COMPARISON OF THE BOUNDS OF THE NUMBER OF COMMON TREATMENTS BETWEEN BLOCKS OF CERTAIN PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction and summary. Agrawal (1964) and Shah (1965) have derived bounds for the number of common treatments between the blocks of PBIB designs. Agrawal's result is more general and is applicable to any (v, b, r, k) design. In this note it is established that Agrawal's bounds are superior to those given by Shah.

2. Result. Let l_{ij} be the number of treatments common between any two blocks of a (v, b, r, k) design. Let $rk, \mu_0, \mu_1, \cdots, \mu_s$ be the distinct characteristic roots of NN', where N is the incidence matrix of the design, $rk > \mu_0 > \mu_1 \cdots > \mu_s$. Then Agrawal (1964) has proved the equivalent of

$$\max [0, 2k - v, (2rk/b) - k, k - \mu_0] \leqq l_{ij} \leqq \min [k, \mu_0 - k + 2(rk - \mu_0)/b]; \quad i \neq j, = 1, 2, \cdots, b. \tag{2.1}$$

In (2.1), above the quantity $(2rk/b) - k$ is introduced to facilitate a more elegent expression of the relations (2.6) and (2.7) below.

After slight manipulation, Shah (1965) results can be put in the form

$$\begin{aligned} [k(r-1)/(b-1)] - [k(b-r)/(b-1)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}} \\ \leqq l_{ij} \leqq [k(r-1)/(b-1)] \\ + [k(b-r)/(b-1)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}}, \\ i \neq j, = 1, 2, \cdots, b; \end{aligned} \tag{2.2}$$

where $\alpha + 1$ is the number of non-zero characteristic roots of NN'.

In the four classes of designs considered by Shah, NN' has rk and only one other characteristic root, namely, μ_0 with multiplicity α. Using the fact that the trace of a matrix is equal to sum of its characteristic roots we get,

$$\begin{aligned} rk + \alpha\mu_0 &= vr = bk, \\ \alpha\mu_0 &= bk - rk, \\ \mu_0 &= k(b-r)/\alpha. \end{aligned} \tag{2.3}$$

To prove that the upper bound is superior we have to show that $\theta_1 \geqq 0$ or

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when $\theta_1 \ngeqq 0$, $\theta_2 \geqq 0$ where

$$\theta_1 = [k(r-1)/(b-1)] + [k(b-r)/(b-1)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}} - \{\mu_0 - k + 2(rk - \mu_0)/b\},$$

and

$$\theta_2 = [k(r-1)/(b-1)] + [k(b-r)/(b-r)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}} - k.$$

Substituting the value of μ_0 and on simplification

(2.4) $\qquad \theta_1 > 0 \quad \text{if} \quad \alpha > \frac{1}{2}b - 1, \quad \text{and} \quad \theta_1 = 0, \quad \text{if} \quad \alpha = \frac{1}{2}b - 1;$ and

(2.5) $\qquad \theta_2 > 0 \quad \text{if} \quad \alpha < \frac{1}{2}b - 1, \quad \text{and} \quad \theta_2 = 0, \quad \text{if} \quad \alpha = \frac{1}{2}b - 1.$

(2.4) and (2.5) together establish the superiority of the upper bound.

To prove that the lower bound is superior we will show that $\theta_3 \geqq 0$ and when $\theta_3 \ngeqq 0$, $\theta_4 \geqq 0$, where

$$\theta_3 = k - \mu_0 - \{[k(r-1)/(b-1)] - [k(b-r)/(b-1)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}}\},$$

and

$$\theta_4 = (2rk/b) - k - \{[k(r-1)/(b-1)] - [k(b-r)/(b-1)][(b-1-\alpha)(b-2)/b\alpha]^{\frac{1}{2}}\}.$$

(2.6) $\qquad \theta_3 > 0, \quad \text{if} \quad \alpha > \frac{1}{2}b, \quad \text{and} \quad \theta_3 = 0, \quad \text{if} \quad \alpha = \frac{1}{2}b;$ and,

(2.7) $\qquad \theta_4 > 0, \quad \text{if} \quad \alpha < \frac{1}{2}b, \quad \text{and} \quad \theta_4 = 0, \quad \text{if} \quad \alpha = \frac{1}{2}b.$

(2.6) and (2.7) establish the superiority of the lower bound.

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