SOME RENEWAL THEOREMS WITH APPLICATION TO A FIRST PASSAGE PROBLEM[1]

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1. Introduction. Let X_i, $i=1, 2, 3, \cdots$ be a sequence of independent and identically distributed random variables with $E|X_i| < \infty$, $EX_i = \mu > 0$. Write $X_i^- = -\min(0, X_i)$, $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1 \leqq k \leqq n} S_k$. In this paper we shall discuss the asymptotic behaviour as $x \to \infty$ of the sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ and $\sum_{n=1}^\infty a_n \Pr(M_n \leqq x)$ for certain classes of positive coefficient sequences $\{a_n\}$ and use the results on the latter sums to investigate the behaviour of the first passage time out of the interval $(-\infty, x]$ for the process S_n as $x \to \infty$.

The analysis that we shall use in obtaining the theorems on asymptotic behaviour follows closely on that of Smith [6] who discussed sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ for a class of coefficient sequences that we shall also discuss and for non-identically distributed random variables. In fact, our Theorem 1 follows directly from a specialization of the analysis of Smith. One of the particularly interesting characteristics of this technique is that it enables us to study the asymptotic behaviour of the sums $\sum_{n=1}^\infty a_n \Pr(S_n \leqq x)$ and $\sum_{n=1}^\infty a_n \Pr(M_n \leqq x)$ in the one operation in spite of essential differences in their behaviour.

2. Renewal theorems. For the first set of positive term coefficient sequences $\{a_n\}$ that we consider we shall suppose (as in [6]) that there exist real numbers $\alpha > 0$, $\gamma \geqq 0$ and some non-negative function of slow growth $L(x)$ such that

$$(1) \qquad \sum_{n=1}^\infty a_n x^n \sim [\alpha/(1-x)^\gamma] L(1-x)^{-1}, \qquad \text{as} \qquad x \to 1^-.$$

This is satisfied, for example, if

$$a_n \sim [\alpha/\Gamma(\gamma)] n^{\gamma-1} L(n) \qquad \text{as} \qquad n \to \infty$$

using an Abelian theorem of Doetsch [3], 460.

In the subsequent work we shall need the following definition:

Definition. The index k of the sequence $\{a_n\}$ is the least real k such that $a_n = O(n^k)$.

Consideration will be restricted to cases where $\sum a_n$ diverges.

Theorem 1. *Suppose* $E|X| < \infty$, $EX = \mu > 0$. *Let k be the index of the sequence $\{a_n\}$ and m be non-negative. In order that*

$$\sum_{n=1}^\infty a_n \Pr(S_n \leqq x) \sim [\alpha L(x)/\Gamma(1+\gamma)](x/\mu)^\gamma \qquad as \qquad x \to \infty$$

for each sequence $\{a_n\}$ such that $k \leqq m$ it is necessary and sufficient that $E|X^-|^{m+2} < \infty$.

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THEOREM 2. *Suppose* $E|X| < \infty$, $EX = \mu > 0$. *Let* k *be the index of the sequence* $\{a_n\}$ *and* m *be non-negative. In order that*

$$\textstyle\sum_{n=1}^{\infty} a_n \Pr(M_n \leqq x) \sim [\alpha L(x)/\Gamma(1+\gamma)](x/\mu)^{\gamma} \quad \text{as} \quad x \to \infty$$

for each sequence $\{a_n\}$ *such that* $k \leqq m$ *it is necessary that* $E|X^-|^{m+1} < \infty$ *and sufficient that* $E|X^-|^{m+2} < \infty$.

I conjecture that the condition $E|X^-|^{m+1} < \infty$ is both necessary and sufficient in Theorem 2. It is certainly known that in the particular case where $a_n = 1$ for all n so that $k = 0$ we only need $E|X^-| < \infty$ (see for example Chow and Robbins [2]).

The proof of the two theorems shall be deferred until we have given four lemmas. The first of these is given in a form which is more general than we shall need subsequently as it has some independent interest.

LEMMA 1. *Suppose the random variables* X_i, $i = 1, 2, 3, \cdots$ *are independent and identically distributed.* Write $S_n = \sum_{i=1}^n X_i$, $M_n = \max_{1\leqq k\leqq n} S_k$, *and, in the case where* $E|X_i| < \infty$, $EX_i = \mu$. *If* $E|X_i|^r < \infty$ *with* $1 \leqq r < 2$ *and* $\mu \geqq 0$, *then*

$$n^{-1/r}(M_n - n\mu) \to_{\text{a.s.}} 0.$$

If $E|X_i|^r < \infty$ *with* $0 < r < 1$ *or* $E|X_i|^r < \infty$ *with* $1 \leqq r < 2$ *and* $\mu < 0$, *then*

$$n^{-1/r} M_n \to_{\text{a.s.}} 0.$$

("a.s." *denotes almost sure convergence*).

The corresponding almost sure convergence versions for the sums S_n have been given by Kolmogorov ($r = 1$) and Marcinkiewicz ($r \neq 1$) (see for example Loève [5] 242, 243).

PROOF. Suppose $E|X_i|^r < \infty$. Let $c_r = \mu$ if $1 \leqq r < 2$; $c_r = 0$ if $0 < r < 1$. If $c_r \geqq 0$, we have

$$\begin{aligned} M_n - nc_r &= \max_{1\leqq j\leqq n} S_j - nc_r \\ &\leqq \max_{1\leqq j\leqq n} (S_j - jc_r) \\ &\leqq \max_{1\leqq j\leqq n} j^{1/r} a_j, \end{aligned}$$

where $a_j = j^{-1/r}|S_j - jc_r|$. Therefore, if $M_n \geqq nc_r$, $0 \leqq M_n - nc_r \leqq \max_{1\leqq j\leqq n} j^{1/r} a_j$, and if $M_n < nc_r$, $0 > M_n - nc_r \geqq S_n - nc_r = -n^{1/r} a_n \geqq -\max_{1\leqq j\leqq n} a_j$, so that $0 \leqq n^{-1/r}|M_n - nc_r| \leqq \max_{1\leqq j\leqq n} n^{-1/r} j^{1/r} a_j$. In order to obtain the desired result in this case it suffices to show that for arbitrary $\epsilon > 0$,

$$\Pr\left(\textstyle\bigcup_{k\geqq n} \{\max_{1\leqq j\leqq k} j^{1/r} k^{-1/r} a_j \geqq \epsilon\}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Now the strong laws of Kolmogorov and Marcinkiewicz ([5], 242, 243) imply that for arbitrary $\epsilon > 0$,

$$\Pr(\max_{j\geqq n} a_j \geqq \epsilon) \to 0 \qquad \text{as} \qquad n \to \infty.$$

Therefore, given $\eta > 0$ arbitrarily small we can choose an N so large that

$$\Pr(\max_{j>N} a_j \geqq \tfrac{1}{2}\epsilon) \leqq \tfrac{1}{2}\eta \tag{2}$$

and then select an $n_0(>N)$ so large that for $n \geqq n_0$,

$$\Pr(N^{1/r}n^{-1/r} \max_{1 \leqq j \leqq N} a_j \geqq \tfrac{1}{2}\epsilon) \leqq \tfrac{1}{2}\eta. \tag{3}$$

Then, for $n \geqq n_0$,

$$\begin{aligned}
\Pr(\textstyle\bigcup_{k \geqq n} \{\max_{1 \leqq j \leqq k} j^{1/r}k^{-1/r}a_j \geqq \epsilon\}) &\\
&= \Pr(\textstyle\bigcup_{k \geqq n} \{\max_{1 \leqq j \leqq N} j^{1/r}k^{-1/r}a_j + \max_{N<j\leqq k} j^{1/r}k^{-1/r}a_j \geqq \epsilon\})\\
&\leqq \Pr(\textstyle\bigcup_{k \geqq n} \{\max_{1 \leqq j \leqq N} j^{1/r}k^{-1/r}a_j + \max_{N<j\leqq k} a_j \geqq \epsilon\})\\
&\leqq \Pr(N^{1/r}n^{-1/r} \max_{1 \leqq j \leqq N} a_j + \max_{j>N} a_j \geqq \epsilon)\\
&\leqq \Pr(N^{1/r}n^{-1/r} \max_{1 \leqq j \leqq N} a_j \geqq \tfrac{1}{2}\epsilon) + \Pr(\max_{j>N} a_j \geqq \tfrac{1}{2}\epsilon)\\
&\leqq \eta,
\end{aligned}$$

using (2) and (3). This completes the proof of this part of the lemma. It remains to consider the case $c_r < 0$.

In the case $c_r < 0$, $M_n^+ = \max(0, M_n)$ actually has a proper limiting distribution (finite with probability one), $M^+ = \lim_{n\to\infty} M_n^+$.

We have

$$\begin{aligned}
\Pr(\textstyle\bigcup_{k \geqq n} \{k^{-1/r}|M_k| \geqq \epsilon\}) &\\
&= \Pr(\textstyle\bigcup_{k \geqq n} \{k^{-1/r}M_k \geqq \epsilon\} \cup \bigcup_{k \geqq n} \{k^{-1/r}M_k \leqq -\epsilon\})\\
&\leqq \Pr(\textstyle\bigcup_{k \geqq n} \{k^{-1/r}M_k \geqq \epsilon\}) + \Pr(\bigcup_{k \geqq n} \{k^{-1/r}M_k \leqq -\epsilon\})\\
&\leqq \Pr(\textstyle\bigcup_{k \geqq n} \{M^+ \geqq k^{1/r}\epsilon\}) + \Pr(\bigcup_{k \geqq n} \{X_1 \leqq -k^{1/r}\epsilon\})\\
&= \Pr(M^+ \geqq n^{1/r}\epsilon) + \Pr(X_1 \leqq -n^{1/r}\epsilon)
\end{aligned}$$

and both these terms approach zero as $n \to \infty$. Therefore, $n^{-1/r}M_n \to_{\text{a.s.}} 0$ as required. This completes the proof of the lemma.

In the subsequent work we shall write $F_n(x) = \Pr(S_n \leqq x)$, $G_n(x) = \Pr(M_n \leqq x)$ and $H_n(x)$ to mean either $F_n(x)$ or $G_n(x)$ (so that if a property holds for both $F_n(x)$ and $G_n(x)$ it holds for $H_n(x)$ and conversely).

LEMMA 2. *$\int_\mu^\infty \{1 - H_n(nx)\}\, dx \to 0$ as $n \to \infty$.*

PROOF. Introduce the new random variables $Y_i = \max(0, X_i)$, $i = 1, 2, 3, \cdots$. Suppose $EY_i = \alpha$. If $M_n > x$ then $\sum_{i=1}^n Y_i > x$ so that $\Pr(\sum_{i=1}^n Y_i > x) \geqq \Pr(M_n > x) \geqq \Pr(S_n > x)$ and hence

$$1 - F_n(x) \leqq 1 - G_n(x) \leqq 1 - K_n(x), \tag{4}$$

where we have written $K_n(x) = \Pr(\sum_{i=1}^n Y_i \leqq x)$.

We show firstly that $\int_\alpha^\infty \{1 - K_n(nx)\}\, dx \to 0$ as $n \to \infty$. We have, by a simple integration by parts,

$$\begin{aligned}
\alpha &= \int_0^\infty \{1 - K_n(nx)\}\, dx\\
&= \int_0^\alpha \{1 - K_n(nx)\}\, dx + \int_\alpha^\infty \{1 - K_n(nx)\}\, dx.
\end{aligned}$$

Further, by the law of large numbers, $K_n(nx) \to 0$ as $n \to \infty$ for $x < \alpha$. It follows from the mean value theorem that $\int_0^\alpha \{1 - K_n(nx)\}\, dx \to \alpha$ as $n \to \infty$ and hence that $\int_\alpha^\infty \{1 - K_n(nx)\}\, dx \to 0$ as $n \to \infty$.

Then, making use of (4),

$$0 \leqq \int_\mu^\infty \{1 - F_n(nx)\}\, dx \leqq \int_\mu^\infty \{1 - G_n(nx)\}\, dx$$
$$\leqq \int_\mu^\alpha \{1 - G_n(nx)\}\, dx + \int_\alpha^\infty \{1 - K_n(nx)\}\, dx.$$

Now, in view of Lemma 1 (case $r = 1, \mu > 0$), $G_n(nx) \to 1$ as $n \to \infty$ for $x > \mu$ so that by the mean value theorem $\int_\mu^\alpha \{1 - G_n(nx)\}\, dx \to 0$ as $n \to \infty$. We have shown that $\int_\alpha^\infty \{1 - K_n(nx)\}\, dx \to 0$ as $n \to \infty$ so the proof is complete.

LEMMA 3. (Smith [6]). *If the non-negative constants* $\{a_n\}$ *satisfy* (1) *then as* $s \to 0+$,

$$\sum_{n=1}^\infty a_n e^{-\mu s n} \sim \alpha(\mu s)^{-\gamma} L(s^{-1}).$$

LEMMA 4. *If the non-negative constants* $\{a_n\}$ *satisfy* (1) *then as* $s \to 0+$,

$$\sum_{n=1}^\infty n a_n e^{-\mu s n} \sim \alpha(\mu s)^{-\gamma-1} L(s^{-1}) \quad \textit{if} \quad \gamma > 0$$
$$\limsup_{s\to 0+} s[L(s^{-1})]^{-1} \sum_{n=1}^\infty n a_n e^{-\mu s n} < \alpha\mu^{-1} \quad \textit{if} \quad \gamma = 0.$$

Smith [6] has established the former result, ignoring the possibility of the case $\gamma = 0$. The result given above for $\gamma = 0$ can be readily extracted from Smith's proof and is adequate in the present context.

PROOF OF THEOREMS 1 AND 2. We follow the methods of [6] but work the proof in terms of $H_n(x)$.

Suppose firstly that $E|X^-|^{m+2} < \infty$. Take β arbitrary with $0 < \beta < \mu$. Consider

$$0 \leqq K_n = \int_{n\beta}^{n\mu} e^{-sx} H_n(x)\, dx \qquad (s \geqq 0)$$
$$= n \int_\beta^\mu e^{-nsx} H_n(nx)\, dx$$
$$\leqq n e^{-n\beta s} \int_\beta^\mu H_n(nx)\, dx.$$

Now using the law of large numbers and the inequality $H_n(y) \leqq F_n(y)$ (or alternatively referring to Lemma 1 as well), we see that $H_n(nx) \to 0$ as $n \to \infty$ for all $x < \mu$. Hence, using the mean value theorem, we may write

$$K_n = n e^{-n\beta s} \delta_n', \tag{5}$$

where $\delta_n' \to 0$ as $n \to \infty$, uniformly in $s \geqq 0$.

Now consider

$$0 \leqq L_n = \int_{n\mu}^\infty e^{-sx} \{1 - H_n(x)\}\, dx \qquad (s \geqq 0)$$
$$= n \int_\mu^\infty e^{-nsx} \{1 - H_n(nx)\}\, dx$$
$$\leqq n e^{-ns\beta} \int_\mu^\infty \{1 - H_n(nx)\}\, dx.$$

In view of Lemma 2, we may write

$$L_n = n e^{-n\beta s} \delta_n'' \tag{6}$$

where $\delta_n'' \to 0$ as $n \to \infty$, uniformly in $s \geqq 0$.

Combining (5) and (6) and putting $\delta_n = \delta_n' - \delta_n''$, we obtain

$$(7) \qquad \sum_{n=1}^{\infty} a_n(L_n - K_n) = -\sum_{n=1}^{\infty} na_n\delta_n e^{-n\beta s}.$$

Now given arbitrary $\epsilon > 0$ we can choose an integer $n_0(\epsilon)$ so large that $|\delta_n| < \epsilon$ for all $n > n_0$. Then,

$$\left|\sum_{n=1}^{\infty} na_n\delta_n e^{-n\beta s}\right| < \sum_{n=1}^{n_0} na_n|\delta_n|e^{-n\beta s} + \epsilon\sum_{n=1}^{\infty} na_n e^{-n\beta s},$$

so that by Lemma 4,

$$\limsup_{s\to 0+} [(\mu s)^{\gamma+1}/L(s^{-1})]\left|\sum_{n=1}^{\infty} na_n\delta_n e^{-n\beta s}\right| \leqq \epsilon\gamma\alpha \text{ if } \gamma > 0$$
$$\leqq \epsilon\alpha \text{ if } \gamma = 0$$

and hence

$$(8) \qquad [(\mu s)^{\gamma+1}/L(s^{-1})]\sum_{n=1}^{\infty} a_n(L_n - K_n) \to 0 \quad \text{as} \quad s \to 0+.$$

Next write

$$(9) \quad \Phi_\beta(x) = \sum_{n=1}^{\infty} a_n H_n(x) U(x - n\beta)$$
$$= \sum_{n=1}^{\infty} a_n U(x - n\mu) - \sum_{n=1}^{\infty} a_n\{U(x - n\mu) - H_n(x)\} U(x - n\beta)$$

where

$$U(x) = 1, \qquad x \geqq 0,$$
$$= 0, \qquad x < 0.$$

If we denote the Laplace transform of a function $A(x)$ by $A^0(s) = \int_0^\infty e^{-sx}A(x)\,dx$ we have

$$(10) \qquad \Phi_\beta^{\,0}(s) = s^{-1}\sum_{n=1}^{\infty} a_n e^{-n\mu s} - \sum_{n=1}^{\infty} a_n(L_n - K_n);$$

the term by term integration being justified by monotone convergence.

From (8), (10) and Lemma 3 it follows that

$$(11) \qquad [(\mu s)^{\gamma+1}/L(s^{-1})]\Phi_\beta^{\,0}(s) \to \alpha\mu \quad \text{as} \quad s \to 0+.$$

Appealing to a Tauberian theorem of Doetsch [3], 511, we then obtain

$$(12) \qquad [1/t^{\gamma+1}L(t)] \int_0^t \Phi_\beta(x)\,dx \to \alpha/\mu^\gamma\Gamma(\gamma + 2).$$

Now for $t > 0$ and $0 < \theta < 1$,

$$\Phi_\beta(\theta t)(t - \theta t) \leqq \int_{\theta t}^t \Phi_\beta(x)\,dx \leqq \Phi_\beta(t)(t - \theta t)$$

so that

$$[1/t^\gamma L(t)]\Phi_\beta(\theta t) \leqq [1/(1 - \theta)]\{[1/t^{\gamma+1}L(t)] \int_0^t \Phi_\beta(x)\,dx$$
$$- [1/t^{\gamma+1}L(t)] \int_0^{\theta t} \Phi_\beta(x)\,dx\} \leqq [1/t^\gamma L(t)]\Phi_\beta(t).$$

Then, for fixed θ and $t \to \infty$, $[1/t^{\gamma+1}L(t)] \int_0^t \Phi_\beta(x)\,dx \to \alpha/\mu^\gamma\Gamma(\gamma + 2)$, $[1/(\theta t)^{\gamma+1}L(t)] \int_0^{\theta t} \Phi_\beta(x)\,dx \to [1/(\theta t)^{\gamma+1}L(\theta t)] \int_0^{\theta t} \Phi_\beta(x)\,dx \to \alpha/\mu^\gamma\Gamma(\gamma + 2)$ by

(12) so that

$$(13)\quad \limsup_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(\theta t) \leqq [\alpha/\mu^{\gamma}\Gamma(\gamma + 2)](1 - \theta^{\gamma+1})/(1 - \theta) \leqq \liminf_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(t).$$

Taking $\theta \to 1$ in the right hand part of inequality (13) gives

$$(14)\quad \liminf_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(t) \geqq \alpha/\mu^{\gamma}\Gamma(\gamma + 1).$$

Further, the left hand part of the inequality (13) can be written

$$\limsup_{t\to\infty} [1/(\theta t)^{\gamma}L(\theta t)]\Phi_{\beta}(\theta t) \leqq [-\theta\alpha/\mu^{\gamma}\Gamma(\gamma + 2)](1 - \theta^{-(\gamma+1)})/(1 - \theta)$$

and the left hand side of this is equal to $\limsup_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(t)$ for θ fixed. Then, taking $\theta \to 1$ in the right hand side, we obtain

$$(15)\quad \limsup_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(t) \leqq \alpha/\mu^{\gamma}\Gamma(\gamma + 1),$$

so that combining (14) and (15),

$$(16)\quad \lim_{t\to\infty} [1/t^{\gamma}L(t)]\Phi_{\beta}(t) = \alpha/\mu^{\gamma}\Gamma(\gamma + 1).$$

Now under the condition $E|X^-|^{m+2} < \infty$ we certainly have for all x, $-\infty < x < \infty$, $\sum n^m H_n(x) < \infty$ in view of Theorem A of Heyde [4] and the inequality $H_n(x) \leqq F_n(x)$. Hence, $\sum a_n H_n(x) < \infty$, since the sequence $\{a_n\}$ has index $k \leqq m$. Write

$$(17)\quad \sum_{n=1}^{\infty} a_n H_n(x) = \Phi_{\beta}(x) + \Psi_{\beta}(x),$$

where

$$(18)\quad \Psi_{\beta}(x) = \sum_{n=1}^{\infty} a_n H_n(x)\{1 - U(x - n\beta)\}.$$

We shall go on to show that $\Psi_{\beta}(x)/x^{\gamma}L(x) \to 0$ as $x \to \infty$.

Define a new sequence of random variables Y_i, $i = 1, 2, 3, \cdots$ by

$$Y_i = X_i - \beta.$$

Then, $EY > 0$ and $E|Y^-|^{m+2} < \infty$ since $E|X^-|^{m+2} < \infty$. It follows from Theorem A of [4] applied to the Y's that for $k \leqq m$, $\sum n^k F_n(n\beta) < \infty$, and since $H_n(n\beta) \leqq F_n(n\beta)$,

$$(19)\quad \sum n^k H_n(n\beta) < \infty.$$

Also, it is clear from (18) that $\Psi_{\beta}(x) \leqq \sum_{n=1}^{\infty} a_n H_n(n\beta)$ and (19) ensures that this upper bound is finite since k is the index of the sequence $\{a_n\}$. We therefore have $\Psi_{\beta}(x)/x^{\gamma}L(x) \to 0$ as $x \to \infty$ and hence, using (16) and (17),

$$\sum_{n=1}^{\infty} a_n H_n(x) \sim [\alpha L(x)/\Gamma(1 + \gamma)](x/\mu)^{\gamma} \quad \text{as} \quad x \to \infty.$$

This result is true for all sequences $\{a_n\}$ with index $k \leqq m$ and hence establishes the sufficiency parts of both Theorems 1 and 2.

The necessity parts of both the theorems are easy to establish. It suffices to note that in particular $\sum n^m H_n(x) < \infty$ for all x, $-\infty < x < \infty$, and hence by Theorem A of [4] in the case $H_n(x) = F_n(x)$ we obtain $E|X^-|^{m+2} < \infty$ and by Theorem A of [4] together with the well known inequality $G_n(0) = \Pr(M_n \leqq 0) \geqq n^{-1} \Pr(S_n \leqq 0)$ in the case $H_n(x) = G_n(x)$ we obtain $E|X^-|^{m+1} < \infty$. This completes the proof of both Theorems 1 and 2.

Some remarks on the possibility that the condition $E|X^-|^{m+1} < \infty$ might be both necessary and sufficient in Theorem 2 would seem in order. To establish that this is the case, it would be adequate to show that if $EX = \mu > 0$ and $E|X^-|^{k+1} < \infty$, then for $0 < \beta < \mu$,

$$\sum n^k G_n(n\beta) = \sum n^k \Pr(M_n \leqq n\beta) < \infty.$$

The nearest approach to this that I have obtained is summarized in the following theorem:

THEOREM 3. *Suppose* $E|X| < \infty$, $EX > 0$, $\Pr(X < 0) > 0$, *and let* k *be a non-negative integer. A necessary and sufficient condition for the convergence of the series*

$$\sum_{n=1}^{\infty} n^k \Pr(M_n \leqq x), \qquad -\infty < x < \infty,$$

is that $E|X^-|^{k+1} < \infty$.

PROOF. It follows from the work of Heyde [4] that a necessary and sufficient condition for the convergence of the series $\sum n^k \Pr(M_n \leqq 0)$ is that $E|X^-|^{k+1} < \infty$. Therefore, in order to complete the proof it is only necessary to show that the convergence of $\sum n^k \Pr(M_n \leqq 0)$ implies that of $\sum n^k \Pr(M_n \leqq x)$, $0 < x < \infty$.

To accomplish this we define a new sequence of random variables:

$$N_0 = S_0 = 0,$$

$$N_1 = \max(S_0, S_1) = X_1^+,$$

$$N_2 = \max(S_0, S_1, S_2) = (X_1 + X_2^+)^+,$$

$$\cdots$$

$$N_n = \max(S_0, S_1, S_2, \cdots, S_n) = (X_1 + (X_2 + \cdots + (X_{n-1} + X_n^+)^+ \cdots)^+$$

$$\cdots$$

Since the X_i are independent and identically distributed we may write $N_0 = 0$, $N_{n+1} \sim (X + N_n)^+$, $n \geqq 1$, where for two random variables X and Y, we write $X \sim Y$ if they have the same distribution. Clearly, $N_n = M_n^+ = \max(0, M_n)$, so that

$$\Pr(N_n \leqq x) = \Pr(M_n \leqq x), \qquad x > 0,$$

$$\Pr(N_n = 0) = \Pr(M_n \leqq 0).$$

Now for $A > 0$, $n \geqq 1$, we have

$$\Pr(M_{n+1} \leqq A) = \int_{-\infty}^{A} \Pr(M_n \leqq A - y)\, d\Pr(X \leqq y)$$

so that for arbitrary $B > 0$,

$$\begin{aligned}\sum_{n=1}^{N} n^{\alpha} \Pr(M_{n+1} \leqq A) &= \int_{-\infty}^{A} \{\sum_{n=1}^{N} n^{\alpha} \Pr(M_n \leqq A - y)\}\, d\Pr(X \leqq y) \\ &\geqq \int_{-\infty}^{-B} \{\sum_{n=1}^{N} n^{\alpha} \Pr(M_n \leqq A - y)\}\, d\Pr(X \leqq y) \\ &\geqq \{\sum_{n=1}^{N} n^{\alpha} \Pr(M_n \leqq A + B)\} \Pr(X \leqq -B).\end{aligned}$$

Then, choosing B so that $\Pr(X \leqq -B) > 0$, we see that if $\sum n^{\alpha} \Pr(M_n \leqq A)$ converges then $\sum n^{\alpha} \Pr(M_n \leqq A + B)$ converges. Let $A \to 0+$; we see that the convergence of $\sum n^{\alpha} \Pr(M_n \leqq 0)$ implies that of $\sum n^{\alpha} \Pr(M_n \leqq rB)$ for all positive integral r and hence that of $\sum n^{\alpha} \Pr(M_n \leqq x)$ for all x, $0 < x < \infty$. This completes the proof of the theorem.

The restriction to non-negative integral k in Theorem 3 is unfortunate. This comes about from the use of the derivatives of the generating function

$$\sum_{n=0}^{\infty} \Pr(M_n \leqq 0) t^n = \exp\{\sum_{n=1}^{\infty} n^{-1} t^n \Pr(S_n \leqq 0)\}$$

in [4].

We now go on to consider coefficient sequences of the form $a_n = e^{rn}$, $r > 0$, and shall establish the following two theorems:

THEOREM 4. *Suppose* $E|X| < \infty$, $EX = \mu > 0$. *In order that*

$$\sum_{n=1}^{\infty} e^{rn} \Pr(S_n \leqq x) \sim r^{-1} e^{xr\mu^{-1}} \qquad \text{as} \qquad x \to \infty$$

for any r in some interval $0 < r < R$, it is necessary and sufficient that X^- should have an analytic characteristic function.

(The term "analytic characteristic function" is used for a characteristic function which is analytic in a strip containing the origin as an interior point.)

THEOREM 5. *Suppose* $E|X| < \infty$, $EX = \mu > 0$. *In order that*

$$\sum_{n=1}^{\infty} e^{rn} \Pr(M_n \leqq x) \sim r^{-1} e^{xr\mu^{-1}} \qquad \text{as} \qquad x \to \infty$$

for any r in some interval $0 < r < R$, it is necessary and sufficient that X^- should have an analytic characteristic function.

The proofs of these theorems, of course, follow markedly similar lines to the proofs of Theorems 1 and 2. A generalized coefficient sequence form along the lines of Theorems 1 and 2 is not, however, convenient in this case.

PROOF OF THEOREMS 4 AND 5. We construct the proofs in parallel fashion as we did with Theorems 1 and 2.

Suppose X^- has an analytic characteristic function. Take β arbitrary with $0 < \beta < \mu$. We consider to begin with

$$\begin{aligned}K_n &= \int_{n\beta}^{n\mu} e^{-sx} H_n(x)\, dx \qquad (s > 0) \\ &\leqq n e^{-n\beta s} \int_{\beta}^{\mu} H_n(nx)\, dx \\ &= n e^{-n\beta s} (\mu - \beta) H_n(n\xi)\end{aligned}$$

for some $\xi = \xi(n)$ in $\beta < \xi < \mu$. $(X - \xi)^-$ has an analytic characteristic function and $E(X - \xi) > 0$. It follows from Theorem B of [4] and the inequality

$H_m \leqq F_m$ that for any r in some interval $0 < r < R$,

$$\sum e^{rm} H_m(m\xi) < \infty, \tag{20}$$

and hence, considering the totality of series (20) as n varies,

$$H_n(n\xi) = o(e^{-rn}) \quad \text{as} \quad n \to \infty. \tag{21}$$

Therefore,

$$\sum_{n=1}^{\infty} e^{rn} K_n \leqq (\mu - \beta) \sum_{n=1}^{\infty} n e^{-n\beta s} H_n(n\xi) e^{rn}$$
$$< \infty \quad \text{as} \quad s \to \mu^{-1} r+$$

in view of (21) and so

$$\lim_{s \to \mu^{-1} r+} (s - \mu^{-1} r) \sum_{n=1}^{\infty} e^{rn} K_n = 0. \tag{22}$$

Next consider

$$L_n = \int_{n\mu}^{\infty} e^{-sx}\{1 - H_n(x)\}\, dx \qquad (s > 0).$$

Using the mean value theorem and the fact that $H_n(nx) \to 1$ as $n \to \infty$ for $x > \mu$, we may write

$$L_n = \delta_n \int_{n\mu}^{\infty} e^{-sx}\, dx = \delta_n s^{-1} e^{-n\mu s},$$

where $\delta_n > 0$ and $\delta_n \to 0$ as $n \to \infty$ uniformly in $s > 0$. Now given arbitrary $\epsilon > 0$ we can choose an integer $n_0(\epsilon)$ so large that $\delta_n < \epsilon$ for all $n > n_0$. Then, for $s > \mu^{-1} r$,

$$\sum_{n=1}^{\infty} e^{rn} L_n = s^{-1} \sum_{n=1}^{\infty} \delta_n e^{-n\mu(s-\mu^{-1}r)}$$
$$< s^{-1} \sum_{n=1}^{n_0} \delta_n e^{-n\mu(s-\mu^{-1}r)} + \epsilon s^{-1} \sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)},$$

and since for $s > \mu^{-1} r$,

$$\sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)} = e^{-\mu(s-\mu^{-1}r)}/(1 - e^{-\mu(s-\mu^{-1}r)}) \sim 1/\mu(s - \mu^{-1} r) \quad \text{as} \quad s \to \mu^{-1} r+,$$

it follows that

$$\lim_{s \to \mu^{-1} r+} (s - \mu^{-1} r) \sum_{n=1}^{\infty} e^{rn} L_n = 0, \tag{23}$$

so that combining (22) and (23),

$$\lim_{s \to \mu^{-1} r+} (s - \mu^{-1} r) \sum_{n=1}^{\infty} e^{rn} (L_n - K_n) = 0. \tag{24}$$

Now consider the function

$$\Phi_\beta(x) = \sum_{n=1}^{\infty} e^{rn} H_n(x) U(x - n\beta)$$
$$= \sum_{n=1}^{\infty} e^{rn} U(x - n\mu) - \sum_{n=1}^{\infty} e^{rn}\{U(x - n\mu) - H_n(x)\} U(x - n\beta).$$

Taking Laplace transforms, we have

$$\Phi_\beta^0(s) = s^{-1} \sum_{n=1}^{\infty} e^{-n\mu(s-\mu^{-1}r)} - \sum_{n=1}^{\infty} e^{rn}(L_n - K_n),$$

from which it follows, using (24), that $(s - \mu^{-1}r)\Phi_\beta^{\,0}(s) \to r^{-1}$ as $s \to \mu^{-1}r+$. Now let

$$\Theta_\beta^{\,0}(s - \mu^{-1}r) = \Phi_\beta^{\,0}(s); \tag{25}$$

we have $w\Theta_\beta^{\,0}(w) \to r^{-1}$ as $w \to 0+$, and appealing to a Tauberian theorem of Widder [7], Theorem 4.3, 192, we obtain

$$t^{-1}\int_0^t \Theta_\beta(x)\,dx \to r^{-1} \quad \text{as} \quad t \to \infty. \tag{26}$$

Then, following through exactly the analysis that we used in the case of Theorems 1 and 2 (we are here dealing with the case $\gamma = 0$, $L(t) = 1$, $\alpha\mu^{-\gamma}[\Gamma(\gamma + 2)]^{-1} = r^{-1}$) we find from (26) that

$$\lim_{t\to\infty} \Theta_\beta(t) = r^{-1}. \tag{27}$$

However, using (25) and the uniqueness theorem for Laplace transformations, $\Theta_\beta(t) = e^{-r\mu^{-1}t}\Phi_\beta(t)$, so that (27) gives

$$\lim_{t\to\infty} e^{-r\mu^{-1}t}\Phi_\beta(t) = r^{-1}. \tag{28}$$

Now write

$$\sum_{n=1}^{\infty} e^{rn}H_n(x) = \Phi_\beta(x) + \Psi_\beta(x), \tag{29}$$

where

$$\Psi_\beta(x) = \sum_{n=1}^{\infty} e^{rn}H_n(x)\{1 - U(x - n\beta)\}, \tag{30}$$

$\sum e^{rn}H_n(x)$ being finite for r in $0 < r < R$ by (20). Also it is clear from (30) and (20) that for all x, $\Psi_\beta(x) \leqq \sum_{n=1}^{\infty} e^{rn}H_n(n\beta) < \infty$. We therefore have $e^{-r\mu^{-1}x}\Psi_\beta(x) \to 0$ as $x \to \infty$, and hence by (28), $\sum_{n=1}^{\infty} e^{rn}H_n(x) \sim r^{-1}e^{r\mu^{-1}x}$ as $x \to \infty$. This establishes the sufficiency parts of Theorems 4 and 5.

The necessity parts of the theorems follow readily from Theorem B of [4]. We have $\sum e^{rn}H_n(x) < \infty$ for all x, $-\infty < x < \infty$ and Theorem B gives the result directly in the case $H_n(x) = F_n(x)$. In the case $H_n(x) = G_n(x)$, we take $x > 0$ and make use of the well known relations $G_n(x) = \Pr(M_n \leqq x) \geqq \Pr(M_n \leqq 0) \geqq n^{-1}\Pr(S_n \leqq 0)$. It is clear that the convergence of $\sum e^{rn}\Pr(M_n \leqq x)$ implies the convergence of $\sum n^{-1}e^{rn}\Pr(S_n \leqq 0)$ and hence that of $\sum e^{sn}\Pr(S_n \leqq 0)$ for $s < r$ and making use of Theorem B again we see that X^- must have an analytic characteristic function. This completes the proofs of both Theorems 4 and 5.

3. Application to a first passage problem. Let X_i, $i=1, 2, 3, \cdots$ be independent and identically distributed random variables with $E|X| < \infty$, $EX = \mu > 0$. Write $S_n = \sum_{i=1}^{n} X_i$ and $M_n = \max_{1\leqq k\leqq n} S_k$. Consider a single boundary at x $(\geqq 0)$ so that if

$$G_0(x) = 1,$$

$$G_n(x) = \Pr(M_n \leqq x), \qquad n \geqq 1,$$

the probability p_n that the first passage time, $M(x)$, out of the interval $(-\infty, x]$ for the process S_n is n is given by $p_n = G_{n-1}(x) - G_n(x)$, $n \geqq 1$.

We introduce the probability generating function $P(\lambda) = \sum_{r=1}^{\infty} \lambda^r p_r$ for the first passage time distribution $\Pr(M(x) = n) = p_n$.

Formally differentiating,

$$P^{(1)}(1) = E[M(x)] = 1 + \sum_{r=1}^{\infty} G_r(x),$$

and for $k > 1$,

$$\begin{aligned} P^{(k)}(1) &= (\alpha)_k \quad \text{(the } k\text{th factorial moment of } M(x)) \\ &= k \sum_{r=k-1}^{\infty} (r)_k G_r(x) = \sum_{r=0}^{k} s(k, r) E\{[M(x)]^r\}, \end{aligned}$$

where $(r)_k = r(r-1)(r-2)\cdots(r-k+1)$ and $s(k, r)$ are the Stirling numbers of the first kind. It is thus clear that $E\{[M(x)]^r\} < \infty$ for some positive integer r if and only if $\sum n^{r-1} G_r(x) < \infty$. Also, the random variable $M(x)$ has an analytic characteristic function if and only if the radius of convergence of $P(\lambda)$ is greater than unity or equivalently if $\sum e^{rn} G_n(x) < \infty$ for some $r > 0$.

As a particular case of Theorem 2 we obtain for integral $k \geqq 1$,

$$kx^{-k} \sum_{n=1}^{\infty} n^{k-1} G_n(x) \to \mu^{-k} \quad \text{as} \quad x \to \infty,$$

so long as $E|X^-|^{k+1} < \infty$. Therefore, in view of the above comments, we see that as $x \to \infty$, $E\{[x^{-1}M(x)]^r\} \to \mu^{-r}$ for integral $r \geqq 1$ so long as $E|X^-|^{r+1} < \infty$. (The $r = 1$ and $r = 2$ cases are included in the results of [2] and [1] respectively). If $E|X^-|^r = \infty$ then Theorem 3 shows us that $\sum n^{r-1} G_n(x)$ diverges and hence that $E\{[x^{-1}M(x)]^r\} = \infty$. If we have the condition that X^- possesses an analytic characteristic function, then it follows from Theorem B of [4] and the inequality, $\Pr(S_n \leqq x) \geqq \Pr(M_n \leqq x) = G_n(x)$, that $M(x)$ possesses an analytic characteristic function. We have, in fact, for $r > 0$ sufficiently small,

$$\begin{aligned} E[e^{rM(x)}] &= \sum_{n=1}^{\infty} e^{rn} P_n \\ &= e^r + (e^r - 1) \sum_{n=1}^{\infty} e^{rn} G_n(x), \end{aligned}$$

so that by Theorem 5,

$$E[e^{rM(x)}] \sim r^{-1}(e^r - 1) e^{r\mu^{-1}x} \quad \text{as} \quad x \to \infty.$$

The function $r^{-1}(e^r - 1)e^{r\mu^{-1}x}$ is the Laplace-Stieltjes transform of the convolution of a rectangular distribution on the interval $(0, 1)$ and a degenerate distribution at $x\mu^{-1}$. We have therefore obtained the following theorem:

THEOREM 6. *Suppose* $EX = \mu > 0$. *If for some integral* $r \geqq 1$, $E|X^-|^{r+1} < \infty$, *then*

$$E\{[x^{-1}M(x)]^r\} \to \mu^{-r} \quad \textit{as} \quad x \to \infty.$$

If X^- *possesses an analytic characteristic function then for* $r > 0$ *sufficiently small*

$$E[e^{rM(x)}] \sim r^{-1}(e^r - 1)e^{r\mu^{-1}x} \quad \textit{as} \quad x \to \infty.$$

One can further use a method due to Doob based on the strong law of large numbers to obtain the following result:

THEOREM 7. *Suppose* $EX = \mu > 0$. *Then,*

$$x^{-1}M(x) \rightarrow_{\text{a.s.}} \mu^{-1} \quad \text{as} \quad x \rightarrow \infty.$$

PROOF. According to the strong law of large numbers, we have for $0 < \epsilon < \mu$ and sufficiently large n, $(\mu - \epsilon)n \leqq S_n \leqq (\mu + \epsilon)n$ with probability one. In particular, if $n = M(x)$ the left hand side implies $M(x)/x \leqq 1/(\mu - \epsilon)$ and if $n = M(x) + 1$ the right side implies $M(x) + 1 \geqq x/(\mu + \epsilon)$. Thus, for large x, $1/(\mu + \epsilon) - 1/x \leqq M(x)/x \leqq 1/(\mu - \epsilon)$ with probability one. The result follows.

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REFERENCES

[1] CHOW, Y. S. (1966). On the moments of some one-sided stopping rules. *Ann. Math. Statist.* **37** 382–387.

[2] CHOW, Y. S. and ROBBINS, H. (1963). A renewal theorem for random variables which are dependent or non-identically distributed. *Ann. Math. Statist.* **34** 390–395.

[3] DOETSCH, G. (1950). *Handbuch der Laplace-Transformation* **1**. Verlag-Birkhauser, Basel.

[4] HEYDE, C. C. (1964). Two probability theorems and their application to some first passage problems. *J. Austral. Math. Soc.* **4** 214–222.

[5] LOÈVE, M. (1960). *Probability Theory* (2nd edition). Van Nostrand, New York.

[6] SMITH, W. L. (1964). On the elementary renewal theorem for nonidentically distributed random variables. *Pacific J. Math.* **14** 673–699.

[7] WIDDER, D. V. (1946). *The Laplace Transform.* Princeton Univ. Press.