NOTES

THE GROWTH OF A RECURRENT RANDOM WALK[1]

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Let $X_1, X_2, \cdots$ denote independent identically distributed non-degenerate random variables and set $S_n = X_1 + \cdots + X_n$. The random walk S_n is called *recurrent* if for some number M $P(|S_n| \leqq M$ for some $n \geqq m) = 1$ for all m. The purpose of this note is to prove the following

Theorem. *If S_n is recurrent, then*

$$P\{\limsup_{n\to\infty} S_n/n^{\frac{1}{2}} = \infty\} = 1. \tag{1}$$

The author was led to this result by a conjecture of Y. S. Chow, namely the

Corollary. *If $E|X_k| < \infty$ and $EX_k = 0$, then* (1) *holds.*

It is plausible that (1) also holds whenever $P\{S_n \geqq 0$ for some $n \geqq m\} = 1$ for all m.

Proof of Theorem. Let μ denote the common distribution of the X_k's. Then we can find positive constants a and b and probability measures ν and φ such that $\mu = a\nu + b\varphi$ and φ is non-degenerate and has zero first moment and finite second moment σ^2. Let $\hat{\mu}$ and $\hat{\nu}$ denote the characteristic function of μ and ν respectively. A direct computation shows that there exist positive constants ϵ, c_1 and c_2 such that

$$0 \leqq c_1 \Re(1 - r\hat{\mu}(\theta))^{-1} \leqq \Re(1 - r\hat{\mu}(\theta))^{-1} \leqq c_2 \Re(1 - r\hat{\mu}(\theta))^{-1}, \quad |\theta| \leqq \epsilon \text{ and } 0 < r < 1. \tag{2}$$

Let $Y_1, Y_2, \cdots$ be independent random variables with common distribution ν and set $T_n = Y_1 + \cdots + Y_n$. It follows from (2) and Theorem 3 of Chung and Fuchs [1] that T_n is a recurrent random walk. (If $E|X_k| < \infty$ and $EX_k = 0$ then this fact follows alternatively from Theorem 4 of [1]). Let Z_n be independent random variables with common distribution φ and set $U_n = Z_1 + \cdots + Z_n$. Let $\xi_1, \xi_2, \cdots$ be independent identically distributed random variables such that $P\{\xi_k = 1\} = a$ and $P\{\xi_k = 0\} = b = 1 - a$, and set $j(n) = \xi_1 + \cdots + \xi_n$ and $k(n) = n - j(n)$. We can assume that the Y_k's, Z_k's, and ξ_k's are mutually independent.

Set $V_n = T_{j(n)} + U_{k(n)}$. Then V_n has the same probabilistic structure as S_n. By the zero-one law for independent random variables, in order to obtain (1) it suffices to show that for any N

$$\lim_{m\to\infty} P\{V_n/n^{\frac{1}{2}} \geqq N \text{ for some } n \geqq m\} > 0. \tag{3}$$

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Choose M such that $P\{|T_n| \leqq M \text{ for some } n \geqq m\} = 1$ for all m. Set $\tau(m) = \min\,[n \mid n \geqq m, j(n) \leqq (a+1)n/2, \text{ and } |T_{j(n)}| \leqq M]$. Now

$$\begin{aligned} P\{V_n/n^{\frac{1}{2}} &\geqq N \text{ for some } n \geqq m\} \\ &\geqq P\{U_{k(\tau(m))}/((\tau(m)))^{\frac{1}{2}} \geqq (M/(\tau(m))^{\frac{1}{2}}) + N\} \\ &\geqq P\{U_{k(\tau(m))}/(k(\tau(m)))^{\frac{1}{2}} \geqq (M/(k(\tau(m)))^{\frac{1}{2}}) + 2^{\frac{1}{2}}b^{-\frac{1}{2}}N\}, \end{aligned}$$

which approaches $1 - \Phi(2^{\frac{1}{2}}N/\sigma b^{\frac{1}{2}}) > 0$ as $m \to \infty$ by the central limit theorem. Here Φ denotes the standard normal distribution function.

To prove the corollary we need only note that if $E|X_k| < \infty$ and $EX_k = 0$, then by Theorem 4 of [1], S_n is a recurrent random walk and the above theorem applies.

REFERENCES

[1] Chung, K. L. and Fuchs, W. H. J. (1951). On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.* No. 6, 1–12.