ABSTRACTS OF PAPERS

Abstracts of papers presented at the Western Regional meeting, Los Angeles, California, August 15–17, 1966. Additional abstracts appeared in the June issue and will appear in future issues.

5. Estimation in mixtures of two normal distributions. A. CLIFFORD COHEN, University of Georgia.

This paper is concerned primarily with the method of moments in dissecting a mixture of two normal distributions. In the general case, with two means, two standard deviations, and a proportionality factor to be estimated, the first five sample moments are required, and it becomes necessary to find a particular solution of a ninth degree polynomial equation that was originally derived by Karl Pearson (1894). A procedure which circumvents solution of the nonic equation and thereby considerably reduces the total computational effort otherwise required, is presented. Estimates obtained in the simpler special case in which the two standard deviations are assumed to be equal, are employed as first approximations in an iterative method for simultaneously solving the basic system of moment equations applicable in the more general case in which the two standard deviations are unequal. Conditional maximum likelihood and conditional minimum chi-square estimation subject to having the first four sample moments equated to corresponding population moments, are also considered. An illustrative example is included. (Received 2 June 1966.)

6. Some percentile estimators for Weibull parameters. SATYA D. DUBEY, Ford Motor Company.

A percentile estimator for the shape parameter of the Weibull distribution, based on the 17th and 97th sample percentiles, is proposed which is asymptotically about 66% efficient when compared with the MLE (maximum likelihood estimator) and is superior to an estimator proposed by Menon (*Technometrics* **5** 175–182) which uses all n observations of the sample. A two-observation percentile estimator, based on the 40th and 82nd sample percentiles, for the scale parameter when the shape parameter is unknown is asymptotically about 82% efficient when compared with the MLE. The 24th and 93rd sample percentiles yield asymptotically about 41% jointly efficient percentile estimators for both the scale and shape parameters in a class of two-observation percentile estimators when compared with their MLE's. Some other simple percentile estimators for these parameters are also briefly discussed. Finally, asymptotic properties of these estimators are investigated and their application in statistical inference problems is mentioned. (Received 30 May 1966.)

7. Maximum-likelihood estimation, from doubly censored samples, of the parameters of the first asymptotic distribution of extreme values. H. LEON HARTER and ALBERT H. MOORE, Aerospace Research Laboratories, Wright-Patterson AFB, and Air Force Institute of Technology, Wright-Patterson AFB.

Let X be a random variable having the first asymptotic distribution of smallest (largest) values, with location parameter u and scale parameter b, $b > 0$. The natural logarithm of the likelihood function of a sample of size n from such a distribution, the lowest r_1 and the highest r_2 sample values having been censored, is written down and its first and second partial derivatives with respect to the parameters are worked out. The likelihood equations, obtained by equating to zero the first partial derivatives, do not have explicit solutions, but an iterative procedure for solving them on an electronic computer is described. The asymptotic variances and covariances of the maximum-likehood estimators of the parameters are obtained by inverting the information matrix, whose elements are the negatives of the limits, as $n \to \infty$, of the expected values of the second partial derivatives, and tabulated for censoring proportions $q_1 = 0.0\ (0.1)\ 0.9$ from below and $q_2 = 0.0\ (0.1)\ (0.9 - q_1)$ from above. The asymptotic variances and covariances are compared with corresponding sample values obtained from a Monte Carlo study of 2000 samples each of sizes $n = 10$ and $n = 20$ from the distribution of smallest values. For single censoring from above, the mean square errors of the sample esimates are also compared with the variances of the best linear unbiased estimators and the mean square errors of the best linear invariant estimators. (Received 20 May 1966.)

8. A two-sided slippage test for Poisson variates. R. B. HORA, Purdue University, Indianapolis.

Several problems concerning the slippage of parameters have been considered by various authors. Among them are Paulson (*Ann. Math. Statist.* (1952)), Doornbos and Prins (*Indag. Math.* (1956) and (1958)), and Karlin and Truax (*Ann. Math. Statist.* (1960)). Consider $k(k \geqq 2)$ Poisson populations. A test which tests the hypothesis that they all have the same scale parameter against the alternatives that in one of them the parameter has slipped to the right while in another to the left is suggested. Bounds for the probability of making a correct decision are also obtained. (Received 10 May 1966.)

9. Bias and variance criteria for estimators and designs for fitting polynomial responses (preliminary report). M. J. KARSON, A. R. MANSON and R. J. HADER, North Carolina State University. (By title)

When models of the form $y_u = \sum_{i=0}^{d+k-1} \beta_i x_u^i + \epsilon_u$ $(u = 1, 2, \cdots, n)$, with the ϵ_u distributed independently with mean 0 and variance σ^2, are fitted by approximating functions of the form $\hat{y}_u = \sum_{i=0}^{d-1} b_i x_u^i$; a bias error arises due to the fact that $\hat{y}_u$ fails to represent the true model exactly. A set of b_i are obtained which minimize the squared bias integrated over a specific region of interest and subject to this requirement, minimize var $(\hat{y}_u)$ integrated over the same region of interest. For a particular d, the b_i and the corresponding minimum bias and minimum variance depend on k. Conditions on the design moments of the x_u are determined which for a given value of d, satisfy these criteria and give exactly the same sets of b_i for several values of k. Designs are generated from these conditions which allow protection against these values of k. The least squares b_i are the subset which are obtained by a particular choice of design moment conditions. Particular designs have been obtained for $d = 2$, $k = 1, 2, 3, 4$; $d = 3$, $k = 1, 2, 3$; $d = 4$, $k = 1, 2$. The extension of the above results to multi-dimensional polynomial responses is now in progress. (Received 3 June 1966.)

10. Linear processes are nearly Gaussian. C. L. MALLOWS, Bell Telephone Laboratories.

Let T denote the set of all integers, and suppose $Y = \{Y_t ; t \in T\}$ is a process of independent, identically distributed variables with common df $G(\cdot)$, where $E(Y_0) = 0$, $E(Y_0^2) = 1$. Let $\{a_u ; u \in T\}$ be a sequence of real numbers with $\sum_u a_u^2 = 1$. Then $X_t = \sum_u a_u Y_{t-u}$ defines a stationary linear process $X = \{X_t ; t \in T\}$ with $E(X_0) = 0$, $E(X_0^2) = 1$. Let $F(\cdot)$ be the df of X_0. We prove that if $\max_u a_u$ is small, then X is close to Gaussian, in the sense that (i) for each t, $\int (F(y) - \Phi(y))^2 \, dy \leqq g \max_u a_u$ where $\Phi(\cdot)$ is the standard Gaussian df, and g depends on $G(\cdot)$; (ii) for each finite set $\{t_1, \cdots, t_k\}$, $\{X_{t_1}, \cdots, X_{t_k}\}$ is close to Gaussian in a similar sense; and (iii) the process X is close to Gaussian in a somewhat restricted sense. (Received 16 May 1966.)

11. Selection procedures based on ranks: scale parameter case.

PREM S. PURI and MADAN L. PURI, University of California, Berkeley, and Courant Institute of Mathematical Sciences, New York University. (By title)

Let X_{ij} $(j = 1, 2, \cdots, n; i = 1, 2, \cdots, c)$ be independent samples from populations with continuous distribution functions $F(x - \mu/\sigma_i)$, $i = 1, 2, \cdots, c$. The paper is concerned with procedures based on the ranks of the observations for selecting from the above c populations (a) the "best t" populations without regard to the order; (b) the "best t" populations with regard to the order, and (c) a subset which contains all populations "as good or better than a standard one." The "bestness" of a population is characterized by its *scale* parameter. Large-sample methods are provided for computing the sample sizes necessary to guarantee a preassigned probability of a correct grouping (or ranking) under specified conditions on the ratios of scale parameters. The asymptotic efficiency of these procedures relative to the normal theory procedure (c.f. Bechhofer and Sobel, *Ann. Math. Statist.* (1954) 273–289) is shown to be the same as that of the associated tests (c.f. P. S. Puri and M. L. Puri, abstract No. 12). If the ratio of the sample sizes equals this efficiency, the two procedures are shown to have the same asymptotic performance. (Received 12 May 1966.)

12. Coding theorems for two-way channels in information theory.

SURESH C. RASTOGI, University of Maryland.

Only discrete finite channels with independent left and right input probability distributions are considered in this paper. A code (n, N_1, N_2, δ) for two-way channels is introduced, where n is the length of the left and right sequences being transmitted through the channel, N_1 and N_2 are the number of messages to be sent from left and right respectively and δ is the probability of the code. When n is one, the existence of a code (N_1, N_2, δ) for a memoryless two-way channel (MTWC) is established by the method of maximal coding and lower bounds for N_1 and N_2 are obtained. Next the existence of a code $(n, 2^{n(C_{Q_{20}}-\epsilon)}, 2^{n(C_{Q_{10}}-\epsilon)}, \delta)$ for MTWC is proved where $\epsilon > 0$ is arbitrary small number, n is sufficiently large, and $C_{Q_{20}}$ and $C_{Q_{10}}$ are certain positive constants for given Q_{20} and Q_{10}, the right and left input probability distributions respectively. Finally the weak converse to the coding theorem for MTWC is established and it is proved that a code (n, N_1, N_2, δ) satisfies $\log N_1 < (nC_{Q_{20}} + 1)/(1 - \delta)$ and $\log N_2 < (nC_{Q_{10}} + 1)/(1 - \delta)$. (Received 20 May 1966.)

13. On some nonparametric generalizations of Wilks' tests for H_M, H_{VC} and H_{MVC}, I.

PRANAB KUMAR SEN, University of North Carolina.

This paper is concerned with the nonparametric generalizations of the well-known likelihood ratio tests, proposed and studied by S. S. Wilks (*Ann. Math. Statist.* **17** (1946), 257) for the problem of compound symmetry of multivariate distributions. In this paper, the problem of testing nonparametrically the identity of locations of p interchangeable variates has been considered, and the theory has been utilized in the formulation of a class of nonparametric tests for analysis of variance in two way layouts. The proposed tests are shown to have some optimal properties as compared to the other nonparametric tests available in the literature. Some simple multiple comparison procedures are also considered. (Received 23 April 1966.)

14. On nonparametric simultaneous confidence regions and tests for the one criterion analysis of variance. PRANAB KUMAR SEN, University of North Carolina. (By title)

This paper deals with the nonparametric generalizations of the well-known T-method and S-method of multiple comparisons for the one criterion analysis of variance problem. Certain rank order statistics are used to formulate suitable nonparametric analogs of T- and S-methods of multiple comparisons and their various properties studied. It has been shown that the asymptotic efficiency of the proposed method is the same as that of the asymptotic Pitman efficiency of the two sample test based on the same rank order statistics, with respect to the Student's t-test. (Received 23 April 1966.)

15. Strong law of large numbers for branching processes. M. M. SIDDIQUI, Colorado State University.

Let Z_n , $n = 0, 1, \cdots$, denote the number of individuals in the nth generation of a branching process. The usual assumptions are made (see T. E. Harris, *The Theory of Branching Processes*, 1963). Consider some characteristic, T, of individuals, for example, the lifetime. If $Z_n = 0$, set $S_n = 0$, and if $Z_n = k \geqq 1$, set $S_n = T_{n1} + \cdots + T_{nk}$. We assume that $\{T_{nj}\}$ are mutually independent rv's each distributed as T_{01} , the T corresponding to the individual in the 0th generation. The following theorem is proved. Let $EZ_1 = m > 1, EZ_1^2 < \infty$, $ET_{01} = \lambda$, $ET_{01}^2 < \infty$, and define $W_n = Z_n m^{-n}$, $V_n = S_n m^{-n}$, $n = 0, 1, \cdots$. Then, the vector $(W_n , V_n) \rightarrow (W, \lambda W)$ almost surely, where W is a non-degenerate r.v. Also $P(\lim S_n/Z_n \rightarrow \lambda \mid W > 0) = 1$). (Received 16 May 1966.)

16. A second order exponential model for multidimensional dichotomous systems (preliminary report). RHETT F. TSAO, Harvard University and International Business Machine Corporation.

This paper is concerned with a second order exponential model for the study of a multidimensional dichotomous system. The model is a discrete multivariate probability distribution:

$$p(x_1 , x_2 , \cdots , x_m) = \exp \left(\sum_{i=1}^{m} \theta_i x_i + \sum\sum_{i=j} x_i x_j \theta_{ij} - \theta\right)$$

where: $x_i = 0$ or 1 for every i, parameters $-\infty < \theta_i , \theta_{ij} < \infty$ and θ is a function of parameters only, such that $\sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \cdots \sum_{x_m=0}^{1} p(x_1 , x_2 , \cdots , x_m) = 1$. Considering a sample from a population specified by this model, we proceed to study the maximum likelihood estimates of the parameters and the asymptotic tests in one-sample and two-sample problems. A numerical index is introduced to measure the divergence between two populations. The maximum likelihood estimates of the index of divergence is derived and its properties are discussed. (Received 2 June 1966.)

17. A Bayes sequential strategy for crossing a field containing absorption points. SHELEMYAHU ZACKS, Kansas State University.

The paper studies the form and characteristics of the optimal strategy for the crossing of N particles in two alternative paths, P. I and P. II where an unknown (random) number of absorption points are planted in each path. The model assumes that if a particle clashes with an absorption point it survives with probability s, $\theta < s < 1$. If a particle is absorbed, both the particle and the absorption point are destroyed. All absorption points act independently. The objective of the optimal strategy is to maximize the expected number of survivors. It is proven that the following sequential strategy of directing the particles to the paths is optimal (Bayes): Given the results of the first n ($n = 0, 1, \cdots, N - 1$) crossings, let M_n denote the number of particles directed to path P. I, $D_n^{(1)}$ – the number of particles absorbed in path P. I, and $D_n^{(2)}$ – the number of particles absorbed in path P. II. The conditional survival probabilities $\zeta_{n+1}^{(i)}$ $(M_n, D_n^{(1)}, D_n^{(2)})$, $i = 1, 2$, are computed for the $(n + 1)$st particle. It is proven that $\zeta_{n+1}^{(i)}$ $(M_n, D_n^{(1)}, D_n^{(2)})$ $(i = 1, 2; n = \theta, 1, \cdots, N - 1)$ are independent of the crossing strategy. These probabilities depend on the prior joint distribution of the number of absorption points (J_1, J_2), in each path before the crossing. It is then shown that the optimal (Bayes) sequential strategy for the $(n + 1)$st particle is to attempt crossing in path I if $\zeta_{n+1}^{(1)}$ $(M_n, D_n^{(1)}, D_n^{(2)}) \geqq \zeta_{n+1}^{(2)}$ $(M_n, D_n^{(1)}, D_n^{(2)})$, and in path II otherwise. The operating characteristics of this Bayes sequential strategy, such as the expected number of M_n, and the expected number of survivors, are studied too. (Received 16 May 1966.)

(Abstracts of papers to be presented at the Annual meeting, New Brunswick, New Jersey, August 30–September 2, 1966. Additional abstracts appeared in the June issue and will appear in future issues.)

5. Statistical determination of certain mathematical constants and functions using computors. Satya D. Dubey, Ford Motor Company. (By title)

With the availability of high-speed electronic computers it is now quite convenient to devise statistical experiments for the purpose of estimating certain mathematical constants and functions. The paper contains statistical formulas for estimating mathematical constants and functions like π, C (Euler's constant), e, $\psi^{(1)}$ (1), $\Gamma(x)$, $\ln x$, $B(x, y)$, arc tan x, $\Psi(p)$, polygamma functions, etc. Statistical estimates of these quantities may be used to construct desired confidence intervals for these parameters. Although numerical techniques are available to approximate these mathematical quantities very satisfactorily, a statistical approach to these problems seems to deserve mention in the scientific literature. Numerical illustrations are given in the paper which also give some indication of the effect of pseudo random numbers on the final results. Statistical procedures, considered in the paper, make extensive use of a high-speed electronic computer which may help to develop a positive attitude among theoreticians, promising students of mathematics, etc., toward the role of computers in the ever expanding scientific work.

6. An application of numerical integration techniques to statistical tolerancing. David H. Evans, General Motors Research Laboratories. (Introduced by John S. White.)

In order to set tolerances in manufacturing when the component distributions are to be specified the distribution of the response of the system is required. More precisely, let the component values be denoted by y_i, let the response be $X = f(y_1, y_2, \cdots, y_n)$, then, if the y_i are random variables from known distributions the lower order moments of X are required. It is assumed that f is known experimentally or analytically. We offer a new method of solution which in many circumstances has advantages over the well known methods. If the y_i are independent and normally distributed the moments of X are n-dimensional, symmetric integrals with the product of normal density functions as weight function. For numerical approximation Gaussian quadrature is indicated. The particular quadrature chosen requires evaluations of f for (1) all components at their means μ_i, (2) all except one at their means, and (3) all except two; the off mean components are for (2) $\mu_i \pm a\sigma_i$, and for (3) $\mu_i \pm b\sigma_i$, $\mu_j \pm b\sigma_j$. There is some latitude in the choice of a, b; $a = b = 3^{\frac{1}{2}}$ is a good choice. There are $2n^2 + 1$ evaluations of f required and the precision is fifth order. Generally, if a so called high-low tolerance evaluation is possible then this method is practicable and superior. (Received 26 May 1966.)

7. Maximum-likelihood estimators of the relative scale parameters of Type III and Type V populations. NILAN NORRIS, Hunter College.

For a Pearson Type III distribution, which designates the frequency with which a random variable X falls in the range dX, a measure of the relative variation of the population is the ratio of the population geometric mean to the population arithmetic mean as given by θ_2/θ_1. In random samples of n, the asymptotically efficient estimator of relative variation is G/A, where G is the sample geometric mean and A is the sample arithmetic mean. For a Pearson Type V distribution, representing the frequency with which a random variable X falls in the range dX, a measure of the relative variation of the population is the ratio of the population harmonic mean to the population geometric mean as given by θ_3/θ_2. In random samples of n from a Type V universe, the asymptotically efficient estimator of relative variation is H/G, where H is the sample harmonic mean and G is the sample geometric mean. Digamma and trigamma functions may be used to derive the variances of the two maximum likelihood estimators for the instances in which either one or both parameters of the respective distributions are to be estimated. (Received 6 June 1966.)

8. On age dependent branching processes. HOWARD J. WEINER, University of California, Davis.

At time $t = 0$ a cell is born, passes through n states, spends a random time in each state, and chooses the states in accord with a Markov chain until a particular state, the mitotic state, is completed. The parent cell is then replaced by k independent and identical new cells with probability $p_k \geqq 0$, $k \geqq 0$, and $m = \sum_{k=0}^{\infty} kp_k > 1$. For the case where the states are entered sequentially, the limiting fraction of cells in a state, the limiting age distribution of cells in a state, and related distributions are explicitly obtained. In the general case the limiting fraction of cells in a state, as $m \downarrow 1$ is evaluated, which also gives a new proof of a theorem by W. Smith. The numbers of cells in the states divided by their corresponding means all tend to the same Bellman-Harris random variable in quadratic mean as $t \to \infty$. The number of births by t, $N(t)$, the number of splits by t, and the number of cells alive at t each divided by their respective means tend to the same Bellman-Harris random variable in quadratic mean. As $t \to \infty$, $E[(N(t))^n] \sim C_n t^{2n-1}$, where the C_n may be recursively obtained. (Received 17 May 1966.)

9. Asymptotically admissible linear unbiased estimators (AALUE) for robust estimation of location and scale parameters (preliminary report). E. F. YHAP, IBM Research Center.

Let $f_\lambda(x - \theta/\sigma)/\sigma$, $\lambda = 1, 2, \cdots, m$, be m specified probability density functions for the random variable X. Let $(x_1, x_2, \cdots, x_n)$ be a random sample, and $(x_{(1)}, x_{(2)}, \cdots, x_{(n)})$ be the ordered random sample such that $x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(n)}$. Continuous weight functions $h^{(i)}(u)$, $0 \leqq u \leqq 1$, $i = 1, 2$, are derived under conditions guaranteeing the asymptotic normality and unbiasedness of the linear estimators: $\hat{\theta} = n^{-1} \sum_{j=1}^{n} h^{(1)}(j/n + 1)X_{(j)}$, and $\hat{\sigma} = n^{-1} \sum_{j=1}^{n} h^{(2)}(j/n + 1)X_{(j)}$. These estimators are shown to be admissible with asymptotic variance, over respective possible pdf "shapes" λ, as risk function. They are generalizations of the functions obtained by J. Jung (*Ark. für Mat.* Band 3, Nr. 15,199) for a single "shape" λ. The problem is formulated as a generalization of Jung's using the general method for constructing robust estimators given by A. Birnbaum (abstract, *Ann. Math. Statist.* **32** 622). Examples are given and compared with other estimators developed for efficiency-robust estimation. (Received 24 May 1966.)

(*Abstracts of papers to be presented at the European Regional meeting, London, England, September 5–10, 1966. Additional abstracts will appear in future issues.*)

2. Statistical estimation procedures for the burn-in process (preliminary report). Richard E. Barlow, Frank Proschan and Ernest M. Scheuer, University of California, Berkeley, Boeing Scientific Research Laboratories and The RAND Corporation.

The phenomenon of "infant mortality" has been observed in the analysis of life test data; it refers to a situation in which the failure rate is high during the early period of life and lower for a long period thereafter. The higher failure rate over the early period may result from contamination of the population of standard items by a small percentage of poor quality or defective items which tend to fail soon after they are put into operation. To identify and eliminate the contaminating defective items, a practical measure often adopted is to "burn-in" (i.e., put on test) all items for the period of infant mortality. Thus it is of importance to determine the period of time over which infant mortality persists and, more generally, to estimate the failure rate function. In this paper we exhibit the maximum likelihood estimate of the failure rate function assuming only that it be non-increasing (Grenander, Marshall and Proschan). We obtain a conservative upper confidence limit for the "stable" failure rate under even weaker assumptions. We consider complete, censored, and truncated samples. Our approach is to be contrasted with those in which it is assumed that the life distribution has a specified parametric form—Weibull, lognormal, mixture of exponentials, etc. (Received 21 April 1966.)

3. The theory of stationary point processes. Frederick J. Beutler and Oscar A. Z. Leneman, University of Michigan and Massachusetts Institute of Technology.

An axiomatic formulation is presented for point processes which may be interpreted as ordered sequences of points randomly located on the real line. Such concepts as forward recurrence times and number of points in intervals are defined and related in set-theoretic terms, and conditions established under which these random variables are finite valued. Several types of stationarity are defined, and it is shown that these (each requiring a kind of statistical uniformity over the entire real axis) are equivalent to one another. Stationarity does *not* imply that the intervals between points are either independently or identically distributed. Convexity and absolute continuity properties are found for the forward recurrence times of the stationary point process (spp). The moments of the number of points in an interval are described in terms of these distributions, which appear in series whose convergences are necessary and sufficient conditions for the finiteness of the moments. Local and global properties of the moments are related, and it is shown that any existent moment is an absolutely continuous function of the interval length. The distribution functions of the forward recurrences are related to the statistics of the point sequence and the interval times. Moment properties are also determined in terms of the latter. An ergodic theorem relates the behavior of individual realizations of the number of points to their statistical averages. Several classes of point processes are described, and stationarity verified where applicable, using the most convenient of the (equivalent) criteria for each case.

The preceding theory is applied to the problem of calculating moments and other process statistics. (Received 23 April 1966.)

4. Asymptotic distribution of likelihood ratio in testing against stochastic inequality (preliminary report). H. D. Brunk, University of Missouri.

Let independent random samples of sizes $n - d$ and d be taken from populations with distribution functions F and G respectively. The maximum likelihood estimates of F, G subject to $F \geqq G$ have been studied by Franck, Hanson, Hogg and Brunk (University of Missouri Mathematical Sciences Technical Report No. 11, 1965). The likelihood ratio test of $H_0 : F = G$ against the alternative $F \geqq G$, $F \neq G$ can be described in terms of these estimates. Under the null hypothesis, the distribution of the likelihood ratio, Λ, can be expressed in terms of the distribution of cycle lengths in a randomly chosen permutation of $1, 2, \cdots, n$ and multivariate hypergeometric distributions. A limiting distribution is obtained, suggesting the approximation of the distribution of $2 \log \Lambda$ (after a translation) by Bartholomew's linear combination of chi-squares (*Biometrika* **46** (1959) 36–48; page 40). (Received 25 May 1966.)

5. A multi-dimensional linear growth birth and death process. Paul R. Milch, U. S. Naval Postgraduate School, Monterey.

A special two-dimensional (homogeneous) linear growth birth and death process is discussed. The infinitesimal birth and death rates are $(m + n + 1)p$, $(m + n + 1)q$ $(p > 0, q > 0)$ and m, n, respectively, when the process is in state (m, n). This process belongs to the family of reversible stochastic processes. These permit the use of the spectral resolution of the identity of Hermitian operators to obtain a representation formula of the transition probability function. This representation formula is explicitly given as a semi-direct product of classical orthogonal polynomials. Specifically the Krawtchouk, Laguerre, and Meixner polynomials are involved. The probability generating function is also derived and can be used to compute higher moments of the process. The absorption process with infinitesimal birth and death rates $m + 1$, $n + 1$ and $(m + n + 1)p$, $(m + n + 1)q$ when the process is in state (m, n) is closely related to the above process, and permits derivation of results similar to above. All these results are extended to the corresponding multi-dimensional (homogeneous) linear growth birth and death process. The representation formula of the transition probability function is given as a semi-direct product of $N + 1$ orthogonal polynomials. N of these factors are Krawtchouk polynomials of varying arguments whose product can be regarded as a "Krawtchouk polynomial of order N" as it has many of the properties of ordinary Krawtchouk polynomials. The associated absorption process is also discussed. (Received 9 May 1966.)

(Abstracts not connected with any meeting of the institute.)

1. On a complete class of strategies in the "two-armed bandit" problem. Nagata Furukawa, Kyushu University.

Let $Z = (J, U_1)(J, U_2) \times \cdots \times (J, U_N)$ be the state space, where $J = (1, 2)$ and (J, U_n) denotes the state space of $(z_{n1}, z_{n2}) = (x_n, y_n)$, let $p_{\omega,\theta}(z)$ be the probability density of $z = (z_{1j_1}, z_{2j_2}, \cdots, z_{Nj_N})$, where $j = 1$ or 2, wrt measure λ on Z, and let $\xi(\omega, \theta)$ be *a priori* pd of (ω, θ) wrt measure φ on $\Omega = \{(\omega, \theta)\}$. Consider the domains D_{n1} and D_{n2} such that $D_{n1}, D_{n2} \subset (J, U_1) \times (J, U_2) \times \cdots \times (J, U_n)$, $D_{n1} \cap D_{n2} = \phi$, $D_{n1} \cup D_{n2} = (J, U_1) \times (J, U_2) \times \cdots \times (J, U_n)$ for $n = 1, 2, \cdots, N$, and let S_{n1} and S_{n2} be the cylinder sets over D_{n1} and D_{n2} in Z respectively. Let L_1 and L_2 denote the loss when x and y are taken respectively. Then the risk in the "two-armed bandit" problem truncated at the $(N + 1)$th stage is given by

$$\rho_{N+1}(\xi) = \sum_{r=0}^{N} \sum_{i_0=1}^{2} \sum_{i_1=1}^{2} \cdots \sum_{i_r=1}^{2} \int_{S_{0 i_0} \cap S_{1 i_1} \cap \cdots \cap S_{r i_r}} \int_{\Omega} L_{i_r}(\omega, \theta)\xi(\omega, \theta)p_{\omega,\theta}(z)\, d\varphi d\lambda.$$

The main theorems are as follows. THEOREM 1. *The set of strategies depending on a posteriori* pd *of* (ω, θ) *constitutes a complete class.* THEOREM 2. *Let* ρ^* *be the risk function attained by an optimal strategy, and let* $T^X\xi$ *and* $T^Y\xi$ *be a posteriori* pd *given observations* x *and* y *respectively. Then* $\rho_{N+1}^*(\xi) = \min [E(L_1(\omega, \theta)) + E[\rho_N^*(T^X\xi)], E(L_2(\omega, \theta)) + E[\rho_N^*(T^Y\xi)]]$. These results give a reasonable foundation for the works of Bradt, Johnson and Karlin (*Ann. Math. Statist.* **27** (1956) 1060–1074) and of Feldman (*Ann. Math. Statist.* **33** (1962) 847–856). (Received 18 April 1966.)

2. On a monotonicity property associated with the moment inequality. LEON JAY GLESER, Johns Hopkins University.

For a probability space $(\mathfrak{X}, \mathfrak{B}, P)$ having the probability measure P defined on the sigma-algebra $\mathfrak{B}$, and for any measureable function f mapping $\mathfrak{X}$ into the non-negative real numbers, define the rth moment of f to be $\mu_r = \int_{\mathfrak{X}} f^r(x)\, dP(x)$. Then the well-known moment inequality says that $\mu_r^{1/r}/\mu_s^{1/s} \geqq 1$ if $r \geqq s$. A property which apparently has gone unnoticed previously is a certain monotonicity of the ratio $\mu_r^{1/r}/\mu_s^{1/s}$, namely: THEOREM. *For any* $A \in \mathfrak{B}$ *define* $P(\cdot \mid A)$ *as the conditional probability measure given* A. *Define the* rth *conditional moment* $\mu_r(A) \equiv \int_{\mathfrak{X}} f^r(x)\, dP(x \mid A)$. *Then for* $r > s$, $rs > 0$, A *and* $A' \in \mathfrak{B}$, $A' \subset A$, *we have* $\mu_r^{1/r}(A)/\mu_s^{1/s}(A) \geqq \mu_r^{1/r}(A')/\mu_s^{1/s}(A')$. This theorem is proven in the paper. A discussion of one or two of its more-interesting implications for discrete spaces is also included in the paper. (Received 9 May 1966.)

3. A concept of sufficiency for the sequential design of experiments (preliminary report). K. B. GRAY, JR., University of California, Los Angeles.

The problem of sequential design of experiments, with optional stopping, is considered. Conditions are given that allow experiment selection, stopping, and terminal action rules to be based on a sequence $\{T_j\}$ of statistics, where T_j is a function of past observations $\mathbf{X}^j = (X_1, \cdots, X_j)$ and experiment selections $\mathbf{E}^j = (E_1, \cdots, E_j)$. Randomized rules are allowed and it is assumed that all probability distributions are densities over finite sets. Upon stopping, and taking action a, a loss $L(\theta, a)$, where θ is unknown state of nature, is accrued. The sampling cost of stopping at j is $C_j(\theta, \mathbf{X}^j, \mathbf{E}^j)$. Assume that $L \geqq 0$ and $C_j \geqq 0$. Let N be the random stopping time. A selection rule γ, stopping rule ϕ, and terminal action rule δ are defined so that they, together with a population distribution assumption, completely determine the probability structure of the problem. Define $\{T_j\}$ to be *parameter sufficient* (PARS) if, for $j = 1, 2, \cdots$, $\text{Dist}_{\theta,\phi,\gamma}(\mathbf{X}^j, \mathbf{E}^j \mid T_j, N \geqq j)$ is independent of θ for all ϕ, γ and *policy sufficient* (POLS) if, for $j = 0, 1, 2, \cdots$, $\text{Dist}_{\theta,\phi,\gamma}(T_{j+1} \mid T_j, E_{j+1}, N \geqq j + 1)$ is independent of ϕ, γ for all θ. THEOREM. *If* $\{T_j\}$ *is* PARS, *then the class of policies* $\{\phi, \gamma, \delta^0\}$, *where* δ^0 *is based on* $\{T_j\}$, *is essentially complete.* THEOREM. *If* $\{T_j\}$ *is* PARS *and* POLS, *and the sampling cost is of the form* $C_j(\theta, T_j)$, *then the class of policies* $\{\phi^0, \gamma^0, \delta^0\}$, *where* ϕ^0, γ^0, δ^0 *are based on* $\{T_j\}$, *is essentially complete.* (Received 21 April 1966).

4. Investigation of rejection rules for outliers in small samples from the normal distribution (preliminary report). IRWIN GUTTMAN and DENNIS E. SMITH, University of Wisconsin.

Suppose an experimenter takes a sample of n observations, hopefully all from $N(\mu, \sigma^2)$, but where a spurious observation from (i) $N(\mu + a\sigma, \sigma^2)$ or (ii) $N(\mu, (1 + b)\sigma^2)$ may be present in the sample. Three rejection rules, which provide an unbiased estimate in the null case, are investigated via the "premium-protection" approach of Anscombe (*Technometrics* **2** (1960) 123–147), (a) when concerned with estimating μ, with σ^2 known, and (*b*) when concerned with estimating σ^2, with μ known. Each of these rules is composed of a statistic t and a rejection region R. If $t \not\in R$, each rule uses estimator $\bar{y}$, case (a), or Ds^2, case (b), where D is a constant required by the constraint of unbiasedness. If $t \in R$, these rules differ, and may be described as follows: (1) Anscombe's rule: If $t \in R$, the suspect observation is discarded, and estimation proceeds using the remaining $(n - 1)$ observations as a "new" sample. (2) The Winsorization rule: If $t \in R$, the suspect observation is given the value of its nearest neighbor, and esimation proceeds using this "new" sample of n observations. (3) The Semiwinsorization rule: If $t \in R$, the statistic t is given the value of the nearest rejection boundary. Estimation proceeds, subject to the constraint imposed by this procedure. In general, computation of premiums and protections for a sample of size n involves $(n - 1)$-fold integrals. For $n = 3$, the resultant double integrals are evaluated by numerical itegration, while for the other values of n considered $(4 \leqq n \leqq 10)$, Monte Carlo computations are used. (Received 29 April 1966.)

5. A non-parametric test for the bivariate two sample location problem, I: null case (preliminary report). K. V. MARDIA, University of Newcastle. (Introduced by R. L. Plackett.)

Suppose $(x_i, y_i), i = 1, \cdots, m$, and $(x_j', y_j'), j = 1, \cdots, n$, are two independent random samples from bivariate populations with continuous distribution functions $F_1(x, y)$ and $F_2(x, y)$ respectively. An unconditional non-parametric test U^2 is developed to test $F_1(x, y) = F_2(x, y)$ against $F_1(x, y) = F_2(x + a, y + b)$. The test U^2 is based on uniform circular scores and the location vector. It is shown that the test is invariant under the general linear transformation. The complete null distribution of U^2 for $N \leqq 15$, and critical points for $N \leqq 18$, where $N = m + n$, are obtained, using a digital computer (KDF 9). The null distribution of U^2 is shown to be distributed asymptotically as χ_2^2. It is seen that even for small values of m and n, the null distribution is approximated satisfactorily by both the χ_2^2-distribution and the modified F-distribution with d and $(N - 3)d$ degrees of freedom where $d = 1 + [N(N + 1) - 6mn]/N(m - 1)(n - 1)$. However the χ_2^2-distribution gives a better approximation. Extensions of the test to the multivariate k-sample location problem are also considered. (Received 18 April 1966.)

6. A non-parametric test for the bivariate two sample location problem, II: asymptotic power and efficiency relative to Hotelling's T^2 (preliminary report). K. V. MARDIA, University of Newcastle. (Introduced by R. L. Plackett.)

Suppose in the notation of the previous abstract, $Lt(m/N) = k$ where $0 < k < 1$ and $E_1(x, y) = F_2(a + aN^{-\frac{1}{2}}, y + bN^{-\frac{1}{2}})$. Under some very mild conditions it is shown by extending a theorem of Chernoff and Savage (*Ann. Math. Statist.* **29** (1958) 972–974) that U^2 is distributed asymptotically as non-central χ_2^2. The test is shown to be consistent. The relative asymptotic efficiency $e(U^2, T^2)$ of this test compared with the Hotelling bivariate T^2 test is obtained. When the populations are $N[(\mu_1, \nu_1), \Sigma]$, and $N[(\mu_2, \nu_2), \Sigma]$, in which case $e(U^2, T^2) = \pi/4$. For the Cramér type distribution (*Mathematical Methods of Statistics*, p. 279) $e(U^2, T^2)$ becomes equal to $E(x^2)[g(\infty) - g(0) - E(1/x)]^2/4$ which is always greater than 0.25 when $g(\infty) = 0$. It is shown with the help of an example that $e(U^2, T^2)$ can tend to infinity. (Received 18 April 1966.)

7. A non-parametric test for the bivariate two sample location problem, III: small sample power in the normal case (preliminary report). K. V. MARDIA, University of Newcastle. (Introduced by R. L. Plackett.)

The tests U^2 and T^2 are invariant under general linear transformations so that we take alternatives as $N[(\delta, 0), I]$, where for notations see the second abstract above. A formal expression for the power function is obtained in this case but even for small samples, the expression cannot be simplified and computed. An approximation is developed which is expected to be good even for small samples. The empirical power function of U^2 is obtained for various δ, m and n by using pseudo-random numbers. The power of T^2 at non-standard levels was obtained by a routine (procedure) depending on the recurrence relation $I_x(p, q+n+1) = [1 + (n+p+q-1)(1-x)/(n+q)]I_x(p, q+n) - (n+p+q-1) \cdot (1-x)I_x(p, q+n)/(n+q)$ where $I_x(\cdot\,,\cdot)$ is the incomplete beta function. For this bivariate case, our routine is convenient and faster than the general non-central T^2 routine developed by Bargmann and Ghosh (IBM Research Report No. RC-1231). The Lehmann-Hodges efficiency (*Ann. Math. Statist.* **27** (1956), 324–335) is extended to cover the case of $m \neq n$. From the sampling trials on U^2 and T^2, it is found that the efficiency decreases with N and lies between 79 and 94 per cent. (Received 18 April 1966.)

8. Some contributions to contingency-type bivariate distributions (preliminary report). K. V. MARDIA, University of Newcastle. (Introduced by R. L. Plackett.)

Plackett (*J. Amer. Statist. Assoc.* **59** (1965) 516–522) has given a class of (contingency-type) bivariate distributions determined by solving a quadratic equation and discarding one of the roots not satisfying a set of inequalities. We have shown that the relevant root, in the notation of the reference, is given by $H = [1 + (F+G)(\psi - 1) - \sqrt{[1 + (F+G)(\psi-1)]^2 - 4\psi(\psi-1)FG}]/2(\psi - 1)$. For given F and $K = H(y \mid x)$, we obtain a quadratic equation in G and show that the relevant root simply depends on the sign of $(2K - 1)$. We show that under mild conditions, for any continuous distribution, $\text{Cov}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H - FG)\, dx\, dy$ and $V(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} F(y)[1 - F(x)]\, dx\, dy$, and utilize these for obtaining Cor (x, y) for contingency-type normal and rectangular distributions. Higher moments of the rectangular distribution are also obtained on extending the above formulas. We compare, as in the reference, the standard bivariate normal distribution with the contingency-type bivariate normal, using Cor (x, y) as well as Cor (F, G). We observe that the contingency-type normal distribution provides a new method of finding the tetrachoric correlation. We give a new estimate of ψ, dependent on Cor (F, G), and show that in the general case, the asymptotic efficiency of Plackett's estimate relative to ours, in the region of interest, lies between 46 and 56 per cent. Other estimates are also considered. (Received 18 April 1966.)

9. Orthogonal main-effect plans permitting estimation of all two-factor interactions for the 2^n3^m factorial series. BARRY H. MARGOLIN, Harvard University.

Relaxation of the restrictions of equal frequency of the factor levels and orthogonality of all the estimates permits considerable savings in the number of runs required for estimability of all main effects and two-factor interactions under the assumption of no higher order interactions for the 2^n3^m factorial series. Plans are presented for various values of (n, m), $4 \leqq n + m \leqq 10$, which provide orthogonal estimates of all main effects and allow estimation of all two-factor interactions (i.e., orthogonal main-effect resolution V plans).

The plans are constructed by: (i) replacement or collapsing of factors in symmetrical fractional plans, (ii) joining fractional designs with different defining relations, or (iii) combinations of (i) and (ii). The plans have a higher degree of freedom efficiency than the comparable plans existing in the literature, while maintaining reasonable average variances of the estimates. The construction techniques above are also used to construct resolution IV plans providing orthogonal estimates of all main effects for the $2^n 3^m$ series. Blocking, systematic methods of analysis, and extensions are discussed. (Received 23 May 1966.)

10. On a generalized hypergeometric distribution.
A. M. MATHAI and R. K. SAXENA, McGill University.

A general family of statistical probability distributions, in terms of a hypergeometric series (Erdelyi, et al, *Higher Transcendental Functions*, Vol. I, p. 56) is given. Almost all classical continuous univariate probability distributions are obtained as special cases from this general family. The distribution function, the characteristic function, the distribution of the sample, mean of a simple random sample the distributions of some order statistics and the distribution of the ratio of two independent variates having the probability functions in this family of probability distributions are obtained in terms of Fox's H-function (Fox, C. The G & H-functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* **98** 395–429). Some special properties enjoyed by this general distribution are also pointed out. (Received 23 May 1966.)

11. Distributions having the gamma properties.
A. M. MATHAI and R. K. SAXENA, McGill University.

It is well known that the distribution of the sample mean, for a simple random sample from a gamma distribution, has again a gamma distribution with the parameters scaled by the sample size. It is investigated whether a gamma distribution can be characterized by this property and it is shown that this property is enjoyed by a number of classes of probability distributions which come under the categories of the parabolic cylinder function distributions, the generalized hypergeometric distribution, and the Bessel function distributions. (Received 23 May 1966.)

12. On some rank tests for homogeneity of scale parameters.
PREM S. PURI and MADAN L. PURI, University of California, Berkeley, and Courant Institute of Mathematical Sciences, New York University.

Let X_{ij} $(j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, c)$ be independent samples from populations with continuous distribution functions $F_i(x) = F(x - \mu/\sigma_i)$, $i = 1, 2, \cdots, c$. Then for testing the hypothesis of the equality of scale parameters, a class of tests based on statistic $\mathcal{S} = \sum_{i=1}^{c} n_i[(S_{N,i} - \nu_{N,i})/L_N]^2$ is proposed. Here $S_{N,i} = 1/n_i \sum_{l=1}^{N} E_\Psi[V^{(l)} - \bar{V}^{(s)}]Z_{N,l}^{(i)}$ where $V^{(1)} < V^{(2)} < \cdots < V^{(N)}$ is an ordered sample of size $N = \sum_{i=1}^{c} n_i$ from a distribution Ψ; E denotes the expectation and $\bar{V}^{(i)} = 1/n_i \sum_{l=1}^{N} E_\Psi[V^{(l)}]Z_{N,l}^{(i)}$; and where $Z_{N,l}^{(i)} = 1$ if the lth smallest observation from the combined sample of size N is from the ith sample, and zero otherwise ($\nu_{N,i}$ and L_N are certain normalising constants). Under certain regularity conditions, it is shown that $\mathcal{S}$ has asymptotically a noncentral chi-square distribution for nearby alternatives. Efficiency properties of the $\mathcal{S}$-test as compared to Bartlett's test are studied. The case where the location parameter differs from distribution to distribution is also considered. (Received 12 May 1966.)

13. Contributions to the theory of non-normality—II. K. SUBRAHMANIUM, University of Western Ontario.

This paper is a sequel to "Contributions to the theory of Non-Normality-I" (Univariate Case)" by the author (Abstract: *Ann. Math. Statist.* **36** 1906. Paper to appear in *Sankhya Ser. A*, **28**). The distribution functions of quadratic forms of the central and noncentral type have been obtained. These are generalizations of the results obtained by Gurland (*Ann. Math. Statist.* **26** 122–127) and by Shah (*Ann. Math. Statist.* **34** 186–190). Using the notation introduced in the earlier paper the probability density functions (pdf) of χ^{*2} and $\chi^{*2}_{n.c.}$ are: $f_n(\chi^{*2}) = \sum_{i=0}^{3} p_i f_{n+2i}(\chi^2)$, $f_n(\chi^{*2}_{n.c.}) = \sum_{i=0}^{6} p_i' f_{n+2i}(\chi^2_{n.c.})$, where $f_j(\chi^2)$ and $f_j(\chi^2_{n.c.})$ are respectively the pdf's of central and noncentral χ^2 with j degrees of freedom. An elementery property of the χ^2 pdf is used here:

$$(D + 1)^j f_k(x) = f_{k-2j}(x) \qquad (k \geqq 2j + 1)$$

with $D = (d/dx)$ and $x = \chi^2/2$. This result is readily established both for central and noncentral case. In the above formula p_i and p_i' are functions of λ_3, λ_4 and μ_i. (Received 1 June 1966.)