and this interval contains the range of the rv λ_a. Therefore, $\rho_+(\lambda_a) + c_a$ is a rv. We now claim that

$$\rho_+(\lambda_a) + c_a = \text{ess}\inf_{t \varepsilon T_{a+}} E(Z_t \mid F_a).$$

We already have that $\rho_+(\lambda_a) + c_a \leqq E(Z_t \mid F_a)$ a.e. for each t in T_{a+} by the discussion above, and it suffices to show that there is a cv s in T_{a+} such that

$$\rho_+(\lambda_a) + c_a = E(Z_s \mid F_a) \text{ a.e.}$$

This can be done by letting s be the cv which for every point ω treats the *a posteriori* probability $\lambda_a(\omega)$ as an *a priori* probability and proceeds in the same way as the minimizing cv in T_+ for that *a priori* probability would proceed.

We now return to consider the set $D_a = \{\lambda_a \leqq \pi_1 \text{ or } \lambda_a \geqq \pi_2\}$. By the earlier case for $a = \varnothing$, this can be rewritten as

$$D_a = \{h(\lambda_a) \leqq \rho_+(\lambda_a)\} = \{h(\lambda_a) + c_a \leqq \rho_+(\lambda_a) + c_a\}.$$

Since $Z_a = h(\lambda_a) + c_a$ and $\rho_+(\lambda_a) + c_a = \text{ess}\inf_{t \varepsilon T_{a+}} E(Z_t \mid F_a)$, we see the inequality $Z_a \leqq \text{ess}\inf_{t \varepsilon T_{a+}} E(Z_t \mid F_a)$ holds on D_a and fails to hold on D_a'. This completes the verification that the sets $\{D_a,\ a \varepsilon A\}$ satisfy the conditions of Theorem 4.2.

It remains to characterize the experimentation plan. Let $\rho_1, \rho_2, \cdots, \rho_m$ be the functions defined on [0, 1] by

$$\rho_j(\lambda) = \inf_{t \varepsilon T_{(j)}} r_t(\lambda) = \text{ess}\inf_{t \varepsilon T_{(j)}} E(Z_t \mid F_\varnothing).$$

By Lemma 4.1 (b),

$$E(X_{(j)} \mid F_\varnothing) = \text{ess}\inf_{t \varepsilon T_{(j)}} E(Z_t \mid F_\varnothing) = \rho_j(\lambda).$$

Therefore, since $\rho_+(\lambda) = \min_{j \varepsilon M} \rho_j(\lambda)$, the sets $B((k))$ of Theorem 4.1 can be written as follows:

$$B((k)) = \{\rho_+(\lambda) < \rho_j(\lambda) \quad \text{for} \quad 1 \leqq j < k,\ \rho_+(\lambda) = \rho_k(\lambda)\}.$$

Here, each $\rho_+(\lambda)$ and $\rho_j(\lambda)$ is to be regarded as a constant-valued rv for a given value of λ, and the set $B((k))$ is either the whole space Ω or $\emptyset$. By Theorem 4.1 the cv t^* is such that the first experiment to be performed would be the kth one if k is the first integer for which $\rho_+(\lambda) = \rho_k(\lambda)$. It will be helpful below to think of the graphs of the concave functions $\rho_+, \rho_1, \rho_2, \cdots, \rho_m$ on [0, 1]. If $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ are the subsets of (π_1, π_2) defined by

$$\Lambda_k = \{\lambda: \rho_+(\lambda) < \rho_j(\lambda) \quad \text{for} \quad 1 \leqq j < k, \rho_+(\lambda) = \rho_k(\lambda)\},$$

the kth experiment is performed first if and only if $\lambda \,\varepsilon\, \Lambda_k$.

By the proof above that

$$\rho_+(\lambda_a) + c_a = \text{ess inf}_{t\varepsilon T_{a+}} E(Z_t \mid F_a),$$

it is clear that for any j in M, we can also set

$$\rho_j(\lambda_a) + c_a = \text{ess inf}_{t\varepsilon T_{aj}} E(Z_t \mid F_a).$$

Since $E(X_{aj} \mid F_a) = \text{ess inf}_{t\varepsilon T_{aj}} E(Z_t \mid F_a)$ a.e. by Lemma 4.1 (b), the class of sets $\{B(a), a \,\varepsilon\, A, a \neq \emptyset\}$ of Theorem 4.1 are given by

$$B(ak) = \{\rho_+(\lambda_a) < \rho_j(\lambda_a) \quad \text{for} \quad 1 \leqq j < k, \rho_+(\lambda_a) = \rho_k(\lambda_a)\}.$$

Therefore, if the cv t^* is being used, the choice of experiment after the observation of Y^a depends on the *a posteriori* probability λ_a in the same way as the initial choice of experiment depends on the *a priori* probability λ. That is, if the sequence $Y^a(\omega) = y^a$ is observed and if $\pi_1 < \lambda_a(\omega) < \pi_2$ so that experimentation continues, the kth experiment is performed next if and only if $\lambda_a(\omega) \,\varepsilon\, \Lambda_k$.

Although this solves the problem in a theoretical sense, the formidable problem of determining the functions ρ_j for any particular case remains. However, since the numbers π_1, π_2 and the sets $\{\Lambda_k, k \,\varepsilon\, M\}$ completely characterize t^*, we might hope to estimate these by substituting other functions for the functions ρ_j which are more readily computable and which would appear to yield good estimates. For example, if S_j is the class of cv's in $T_{(j)}$ which use the jth experiment exclusively and Q_j is the class of cv's in $T_{(j)}$ which stop after at most two experiments, we might substitute the functions ρ_j' or ρ_j'' where $\rho_j'(\lambda) = \inf_{t\varepsilon S_j} r_t(\lambda)$ and $\rho_j'' = \inf_{t\varepsilon Q_j} r_t(\lambda)$. The thinking in using ρ_j' is that for a given λ the best choice of experiment for the first stage in a sequential design problem might very well coincide with or "be close to" the best choice for the case where the same experiment must be used throughout. Since $\rho_j'(\lambda) \geqq \rho_j(\lambda)$ on [0, 1] for each j, the resulting estimates of π_1 and π_2 will be too high for π_1 and too low for π_2. The computation of ρ_j' reduces to finding probabilities of wrong decisions and expected sample sizes for sequential probability ratio tests, and the Wald approximations are available for this.

As an illustration of the above procedure, let $W_1, W_2, \cdots$ be a sequence of independent rv's on (Ω, F, P), each having a normal $(\mu, 1)$ distribution, and suppose that a test of the hypotheses $H_0 : \mu = -1$, $H_1 : \mu = 1$ is desired where there is an *a priori* probability λ that H_0 is true and the losses for wrong decision are $w_0 = w_1 = 80$. At each stage the experimenter can choose one of the three "dosage levels" $-k$, 0, and k where $k > 0$, and the corresponding experiment at the nth stage consists of observing whether or not W_n exceeds the selected dosage level. Accordingly, for each n we define the rv Y_{1n} by

$$\begin{aligned} Y_{1n}(\omega) &= 1, \quad \text{if} \quad W_n(\omega) \geqq -k, \\ &= 0, \quad \text{if} \quad W_n(\omega) < -k; \end{aligned}$$

the rv's Y_{2n} and Y_{3n} are defined similarly for dosage levels 0 and k respectively.

Consider the family of sequential probability ratio tests for each of the sequences $\{Y_{in}, n \geqq 1\}$ where $i = 1, 2, 3$. The rv Y_{3n} corresponding to dosage level k has a binomial distribution with parameter $\Phi(-k - 1)$ under H_0 and $\Phi(-k + 1)$ under H_1 where Φ denotes the normal (0, 1) distribution function. In order that the exact theory can be used to compute probabilities of wrong decisions and expected sample sizes (see [13], pp. 190–191), we shall assume that k is so chosen that

$$\log [\Phi(-k + 1)/\Phi(-k - 1)] = -2 \log [\Phi(k - 1)/\Phi(k + 1)].$$

This guarantees that the resulting random walks for sequential probability ratio tests using the sequence $\{Y_{3n}, n \geqq 1\}$ are such that the step to the right at the nth stage for $Y_{3n}(\omega) = 1$ is twice the step to the left for $Y_{3n}(\omega) = 0$. (The approximate value of k is .4642.) From the graph of $h = h(\lambda) = \min [80\lambda, 80(1 - \lambda)]$ and the graphs of ρ_1', ρ_2', and ρ_3' for this case, the resulting estimates of π_1 and π_2 are approximately .0243 and .9757 respectively. From the estimates of the sets Λ_i, level k would be used at the initial stage if $.0796 < \lambda < .1628$, level $-k$ if $.8372 < \lambda < .9204$, and level 0 if $.0243 < \lambda < .0796$ or $.1628 < \lambda < .8372$ or $.9204 < \lambda < .9757$.

It is of interest to compare this procedure for choosing experiments with the plan suggested by Bradt and Karlin in [2]. They recommended choosing that experiment at the initial stage for which the weighted average (using the *a priori* probability λ) of the Kullback-Leibler information numbers is greatest. At any later stage the *a posteriori* probability λ_a is used in the same way as the *a priori* probability λ is used at the initial stage. In the example above, level k would be used under this plan at the initial stage if $\lambda \leqq .3535$, level $-k$ if $\lambda \geqq .6465$, and level 0 if $.3535 < \lambda < .6465$.

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