# BOUNDED LENGTH CONFIDENCE INTERVALS FOR THE $p$-POINT OF A DISTRIBUTION FUNCTION, III[1]

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**1. Introduction.** Let $0 < p < 1$. A number $\gamma_{p,F}$ is a $p$-point of the distribution function $F$ if $F(\gamma_{p,F}) \geqq p$ and $F(\gamma_{p,F}-) \leqq p$. Given $L > 0$ and $0 < \alpha < 1$, a $L$-$\alpha$ bounded length confidence interval procedure operates successfully for $F$ means that when sampling stops an interval of length not exceeding $L$ is given which covers $\gamma_{p,F}$ with probability at least $1 - \alpha$. It is the purpose of this paper to give the construction of two $L$-$\alpha$ bounded length confidence interval procedures which operate successfully for all $F \varepsilon \mathbf{F}$. $\mathbf{F}$ will be the set of all distribution functions $F$ such that $\epsilon_F > 0$. $\epsilon_F$ is defined below in (1.10).

Throughout we let $\{X_n, n \geqq 1\}$ be a sequence of independently and identically distributed random variables such that $F$ is the distribution function of $X_1$. As we allow different choices of $F$, we indicate the choice when computing expectations by use of "$F$" as a subscript.

The procedures constructed are measurable functions of the random variables $\{X_n, n \geqq 1\}$. Two sets of functions $\{u_{i,n}, n \geqq 1\}$ and $\{v_{i,n}, n \geqq 1\}$, $i = 1, 2$, are constructed. Properties of the procedures are as follows:

(1.1) If $i = 1, 2$ and if $n \geqq 1$ then $u_{i,n}$ and $v_{i,n}$ are real valued Borel measurable functions defined on Euclidean $n$-space, and $u_{i,n}(x_1, \cdots, x_n) \leqq v_{i,n}(x_1, \cdots, x_n)$ for all $(x_1, \cdots, x_n)$ in $n$-space.

(1.2) If $i = 1, 2$ and if $n \geqq 1$ let random variables be defined by $U_{i,n} = u_{i,n}(X_1, \cdots, X_n)$ and $V_{i,n} = v_{i,n}(X_1, \cdots, X_n)$. There exist real numbers $\alpha$ and $\beta$ such that $0 < \alpha < 1$ and $0 < \beta < 1$ and such that if $F \varepsilon \mathbf{F}$ then

$$P_F(\text{all } n \geqq 1, U_{i,n} \leqq \gamma_{p,F}) \geqq 1 - \beta;$$

$$P_F(\text{all } n \geqq 1, V_{i,n} \geqq \gamma_{p,F}) \geqq 1 - \alpha.$$

(1.3) If $F$ is a distribution function and $\gamma_{p,F}$ is the unique $p$-point of $F$ then if $i = 1, 2$,

$$P_F(\lim_{n\to\infty} (V_{i,n} - U_{i,n}) = 0) = 1.$$

(1.4) If $f: (-\infty, \infty) \to (-\infty, \infty)$ is strictly increasing, then for $i = 1, 2$, $n \geqq 1$, and all $x_1, \cdots, x_n$,

$$u_{i,n}(f(x_1), \cdots, f(x_n)) = f(u_{i,n}(x_1, \cdots, x_n));$$

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$$v_{i,n}(f(x_1), \cdots, f(x_n)) = f(v_{i,n}(x_1, \cdots, x_n)).$$

(1.5) If $i = 2$ and $n \geqq 1$ then $u_{i,n} \leqq u_{i,n+1} \leqq v_{i,n+1} \leqq v_{i,n}$.

In this paper we will discuss various stopping variables. An integer valued random variable $N$ will be called a stopping variable (relative to $\{X_n, n \geqq 1\}$ if $N \geqq 1$ and the event $\{N \geqq i\}$ is independent of $\{X_n, n \geqq i\}$, $i \geqq 1$. A basic theorem is as follows.

THEOREM 1.1. *Let N be a stopping variable. If F is a distribution function then* $P_F(U_{i,N} \leqq \gamma_{p,F} \leqq V_{i,N}) \geqq 1 - \alpha - \beta$. *If F is a continuous distribution function then the numbers* $P_F(\text{all } n \geqq 1, U_{i,n} \leqq \gamma_{p,F})$ *and* $P_F(\text{all } n \geqq 1, V_{i,n} \geqq \gamma_{p,F})$ *do not depend on F. If N is a stopping variable and if G is obtained from F by translation of the random variables then*

$$P_G(U_{i,N} \leqq \gamma_{p,G} \leqq V_{i,N}) = P_F(U_{i,N} \leqq \gamma_{p,F} \leqq V_{i,N}).$$

PROOF.

$$1 - P_F(U_{i,N} \leqq \gamma_{p,F} \leqq V_{i,N}) \leqq P_F(\text{some } n \geqq 1, U_{i,n} > \gamma_{p,F}) + P_F(\text{some } n \geqq 1, V_{i,n} < \gamma_{p,F}) < \alpha + \beta.$$

To prove that various probabilities are independent of $F$, if $F$ has a continuous distribution function then we may choose a strictly increasing function $f$ such that if $Y_1$ has a uniform distribution on [0, 1] then $f(Y_1)$ has $F$ as distribution function. Then, if $\{Y_n, n \geqq 1\}$ is a sequence of independently and uniformly distributed random variables, if $i = 1, 2$,

$$\begin{aligned} P(\text{all } n \geqq 1, u_{i,n}(Y_1, \cdots, Y_n) \leqq p) &= P(\text{all } n \geqq 1, u_{i,n}(f(Y_1), \cdots, f(Y_n)) \leqq f(p)) \\ &= P_F(\text{all } n \geqq 1, U_{i,n} \leqq \gamma_{p,F}). \end{aligned}$$

Similarly for the proofs of the remaining parts of the theorem.

We shall usually choose $N$ to be the least integer $n$ such that $V_{i,n} - U_{i,n} \leqq L$, $i = 1, 2$. By virtue of (1.3), $P_F(N < \infty) = 1$ if $N$ is defined in this manner.

The functions $u_{i,n}$ and $v_{i,n}$, $i = 1, 2$, and $n \geqq 1$ are constructed from the order statistics. If $\{x_n, n \geqq 1\}$ is a real number sequence we let $x_{n,1} \leqq x_{n,2} \leqq \cdots \leqq x_{n,n}$ be the rearrangement of $x_1, \cdots, x_n$ into ascending order. We will use integer sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ satisfying

(1.6) if $n \geqq 1$ then $a_n > b_n$; $\lim_{n\to\infty} a_n/n = \lim_{n\to\infty} b_n/n = p$.

As an indication of method, we define $u_{1,n}$ and $v_{1,n}$, $n \geqq 1$, as follows:

$$(1.7) \qquad \begin{aligned} u_{1,n}(x_1, \cdots, x_n) &= x_{n,b_n+1}; \\ v_{1,n}(x_1, \cdots, x_n) &= x_{n,a_n}. \end{aligned}$$

The functions so defined clearly satisfy (1.1) and (1.4). (1.3) follows from Theorem 1.2 below.

THEOREM 1.2. *Let* $\gamma_{p,F,+} = \inf \{\gamma \mid F(\gamma) > p\}$ *and* $\gamma_{p,F,-} = \sup \{\gamma \mid F(\gamma) < p\}$. *Then* $1 = P_F(\liminf_{n\to\infty} X_{n,b_n+1} \geqq \gamma_{p,F,-}) = P_F(\limsup_{n\to\infty} X_{n,a_n} \leqq \gamma_{p,F,+})$.

PROOF. The following notation will be used throughout the paper:

(1.8) If $t$ is a real number and if $n \geqq 1$, then $S_n(t)$ is the number of integers $i$ such that $1 \leqq i \leqq n$ and $X_i \leqq t$.

Then $\{X_{n,b_n+1} \leqq t\}$ is the same event as $\{S_n(t) > b_n\}$ and $\{X_{n,a_n} > t\}$ is the same event as $\{S_n(t) < a_n\}$. Therefore $P_F(\liminf_{n\to\infty} X_{n,b_n+1} \geqq t) \geqq P_F(\text{all but a finite number of } n, S_n(t) \leqq b_n)$. $S_n(t)$ is a sum of $n$ independently and identically distributed Bernoulli random variables such that $P_F(S_1(t) = 1) = F(t)$. Therefore, if $t < \gamma_{p,F,-}$, then $F(t) < p$, and since $\lim_{n\to\infty} b_n/n = p$, it follows from the law of large numbers that $P_F(\text{all but a finite number of } n, S_n(t) \leqq b_n) = 1$. Since this holds for all $t < \gamma_{p,F,-}$, the first half of Theorem 1.2 follows. A similar argument proves the other half.

As observed earlier, if $N$ is the least integer $n$ such that $V_{1,n} - U_{1,n} \leqq L$ then $P_F(N < \infty) = 1$ will follow if (1.6) holds and if $\gamma_{p,F,+} - \gamma_{p,F,-} < L$.

In our introduction to methods, the last item is verification of (1.2). This has to do with proper choice of the integer sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$. The basic discussion is contained in Farrell [2] and [3]. It has been shown there that it is possible to choose sequences satisfying (1.6) such that for all distributions $F$,

$$\text{(1.9)} \qquad \begin{aligned} &\text{if } F(t) \leqq p \text{ then } P_F(\text{some } n \geqq 1, S_n(t) \geqq a_n) \leqq \alpha; \\ &\text{if } F(t) \geqq p \text{ then } P_F(\text{some } n \geqq 1, S_n(t) \leqq b_n) \leqq \beta. \end{aligned}$$

All that is needed in the sequel is (1.9). Therefore we refer the reader to Farrell [2] and [3] for method.

For method 1 we may now calculate that

$$\begin{aligned} P_F(\text{some } n \geqq 1, U_{1,n} > \gamma_{p,F}) &= P_F(\text{some } n \geqq 1, X_{n,b_n+1} > \gamma_{p,F}) \\ &= P_F(\text{some } n \geqq 1, S_n(\gamma_{p,F}) \leqq b_n) \\ &\leqq \beta. \end{aligned}$$

Similarly,

$$\begin{aligned} P_F(\text{some } n \geqq 1, V_{1,n} < \gamma_{p,F}) &= P_F(\text{some } n \geqq 1, X_{n,a_n} < \gamma_{p,F}) \\ &= P_F(\text{some } n \geqq 1, S_n(\gamma_{p,F}) \geqq a_n) \\ &\leqq \alpha. \end{aligned}$$

Therefore (1.2) follows.

The discussion shows that the first method will always give an interval covering $\gamma_{p,F}$ with probability not less than $1 - (\alpha + \beta)$. But closure of the stopping rule and more generally the expected sample size, depends on the flatness of $F$ about its $p$-points. In Farrell [3] we have defined

(1.10) $\epsilon_F = \sup_{0<\rho<1} \min (F(\gamma_{p,F} + \rho L) - p, p - F(\gamma_{p,F} + (\rho - 1)L))$.

The following theorem was proven there:

THEOREM 1.3. *Suppose* **F** *contains all* $F$ *having bimodal density functions which are continuous and everywhere positive. Suppose* $L > 0$ *and* $0 < \alpha < 1$, *and a* $L$-$\alpha$ *bounded length confidence interval procedure for* $\gamma_{p,F}$ *is given which works successfully for all* $F \varepsilon \mathbf{F}$. *If* $N$ *is the stopping variable for the confidence interval procedure then*

$$\limsup_{\epsilon_F \to 0+, F \varepsilon \mathbf{F}} (\epsilon_F)^2 (\log (\log (\epsilon_F)^{-1}))^{-1} E_F N \geqq 2(1 - 2\alpha)p(1 - \rho).$$

In Section 2 we detail the construction of the second procedure. In Section 3 we examine the question of the expected sample size. In particular we prove the following result.

THEOREM 1.4. *There exist choices of the sequences* $\{a_n, n \geqq 1\}$ *and* $\{b_n, n \geqq 1\}$ *such that if* $\mathbf{F} = \{F \mid \epsilon_F > 0$ *and* $F$ *is continuous*$\}$, *then*

$$\limsup_{\epsilon_F \to 0+} (\epsilon_F)^2 (\log (\log (\epsilon_F)^{-1}))^{-1} E_F N \leqq 4(1 - \alpha)p(1 - p).$$

It should be noted that the discrepancy between the constants in Theorems 1.3 and 1.4 is between $2(1 - 2\alpha)$ and $2(2 - 2\alpha)$. Therefore, the method of Section 2 is close to being minimax.

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**2. The second procedure.** We begin by considering tests of $\gamma_{p,F} < t$ against the alternative $\gamma_{p,F} > t$. Let the sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ satisfy (1.6). Take for a sequential procedure the following: Let $N_t$ be the least integer $n$ such that $S_n(t) \leqq b_n$ or $S_n(t) \geqq a_n$, $S_n(t)$ as in (1.8). If $S_{N_t}(t) \leqq b_{N_t}$ decide $t < \gamma_{p,F}$; if $S_{N_t}(t) \geqq a_{N_t}$ decide $t > \gamma_{p,F}$. By the law of large numbers,

$$\lim_{n\to\infty} S_n(t)/a_n = \lim_{n\to\infty} S_n(t)/b_n = F(t)/p.$$

Therefore, if $F(t) \neq p$, $P_F(N_t < \infty) = 1$.

We now consider performing simultaneously for all $t$, $-\infty < t < \infty$, the tests described above. As is easily seen, if $w < t$ and if $S_{N_t}(t) \leqq b_{N_t}$ then $S_{N_w}(w) \leqq b_{N_w}$; if $S_{N_w}(w) \geqq a_{N_w}$ then $S_{N_t}(t) \geqq a_{N_t}$. We define random variables

$$(2.1) \qquad U_{2,n} = \sup \{t \mid N_t \leqq n, S_{N_t}(t) \leqq b_{N_t}\};$$

$$V_{2,n} = \inf \{t \mid N_t \leqq n, S_{N_t}(t) \geqq a_{N_t}\}.$$

It is clear that the definitions (2.1) can be given in terms of a real number sequence $\{x_n, n \geqq 1\}$. Consequently explicit definitions of $u_{2,n}$, $v_{2,n}$, $n \geqq 1$ can be given by a parallel construction.

THEOREM 2. *Let the sequences* $\{a_n, n \geqq 1\}$ *and* $\{b_n, n \geqq 1\}$ *satisfy* (1.6).

*Using the above definitions,*

(2.2) *if* $n \geqq 1$ *then* $U_{2,n}$ *and* $V_{2,n}$ *are random variables*;

(2.3) *if* $n \geqq 1$ *then* $U_{2,n} \leqq U_{2,n+1} \leqq V_{2,n+1} \leqq V_{2,n}$ ;

(2.4) *if* $n \geqq 1$ *then* $U_{2,n} < V_{2,n}$ *if and only if*

$$U_{2,n} = \max_{1 \leqq i \leqq n} X_{i,b_i+1} < \min_{1 \leqq i \leqq n} X_{i,a_i} = V_{2,n} ;$$

(2.5) *if the p-point of F is unique then* $P_F(\lim_{n\to\infty} (V_{2,n} - U_{2,n}) = 0) = 1$;

(2.6) *given* $0 < \alpha < 1$ *and* $0 < \beta < 1$, *if in addition to* (1.6) *the sequences* $\{a_n , n \geqq 1\}$ *and* $\{b_n , n \geqq 1\}$ *satisfy* (1.9), *then*

$$P_F(\text{all } n \geqq 1, U_{2,n} \leqq \gamma_{p,F}) \geqq 1 - \beta;$$

$$P_F(\text{all } n \geqq 1, V_{2,n} \geqq \gamma_{p,F}) \geqq 1 - \alpha.$$

(2.7) *If* $f: (-\infty, \infty) \to (-\infty, \infty)$ *is a strictly increasing function, if for* $n \geqq 1$, $X_n^* = f(X_n)$, *and if for* $n \geqq 1$, $U^*_{2,n}$ *and* $V^*_{2,n}$ *are defined by* (2.1) *using* $\{X_n^*, n \geqq 1\}$ *instead of* $\{X_n , n \geqq 1\}$, *then if* $n \geqq 1$, $U^*_{2,n} = f(U_{2,n})$ *and* $V^*_{2,n} = f(V_{2,n})$.

PROOF OF THEOREM 2. We prove (2.3) first. By the definitions it is immediate that $V_{2,n} \geqq V_{2,n+1}$ and $U_{2,n} \leqq U_{2,n+1}$. Suppose that $N_t \leqq n$ and that $S_{N_t}(t) \leqq b_{N_t}$. Then if $i \leqq N_t$ it follows that $S_i(t) < a_i$. Since $S_i(\cdot)$ is a nondecreasing function, if $w < t$ then $S_i(w) < a_i$, $i = 1, \cdots, N_t$, and $S_{N_t}(w) \leqq b_{N_t}$. Therefore if $w < t$, $N_w \leqq N_t \leqq n$ and $S_{N_w}(w) \leqq b_{N_w}$. Similarly if $N_t \leqq n$ and $w > t$ then $N_w \leqq N_t$ and $S_{N_w}(w) \geqq a_{N_w}$. Therefore $U_{2,n} \leqq V_{2,n}$.

To prove (2.2), that is, the measurability of $U_{2,n}$ and $V_{2,n}$, let $A_1(t) = \{\text{sample points} \mid N_t \leqq n, S_{N_t}(t) \leqq b_{N_t}\}$. In the preceding paragraph we have shown that if $w < t$ then $A_1(w) \supset A_1(t)$. Then

$$\begin{aligned}\{\text{sample points} \mid U_{2,n} > a\} &= \{\text{sample points} \mid \text{some } t > a, N_t \leqq n \text{ and } S_{N_t}(t) \leqq b_{N_t}\} \\ &= U_{t>a,t \text{ rational}} A_1(t).\end{aligned}$$

Since this holds for all real numbers $a$, it follows $U_{2,n}$ is measurable, $n \geqq 1$. By a similar argument, $V_{2,n}$ is measurable, $n \geqq 1$.

To prove (2.4) it is first shown that $U_{2,n} \leqq \max_{1 \leqq i \leqq n} X_{i,b_i+1}$ and $V_{2,n} \geqq \min_{1 \leqq i \leqq n} X_{i,a_i}$. Suppose $t < U_{2,n}$. Then for some $i \leqq n$, $S_i(t) \leqq b_i$. Therefore, for some $i \leqq n$, $X_{i,b_i+1} > t$, i.e., $\max_{1 \leqq i \leqq n} X_{i,b_i+1} > t$. Since this holds for all $t < U_{2,n}$, the first inequality follows. Suppose $t > V_{2,n}$. Then for some $i \leqq n$, $S_i(t) \geqq a_i$, and $X_{i,a_i} \leqq t$. Therefore $\min_{1 \leqq i \leqq n} X_{i,a_i} \leqq t$. As this holds for all $t > V_{2,n}$, the second inequality follows. Suppose $U_{2,n} < t < V_{2,n}$. Then, if $1 \leqq i \leqq n$ it follows that $b_i < S_i(t) < a_i$, therefore that $X_{i,b_i+1} \leqq t < X_{i,a_i}$, and therefore that $\max_{1 \leqq i \leqq n} X_{i,b_i+1} \leqq t \leqq \min_{1 \leqq i \leqq n} X_{i,a_i}$. Since this holds for all $t$ satisfying $U_{2,n} < t < V_{2,n}$ it follows that $\max_{1 \leqq i \leqq n} X_{i,b_i+1} \leqq U_{2,n} < V_{2,n} \leqq \min_{1 \leqq i \leqq n} X_{i,a_i}$. That completes the proof of (2.4).

To prove (2.5) assume $\gamma_{p,F}$ is unique. If $U_{2,i} = V_{2,i}$ for some $i$ then for all $n \geqq i$, $U_{2,n} = V_{2,n}$. On the complementary event, for all $n \geqq 1$, $U_{2,n} < V_{2,n}$. Conditional on $U_{2,n} < V_{2,n}$ for all $n \geqq 1$ we have for $1 \leqq i \leqq n$ by (2.4) that $X_{i,b_i+1} \leqq U_{2,n} < V_{2,n} \leqq X_{i,a_i}$. By Theorem 1.2, with probability one, $\lim_{i\to\infty} X_{i,b_i+1} = \lim_{i\to\infty} X_{i,a_i} = \gamma_{p,F}$. Therefore conditional on $U_{2,n} < V_{2,n}$ for all $n \geqq 1$, with probability one, $\lim_{n\to\infty} (V_{2,n} - U_{2,n}) = 0$.

To prove (2.6), observe that $P_F(\text{for all } n \geqq 1, U_{2,n} \leqq \gamma_{p,F}) = 1 - P_F(\text{some } n \geqq 1, U_{2,n} > \gamma_{p,F})$. Since if $n \geqq 1$, $U_{2,n+1} \geqq U_{2,n}$ we find $P_F(\text{some } n \geqq 1, U_{2,n} > \gamma_{p,F}) = \lim_{n\to\infty} P_F(U_{2,n} > \gamma_{p,F})$. Next,

$$\begin{aligned} P_F(U_{2,n} > \gamma_{p,F}) &= P_F(\text{some } t > \gamma_{p,F}, N_t \leqq n, S_{N_t}(t) \leqq b_{N_t}) \\ &\leqq P_F(\text{some } t > \gamma_{p,F}, N_t < \infty, S_{N_t}(t) \leqq b_{N_t}) \\ &= \lim_{t\to\gamma_{p,F}^+} P_F(N_t < \infty, S_{N_t}(t) \leqq b_{N_t}) \\ &\leqq \lim_{t\to\gamma_{p,F}^+} P_F(\text{some } n \geqq 1, S_n(t) \leqq b_n). \end{aligned}$$

Using (1.9), we see that if $t \geqq \gamma_{p,F}$ then $F(t) \geqq p$ and $P_F(\text{some } n \geqq 1, S_n(t) \leqq b_n) \leqq \beta$. Therefore $\lim_{n\to\infty} P_F(U_{2,n} > \gamma_{p,F}) \leqq \beta$.

By a similar argument,

$$P_F(\text{some } n \geqq 1, V_{2,n} < \gamma_{p,F}) = \lim_{t\to\gamma_{p,F}^-} P_F(\text{some } n \geqq 1, S_n(t) \geqq a_n) \leqq \alpha.$$

The proof of (2.7) is obvious and details are omitted.

**3. The expected sample size.** In this section it will be convenient to distinguish between the two methods, method one of Section 1 and method two of Section 2. We suppose for the discussion that the same pair of sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ are used satisfying (1.6) and (1.9). We let $N_1$ be the least integer $n$ such that $X_{n,a_n} - X_{n,b_n+1} \leqq L$, and $N_2$ be the least integer $n$ such that $V_{2,n} - U_{2,n} \leqq L$. From (2.4) it follows at once that if $N_1 = n$ then either $U_{2,n} = V_{2,n}$ or $\min_{1 \leqq i \leqq n} X_{i,a_i} - \max_{1 \leqq i \leqq n} X_{i,b_i+1} \leqq X_{n,a_n} - X_{n,b_n+1} \leqq L$. Therefore $N_2 \leqq N_1$.

A basic theorem is

THEOREM 3.1. *Suppose there exists a number $\delta$, $0 < \delta < 1$, such that if $t_{-1} = \gamma_{p,F} - \delta L$ and $t_0 = \gamma_{p,F} + (1 - \delta)L$, then $F(t_{-1}) < p$ and $F(t_0) > p$. Then $N_1$ has a moment generating function in a neighborhood of 0.*

PROOF.

$$\begin{aligned} P_F(N_1 \geqq n) &\leqq P_F(\text{all } i, 1 \leqq i \leqq n, X_{i,b_i+1} \leqq t_{-1} \text{ or } X_{i,a_i} > t_0) \\ &\leqq P_F(X_{n,b_n+1} \leqq t_{-1} \text{ or } X_{n,a_n} > t_0) \\ &\leqq P_F(S_n(t_{-1}) > b_n) + P_F(S_n(t_0) < a_n). \end{aligned}$$

Under the hypothesis made, since $S_n(t_{-1})$ is a sum of independently and identically distributed Bernoulli random variables, $P(S_1(t_{-1}) = 1) = F(t_{-1}) < p$, and since similarly $P_F(S_1(t_0) = 1) = F(t_0) > p$, and since (1.6) holds, the sequences $\{P_F(S_n(t_{-1}) > b_n), n \geqq 1\}$ and $\{P_F(S_n(t_0) < a_n), n \geqq 1\}$ decrease at least geometrically. Therefore $\{P_F(N_1 \geqq n), n \geqq 1\}$ decreases at least geometrically which implies the existence of a moment generating function near zero.

In the remainder of this section we will suppose $F$ is a continuous distribution function. Then from the definition of $\epsilon_F$ in (1.10) it follows that for some $\rho_0$ satisfying $0 < \rho_0 < 1$,

$$\epsilon_F = F(\gamma_{p,F} + \rho_0 L) - p = p - F(\gamma_{p,F} + (\rho_0 - 1)L). \tag{3.1}$$

We will define numbers

$$t_0 = \gamma_{p,F} + \rho_0 L, \qquad t_{-1} = \gamma_{p,F} + (\rho_0 - 1)L. \tag{3.2}$$

The event $N_2 > n$ is the event $V_{2,n} - U_{2,n} > L$ and this implies the event $U_{2,n} < t_{-1}$ or $V_{2,n} > t_0$. Therefore

$$\begin{aligned} P_F(N_2 > n) &\leqq P_F(U_{2,n} < V_{2,n},\ U_{2,n} < t_{-1} \text{ or } V_{2,n} > t_0) \\ &\leqq P_F(\max_{1 \leqq i \leqq n} X_{i,b_i+1} \leqq t_{-1}) + P_F(\min_{1 \leqq i \leqq n} X_{i,a_i} > t_0) \\ &= P_F(\text{all } i,\ 1 \leqq i \leqq n,\ S_i(t_{-1}) > b_i) \\ &\quad + P_F(\text{all } i,\ 1 \leqq i \leqq n,\ S_i(t_0) < a_i). \end{aligned} \tag{3.3}$$

Therefore, if $N_a$ is the least integer $n$ such that $S_n(t_0) \geqq a_n$, and $N_b$ is the least integer $n$ such that $S_n(t_{-1}) \leqq b_n$, we find

$$P_F(N_2 > n) \leqq P_F(N_a > n) + P_F(N_b > n). \tag{3.4}$$

Summing this expression on $n$ gives

$$E_F N_2 \leqq E_F N_a + E_F N_b. \tag{3.5}$$

If $n \geqq 1$, $S_n(t)$ is a sum of $n$ independently and identically distributed Bernoulli random variables such that $P(S_1(t) = 1) = F(t)$. We bring to bear the results of Farrell [3]. The sequences $\{a_n, n \geqq 1\}$ and $\{b_n, n \geqq 1\}$ are chosen so that (1.6) and (1.9) hold.

By Theorem 1 of Farrell, op. cit., the sequences may be chosen in such a way that

$$\begin{aligned} &\lim_{\epsilon_F \to 0} (\epsilon_F)^2 (\log\ (\log\ (\epsilon_F)^{-1}))^{-1} EN_a = 2p(1 - p)(1 - \alpha); \\ &\lim_{\epsilon_F \to 0} (\epsilon_F)^2 (\log\ (\log\ (\epsilon_F)^{-1}))^{-1} EN_b = 2p(1 - p)(1 - \beta). \end{aligned} \tag{3.6}$$

Therefore the conclusion of Theorem 1.4, Section 1 follows by taking $\alpha = \beta$.

## REFERENCES


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