K. Ito and H. P. Mc Kean, Jr., *Diffusion Processes and their Sample Paths*, Academic Press, New York and Springer-Verlag, Berlin, 1965. xvi + 321 pp. $14.50.

Review by F. Spitzer

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This is the first systematic account of the theory of Brownian motion and its extensions and applications since the appearance of P. Lévy's monograph twenty years earlier. The subject then seemed exciting and important as an intuitive model of many phenomena in semigroups, differential equations and potential theory. Today this is no longer a mere article of faith among Brownian motion enthusiasts, but rather the result of a unified theory which cuts across the borderlines of formerly separate areas of analysis. Therefore the organization of the material for this book was a non-trivial task which the authors accomplished with skill and judgement. In outline, the book consists of three parts. One-dimensional Brownian motion is treated in Chapters 1–2, general one-dimensional diffusion in Chapters 3–6, and in Chapter 7 classical potential theory is developed via the theory of n-dimensional Brownian motion.

The book begins with random walk on the integers (a good opportunity, even for the expert, to learn the notation), introduces Brownian motion, its zeros, continuity properties, absorption probabilities, and Chapter 1 terminates with F. Knights approximation of Brownian motion by random walks, which implies Donsker's invariance principle. Chapter 2 further develops the sample function properties, with emphasis on Lévy's local time, Trotter's theorem and recent refinements. Chapter 3 introduces one-dimensional diffusion (defined as a real valued, continuous, strictly Markovian process), decomposes its state space into intervals with singular end points, and introduces Dynkin's useful infinitesimal generator (in $(T_t - I)/t$ the time t is replaced by the expectation of a family of absorption times which tend to zero). In Chapter 4 these generators are explicitly calculated, resulting in the classification of all possible diffusions according to their speed measure and scale, which in turn determine the generator. The generators are shown to be generalized second derivative operators with respect to the speed and scale—this is the fundamental result of Feller, whose diffusion theory is here simplified by full use of the probability interpretation of the speed and scale. In Chapter 5 the principal result, which here appears for the first time, is that ordinary Brownian motion can be transformed into any diffusion whatever, by a suitable random time change which depends on the local time studied in Chapter 2. This beautiful result has many interesting applications. In particular it permits the complete solution of old problems concerning the most general Brownian motion with elastic barriers, which in part served as the motivation for Feller's diffusion theory. Finally Chapter 6 is devoted to the fine structure of the sample paths of the general linear diffusion.

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While linear diffusion theory is a more or less closed subject, apart from its applications, the classification of n-dimensional diffusions gives rise to entirely new, largely unsolved problems, which are sketched in Chapter 8. There is however a highly developed theory directly associated with n-dimensional Brownian motion, that is the classical Newtonian potential theory when $n \geqq 3$, and the logarithmic theory when $n = 2$. These form the subject of Chapter 7, in the reviewer's opinion, the most beautiful part of the book. The treatment includes all the famous classical theorems of the Newtonian theory and even the essentials for the logarithmic case (Hunt's proof of the existence of a Green function for bounded domains). Included are classical counter examples (Lebesgue's thorn and Littlewood's crocodile), Ikeda's solution of the Neumann problem, and interesting applications of Brownian motion on Riemann surfaces. All this is done in sixty pages, and while the proofs may be hard, every result has an immediate intuitive probability interpretation. It would be a worthwhile job for someone to expand this chapter so as to make it accessible to a wider audience, since, in fact, only minor additions and modifications are required to render it entirely independent from Chapters 2–6.

The reviewer is sufficiently awed by the concise and unusual style of this book to feel that some comment is required. In the interest of speed, elegance, or economy of expression the authors have refrained from the customary formal labelling of key statements as definitions, theorems, or lemmas. Instead every statement and formula is labelled by simple consecutive numbering. Thus a great deal of the usual redundancy is eliminated, without any resulting tendency toward ambiguity, in particular since the authors have devised an extremely purposeful but sophisticated system of notation. There are drawbacks however, apart from the obvious one that the novice will find it hard to guess which of the results he encounters are important for the sequel. To illustrate, consider the scale and the speed measure, defined in Sections 4.1 and 4.2 in the course of the classification of diffusions. Their principal properties are derived in the process of this classification, thus not as a goal in itself but rather as a casual by-product of the calculations in progress. In the course of later developments, however, one often has occasion to use the scale and speed measure. Typically one has a diffusion whose generator is known and requires certain absorption probabilities or expectations which are determined by the speed and scale. This occurs twice in Motoo's proof of the iterated logarithm theorem (Section 4.12) and at the beginning of Section 7.1 where one requires the expected exit time of Brownian motion from the n-dimensional sphere. Yet it is hard to find explicit statements in Chapter 4 which explain how the speed and scale determine the expectations in questions, nor is it stated how one can determine the speed and scale of a diffusion when one knows the form of the generator as a differential operator.

The above remarks indicate that this is not an easy reference book for all those whose work requires calculation with diffusion processes. Nevertheless the book is indeed immensely valuable as a reference because the authors have included a wealth of valuable information which is not essential to the logical development of the subject. In fact, the scope of this work is such that every single important contribution to the subject of Brownian motion seems to have found a natural place, either as a short section, or as one of the many problems with a hint toward the solution, and often in a setting and form far superior to that of the original article.