LINEAR COMBINATIONS OF NON-CENTRAL CHI-SQUARE VARIATES[1]

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1. Introduction. Let $\chi^2_{m,d}$ denote a non-central chi-square variate with m degrees of freedom and non-centrality parameter d, whose probability density function is given by

$$p(x) = [x^{(m-2)/2}/2^{m/2}(\pi)^{\frac{1}{2}}]e^{-(d+x)/2}\sum_{j=0}^{\infty}[(x\,d)^j\Gamma(j+\tfrac{1}{2})/(2j)!\,\Gamma(j+(m/2))],$$

for $x > 0$, and zero otherwise. Define

$$T = U - V, \tag{1.1}$$

$$U = \alpha[\chi^2_{m_0,d_0} + \textstyle\sum_{i=1}^{r} a_i\chi^2_{m_i,d_i}], \tag{1.2}$$

$$V = \beta[\chi^2_{n_0,g_0} + \textstyle\sum_{j=1}^{s} b_j\chi^2_{n_j,g_j}], \tag{1.3}$$

where $\alpha > 0$, $\beta > 0$, $a_i \geqq 1$, $b_j \geqq 1$, $d_i \geqq 0$, $d_0 \geqq 0$, $g_j \geqq 0$, $g_0 \geqq 0$, for all i and j, and all chi-square variates are independent. The problem considered in this paper is the determination of the distribution of T for fixed known values of the parameters. Clearly if $\sum_1^p \lambda_i w_i^2$ is an arbitrary quadratic form in the w_i's in which the constants λ_i are real numbers, and the w_i's are independent random variables each of which has a normal distribution with non-zero mean and unit variance, by a minor bookkeeping change in notation, the form can be given the representation of (1.1)–(1.3). Hence, we are equivalently concerned with indefinite quadratic forms in non-central normal variates. Note that U and V are each expressible as some positive definite quadratic form.

Development of the distribution for an indefinite quadratic form in non-central normal variates was motivated by a classification problem. Consider the problem of classifying an unknown vector observation into one of two multivariate normal populations which have unequal means and covariance matrices. It may be shown (see [5]) that likelihood ratio procedures for this problem lead to consideration of the distribution developed in this paper.

Section 2 will be devoted to the distribution of positive definite quadratic forms in non-central normal variates.

In Section 3, the probability density function of T is found in terms of the results developed in Section 2.

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Finally, in Section 4 the cdf of T is developed for purposes of percentage point computations.

2. Positive definite forms. Let $F_0(x)$, $F_1(x)$, $\cdots$ be any sequence of distribution functions. Let c_0, c_1, $\cdots$ be any sequence of non-negative constants such that $\sum_j c_j = 1$, where unless stated otherwise, it will be understood that all summations are to be taken from zero to infinity. Then, $F(x) = \sum c_j F_j(x)$ is called a mixture of distribution functions (see [6]).

Consider the positive definite quadratic form, U, given in (1.2). It was shown in [6] that when all the non-centrality parameters, d_i, vanish, the cdf of U is expressible as a mixture of cdf's of central chi-squared variates. They also found a related result for the ratio of such forms. In this section, both of those results are extended to the non-central case.

Shah and Khatri [10] have considered the non-central case and have developed a series expansion for the cdf of U in powers of x.

Imhof [3] has inverted the characteristic function of U and evaluated it numerically using the trapezoidal and Simpson rules.

Ruben [7], [8] has also developed a mixture representation for the cdf although the weighting coefficients have quite a different form than those which will be given here, and the corresponding proofs are quite different.

THEOREM 2.1A. *Let U be defined as in (1.2). Then if $F(x)$ is the* cdf *of U, and $F_\nu(x)$ denotes the* cdf *of a central chi-square variate with ν degrees of freedom, one possible*[3] *representation is given by*

$$F(x) = \textstyle\sum_{i=0}^{\infty} q_i F_{M+2i}\,(x/\alpha), \tag{2.1}$$

where $M = \sum_{i=0}^{r} m_i$, $q_i > 0$, $\sum_{i=0}^{\infty} q_i = 1$, and the q_i are constants depending on the (m_i, d_i, a_i).

THEOREM 2.1B. *The weighting constants, q_i, of Theorem* 2.1A *are given explicitly by*

$$q_0 = \left(\textstyle\prod_{j=1}^{r} a_j^{-m_j/2}\right) \exp\left(-\tfrac{1}{2}\textstyle\sum_{j=0}^{r} d_j^2\right), \tag{2.2}$$

and for all $\gamma \neq 0$, by

$$q_\gamma = \textstyle\sum_{\alpha=0}^{\gamma} [e^{-d_0^2/2}(d_0^2/2)^{\gamma-\alpha}/(\gamma-\alpha)!]K_\alpha(r). \tag{2.3}$$

The $K_\alpha(r)$ satisfy $K_\alpha(r) \leqq 1$, and

$$K_\alpha(1) = h_\alpha^{(1)}, \qquad K_\alpha(2) = \textstyle\sum_{i_1=0}^{\alpha} h_{\alpha-i_1}^{(2)} h_{i_1}^{(1)}, \tag{2.4}$$

$$K_\alpha(3) = \textstyle\sum_{i_1=0}^{\alpha} \sum_{i_2=0}^{i_1} h_{\alpha-i_1}^{(3)} h_{i_1-i_2}^{(2)} h_{i_2}^{(1)}, \qquad \text{etc.}$$

The $h_\alpha^{(i)}$ are defined by

$$h_\alpha^{(i)} = \textstyle\sum_{\beta=0}^{\alpha} \sum_{k=0}^{\beta} [e^{-d_i^2/2}(d_i^2/2)^{\beta-k}/(\beta-k)!]c_{\alpha-\beta}^{(i)} g_k^{(i,\beta-k)}, \tag{2.5}$$

where the $c_\alpha^{(i)}$, and the $g_\beta^{(i,k)}$ are defined by

[3] See Ruben [7] for other representations.

(2.6) $$c_\alpha^{(i)} = a_i^{-m_i/2}(1 - a_i^{-1})^\alpha \Gamma(m_i/2 + \alpha)/\Gamma(\alpha + 1)\Gamma(m_i/2),$$

(2.7) $$g_\beta^{(i,k)} = a_i^{-k}(1 - a_i^{-1})^\beta \binom{\beta+k-1}{\beta}, \qquad \beta \geqq 1,\ g_0^{(i,0)} = 1.$$

Proof. Let $\phi_Y(t)$, $\Psi_n(t)$ denote the characteristic functions of a variable Y, and a $\chi^2_{n,0}$ variate, respectively; i.e. $\Psi_n(t) = (1 - 2it)^{-n/2}$. By independence in (1.2),

(2.8) $$\phi_{U/\alpha}(t) = \phi_{\chi^2_{m_0,d_0}}(t) \prod_{i=1}^r \phi_{a_i \chi^2_{m_i,d_i}}(t).$$

Let $p_j(d_0)$ denote the Poisson coefficient $(j!)^{-1}(d_0^2/2)^j \exp(-d_0^2/2)$. Then if $f_j(x)$ denotes the density of a central chi-square variate with j degrees of freedom, it is well known that $\phi_{\chi^2_{m,d}}(t) = \sum_j p_j(d)\Psi_{m+2j}(t)$. Thus,

(2.9) $$\phi_{a\chi^2_{m,d}}(t) = \sum_j p_j(d)\Psi_{m+2j}(at) = \sum_j p_j(d)\phi_{a\chi^2_{m+2j,0}}(t).$$

The characteristic function of a constant times a central chi-square variate can be expressed (see [6], Equation (16)) as

(2.10) $$\phi_{a_j\chi^2_{m_j,0}}(t) = \Psi_{m_j}(a_j t) = (1 - 2ia_j t)^{-m_j/2} \equiv \sum_k c_k^{(j)} \Psi_{m_j+2k}(t),$$

where the $c_k^{(j)}$ are defined by

(2.11) $$\sum_k c_k^{(j)} z^k \equiv a_j^{-m_j/2}[1 - (1 - a_j^{-1})z]^{-m_j/2}, \qquad |z| < 1,$$

and are given explicitly by

$$c_k^{(j)} = a_j^{-m_j/2}(-1)^k(1 - a_j^{-1})^k \binom{-m_j/2}{k} = a_j^{-m_j/2}(1 - a_j^{-1})^k \Gamma(k + m_j/2)/\Gamma(k + 1)\Gamma(m_j/2).$$

From (2.8) it is clear that $\phi_{U/\alpha}(t)$ depends on the product of functions, each of which has the form (2.9). In turn, the function in (2.9) is expressible in terms of functions each of which has the form (2.10). These results are now combined. Note that since $\Psi_n(t) = \Psi_2^{n/2}(t)$,

$$\phi_{a\chi^2_{m,d}}(t) = \sum_j p_j(d) \sum_k \bar{c}_k^{(j)} \Psi_2^{(m+2j+2k)/2}(t),$$

where the $\bar{c}_k^{(j)}$ denotes the values of $c_k^{(j)}$ for $m_j = m + 2j$, and $a_j = a$. Using (2.11) gives

(2.12) $$\phi_{U/\alpha}(t) = \sum_j p_j(d_0)\Psi_2^{(m_0+2j)/2}(t) \prod_{i=1}^r \cdot \{\sum_k p_k(d_i) a_i^{-(m_i+2k)/2} \Psi_2^{(m_i+2k)/2}(t)[1 - (1 - a_i^{-1})\Psi_2(t)]^{-(m_i+2k)/2}\}.$$

Now using (2.7), and $M = \sum_0^r m_i$, it is clear that we can obtain the compact form

(2.13) $$\phi_{U/\alpha}(t) = \sum_j p_j(d_0)\Psi_2^{(M+2j)/2}(t) \prod_{i=1}^r Q_i(t),$$

where

$$Q_i(t) = \sum_\alpha \sum_k \sum_\beta c_\alpha^{(i)} p_k(d_i) g_\beta^{(i,k)} \Psi_2^{\alpha+\beta+k}(t).$$

Note from (2.7) that $g_0^{(i,0)} = 1$, $g_\beta^{(i,0)} = 0$ for $\beta \geqq 1$. Equation (2.13) is simplified by first reducing $Q_i(t)$ to a single series, and finally combining the two remaining series into one by repeated use of Cauchy products.

It is straightforward to verify that all the series in question converge absolutely so that terms can always be rearranged. Define $\gamma_\beta^{(i)} = \sum_{k=0}^{\beta} p_{\beta-k}(d_i) g_k^{(i,\beta-k)}$. Then it is found that $Q_i(t)$ reduces to

$$Q_i(t) = [\sum_\alpha c_\alpha^{(i)} \Psi_2^{\alpha}(t)][\sum_\beta \gamma_\beta^{(i)} \Psi_2^{\beta}(t)].$$

By letting $h_\alpha^{(i)} = \sum_{\beta=0}^{\alpha} \gamma_\beta^{(i)} c_{\alpha-\beta}^{(i)}$, the product can be collapsed into a single series as $Q_i(t) = \sum_\alpha h_\alpha^{(i)} \Psi_2^{\alpha}(t)$. Using the definition of $K_\alpha(r)$ given in (2.4) it is easy to show that

$$\phi_{U/\alpha}(t) = \Psi_2^{M/2}(t) \sum_j \sum_\alpha p_j(d_0) K_\alpha(r) \Psi_2^{j+\alpha}(t).$$

Finally, define $q_i = \sum_{\alpha=0}^{i} K_\alpha(r) p_{i-\alpha}(d_0)$. Then $\phi_{U/\alpha}(t) = \sum_i q_i \Psi_{M+2i}(t)$. Inverting termwise yields the desired result. □

A possible probabilistic interpretation of the coefficients q_i is given in [5].

The next theorem yields an expression for the distribution of the ratio of a positive definite quadratic form in non-central variates to an independent one in central variates.

THEOREM 2.2. *Let U be defined as in* (1.2) *and let V^* be the value of V (defined in* (1.3)) *when $\beta = 1$, and $g_i = 0$, $i = 0, 1, \cdots, s$. Let $F_{m,n}(x)$ denote the* cdf *of the ratio of two independent central chi-square variates $\chi^2_{m,0}/\chi^2_{n,0}$. If $X = U/V^*$, then*

$$P\{X \leqq x\} = \sum_i \sum_k q_i \theta_k F_{M+2i,N+2k}(x/\alpha), \tag{2.14}$$

where $M = \sum_{i=0}^{r} m_i$, $N = \sum_{i=0}^{s} n_i$; the constants q_i are defined by $P\{U \leqq x\} = \sum_i q_i F_{M+2i}(x/\alpha)$, in Theorem 2.1, and the constants θ_k are defined by

$$\sum_k \theta_k z_k \equiv \prod_{i=1}^{s} b_i^{-n_i/2}[1 - (1 - b_i^{-1})z]^{-n_i/2}, \qquad |z| < 1. \tag{2.15}$$

PROOF. It is shown in Theorem 1 of [6] that $P\{V^* \leqq x\} = \sum_k \theta_k F_{N+2k}(x)$, where θ_k is defined by (2.15). Thus, U and V^* are both mixtures. Since U/V^* is a simple Borel function, its cdf can be found just as the cdf of a single non-central F-variate is found. Hence

$$P\{X \leqq x\} = \int\int_{u/v^* \leqq x} d(\sum_i q_i F_{M+2i}(u/\alpha))\, d(\sum_k \theta_k F_{N+2k}(v^*)),$$

or $P\{X \leqq x\} = \sum_i \sum_k q_i \theta_k P\{U_i/V_k^* \leqq x\}$, where the (U_i, V_k^*) are random variables having distribution functions $F_{M+2i}(x/\alpha)$, $F_{N+2k}(x)$, respectively. But U_i/V_k^* is just the ratio of two independent chi-square variates. Hence, the desired result follows. □

Note that $F_{m,n}(x)$ is related to the cdf of a central F-variate. However, in computational work $F_{m,n}(x)$ is often more convenient since $F_{m,n}(x) = I_{x/(1+x)}(m/2, n/2)$, where $I_x(m, n)$ is the incomplete beta function, which is well tabulated.

3. Probability density for indefinite forms. In this section we derive the probability density function for $T = U - V$. Gurland [2] in considering the problem for central variates found an infinite series expansion in terms of Laguerre polynomials for the case in which the number of positive (or negative) coefficients is even. Shah [9] then extended his work to the non-central case.

First consider the weighted difference of two independent, central chi-square variates. Define $Z = \alpha X_1 - \beta X_2$, where $\mathfrak{L}(X_1) = \chi^2_{m,0}$, $\mathfrak{L}(X_2) = \chi^2_{n,0}$, X_1 and X_2 are independent, $\alpha > 0$, $\beta > 0$, and $\mathfrak{L}$ denotes "law." Pachares [4] considered this problem for $m = n$, $\alpha = \beta$, and found the distribution of Z expressible in terms of modified Bessel functions of the second kind. It will be seen that the form of the solution developed here will be more suited to percentage point calculations.

To simplify the development below we introduce the function

$$\psi(a, b; x) \equiv (\Gamma(a))^{-1} \int_0^\infty e^{-xt} t^{a-1}(1 + t)^{b-a-1}\, dt, \tag{3.1}$$

for $a > 0$, $x > 0$. This function satisfies the confluent hypergeometric differential equation of Kummer: $x\, d^2y/dx^2 + (b - x)\, dy/dx - ay = 0$ (see Erdélyi [1]) and is identical with the function $U(a, b; x)$ discussed by Slater [11]. It is expressible in terms of the more common ${}_1F_1$ hypergeometric function as

$$\psi(a, b; x) = [\Gamma(1 - b)/\Gamma(1 + a - b)]\, {}_1F_1(a, b; x) + [\Gamma(b - 1)/\Gamma(a)] x^{1-b}\, {}_1F_1(1 + a - b, 2 - b; x).$$

However, the reason for using the ψ function instead of the ${}_1F_1$ function is the fact that ψ is finite for all x.

THEOREM 3.1. *If $Z = \alpha X_1 - \beta X_2$, $\alpha > 0$, $\beta > 0$, where $L(X_1) = \chi^2_{m,0}$, $L(X_2) = \chi^2_{n,0}$, and X_1 and X_2 are independent, the probability density function of Z is given by*

$$\begin{aligned} p_{m,n}(t) &= [c(m, n)/\Gamma(m/2)] t^{(m+n-2)/2} e^{-t/2\alpha} \psi(n/2, (m + n)/2; \\ &\quad [(\alpha + \beta)/2\alpha\beta] t), \qquad t \geqq 0, \\ &= [c(m, n)/\Gamma(n/2)] (-t)^{(m+n-2)/2} e^{t/2\beta} \psi(m/2, (m + n)/2; \\ &\quad [-(\alpha + \beta)/2\alpha\beta] t), \qquad t \leqq 0, \end{aligned} \tag{3.2}$$

where $c^{-1}(m, n) = 2^{(m+n)/2} \alpha^{m/2} \beta^{n/2}$, and ψ is defined in (3.1).

PROOF. Define $R = Z/\alpha$, $\gamma = \beta/\alpha$. Let $p_0(t)$ denote the density of R, and let $s_1(t)$, $s_2(t)$ denote the densities of X_1, X_2, respectively. Then,

$$p_0(t) = \int_{-\infty}^{\infty} s_1(t + \gamma x) s_2(x)\, dx.$$

If $t \geqq 0$, p_0 reduces to

$$p_0(t) = \int_0^\infty [x^{n/2-1}(t + \gamma x)^{m/2-1}/2^{(m+n)/2}\Gamma(m/2)\Gamma(n/2)] e^{-\frac{1}{2}(t+\gamma x+x)}\, dx.$$

Introduction of y by $\gamma x = ty$ and straightforward algebra yields

$$p_0(t) = [t^{(m+n-2)/2} e^{-t/2}/\gamma^{n/2} 2^{(m+n)/2}\Gamma(m/2)] \psi(n/2, (m + n)/2; [(\gamma + 1)/2\gamma] t).$$

If $t \leqq 0$, the predicted result can be found by letting $xt = -\theta$, $\theta > 0$ and using a symmetrical argument.

By simple scaling, we obtain Equation (3.2).

THEOREM 3.2. *If U is a linear combination of non-central chi-square variates as in* (1.2), *and if $f(x)$ denotes the probability density function of U, then, by using the constants $q_i > 0$, $\sum_0^\infty q_i = 1$ defined in Theorem* 2.1B, *$f(x)$ may be represented as*

$$f(x) = \textstyle\sum_i (q_i/\alpha) f_{M+2i}(x/\alpha), \tag{3.3}$$

where $M = \sum_0^r m_i$, and $f_k(x)$ denotes the density of a central chi-square variate with k degrees of freedom.

PROOF. It is clear that all the F's defined in Theorem 2.1B are differentiable. Let $f_k(\cdot) \equiv F_k'(\cdot)$. Since the series $\sum q_i f_{M+2i}(x)$ certainly converges uniformly on every finite interval of x, it must converge to the required function, $f(x)$. ▯

THEOREM 3.3. *If T, U, V are quadratic forms in non-central variates, as defined in* (1.1)–(1.3), *and if $h(t)$ denotes the probability density function of T,*

$$h(t) = \textstyle\sum_{i=0}^\infty \sum_{j=0}^\infty q_i q_j^* p_{M+2i,N+2j}(t), \tag{3.4}$$

where the q_i, q_j^* are the constants defined in Theorem 2.1B *corresponding to U, V, respectively, $p_{m,n}(t)$ is the "difference of two chi-squares" function defined in* (3.2), *and $M = \sum_{i=0}^r m_i$, $N = \sum_{j=0}^s n_j$.*

PROOF. Let $f(t)$, $g(t)$ denote the densities of U, V, respectively. By definition,

$$h(t) = \textstyle\int_{-\infty}^\infty f(t + x) g(x)\, dx. \tag{3.5}$$

From Theorem 3.2, there are constants q_i, q_j^* for which

$$f(t) = \textstyle\sum (q_i/\alpha) f_{M+2i}(t/\alpha), \qquad t \geqq 0 \tag{3.6}$$

$$g(t) = \textstyle\sum (q_j^*/\beta) f_{N+2j}(t/\beta), \qquad t \geqq 0. \tag{3.7}$$

Define $I_{ij}(t) \equiv \int_0^\infty f_{M+2i}((t + x)/\alpha) f_{N+2j}(x/\beta)\, dx$. Then, by substituting (3.6), (3.7) into (3.5) and noting that both series converge uniformly (permitting interchange of integration and summation), we obtain

$$h(t) = \textstyle\sum_i \sum_j [q_i q_j^*/\alpha\beta] I_{ij}(t).$$

Change variables and identify $(m, n) \equiv (M + 2i, N + 2j)$, where (m, n) are defined in Theorem 3.1. Then by Theorem 3.1, $I_{ij}(t) = \alpha\beta p_{m,n}(t)$. If these operations are carried out separately for $t \geqq 0$, and $t \leqq 0$, (3.4) is readily obtained. ▯

4. Percentage points for indefinite forms. The expression for the density of an indefinite quadratic form in non-central normal variates given in (3.4) is exact, and useful in many theoretical problems. However, its indefinite integral is too complicated for numerical computations of the percentage points of the distribution. Fortunately, by using the asymptotic properties of the finite version of the hypergeometric function, ψ, defined above, the integration can be carried out for the tail probabilities of the distribution, which are all that are usually required.

Let $H(t)$ denote the cdf of T, defined in (1.1). We then obtain

THEOREM 4.1. *If* T, q_i, q_j^*, α, β, M, N *are defined as in Theorems* 2.1B *and* 3.3, *and* (1.1)–(1.3), *then for large values of* t,

$$H(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i q_j^* [\beta/(\alpha+\beta)]^{(M+2i)/2} [1 - F_{N+2j}(-t/\beta)], \quad t \leqq 0, \tag{4.1}$$

$$1 - H(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i q_j^* [\alpha/(\alpha+\beta)]^{(N+2j)/2} [1 - F_{M+2i}(t/\alpha)], \quad t \geqq 0, \tag{4.2}$$

where $F_j(t)$ *is the* cdf *of* $\chi^2_{j,0}$.

PROOF. Define, for $t > 0$, $J_{ij}(t) \equiv \int_{-\infty}^{t} (-x)^{(m+n-2)/2} e^{x/2\beta} \psi(m/2, (m+n)/2; -[(\alpha+\beta)/2\alpha\beta]x)\, dx$, and for $t < 0$, $J_{ij}^*(t) \equiv \int_t^{\infty} x^{(m+n-2)/2} e^{-x/2\alpha} \psi(n/2, (m+n)/2; [(\alpha+\beta)/2\alpha\beta]x)\, dx$, where $(m, n) \equiv (M + 2i, N + 2j)$. Since the series in (3.4) converge uniformly, substitution into the defining integral of $H(t)$ gives

$$H(t) = \sum_i \sum_j [q_i q_j^* c(M+2i, N+2j)/\Gamma((N+2j)/2)] J_{ij}(t), \qquad t \leqq 0,$$
$$= 1 - \sum_i \sum_j [q_i q_j^* c(M+2i, N+2j)/\Gamma((M+2j)/2)] J_{ij}^*(t), \quad t \geqq 0.$$

From the fact that for large y, $\psi(a, b; y) \cong y^{-a}$ (see Slater [11]), it is not hard to find that for $\theta > 0$,

$$J_{ij}(-\theta) = [2^{(m+n)/2} \alpha^{m/2} \beta^{(m+n)/2} \Gamma(n/2)/(\alpha+\beta)^{m/2}][1 - F_n(\theta/\beta)].$$

It may also be seen that $J_{ij}^*(t; \alpha, \beta, m, n) = J_{ij}(-t; \beta, \alpha, n, m)$. The above theorem is now immediate. □

Various techniques have been developed for approximating and simplifying the formulas given in (4.1), (4.2). This subject, along with some applications, is planned for a later paper.

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