CHARACTERIZATION OF NORMAL AND GENERALIZED TRUNCATED NORMAL DISTRIBUTIONS USING ORDER STATISTICS[1]

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1. Introduction and summary. Many contributions have been made to the problem of characterizing the normal distribution using the property of independence of sample mean and sample variance, maximum likelihood, etc. In this paper, using certain identities among the product (linear) moments of order statistics in a random sample, the generalized truncated (both from below and above) normal distributions, the negative normal and the positive normal distributions are characterized in the class of arbitrary distributions having finite second moments. In Theorem 3.3, the normal distribution is characterized in the class of arbitrary distributions having mean zero and finite second moments. Bennett's [1] characterization of the normal distribution without the assumption of absolute continuity is a special case of Theorem 3.3, namely Corollary 3.3.2.

2. Notation and assumptions. Let $X_1, X_2, \cdots$ be independent nontrivial random variables each with the distribution function F. Also, let $X_{1,N} \leqq X_{2,N} \leqq \cdots \leqq X_{N,N}$ denote the ordered values of $X_1, X_2, \cdots, X_N$. Assume that $EX_1^2 < \infty$. Further, let $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} dx$, $-\infty < x < \infty$, $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$.

3. Main results. In this section, we will state and prove the main characterization theorems.

Theorem 3.1.

$$E(X_{N,N}^2 - X_{N-1,N}X_{N,N}) = 1, \qquad N = 2, 3, \cdots, \tag{3.1}$$

if an only if there exists an extended real number A $(-\infty \leqq A < \infty)$ *such that*

$$F(x) = [\Phi(x) - \Phi(A)/[1 - \Phi(A)], \qquad A < x < \infty. \tag{3.2}$$

Proof. Consider

$$E(X_{N,N}^2) - E(X_{N-1,N}X_{N,N}) = N \int\int_{-\infty < z \leqq w < \infty} wF^{N-1}(z)dF(w)dz,$$

after integrating by parts once in the double integral. Now, it is easy to see that for F given by (3.2), $E(X_{N,N}^2 - X_{N-1,N}X_{N,N}) = 1$.

In order to prove the 'only if' part of the theorem, let $Y_1, Y_2, \cdots$ be independent, each being uniformly distributed over the open interval $(0, 1)$. Define

$$H(u) = \inf [x \mid F(x) \geqq u], \qquad 0 < u < 1, \tag{3.3}$$

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where F is taken to be right continuous. Then, for $0 < u < 1$,

$$H(u) \leqq x \Leftrightarrow u \leqq F(x), \tag{3.4}$$

and the distribution of $H(Y_1), H(Y_2), \cdots$ is the same as that of $X_1, X_2, \cdots$. Also, $H(Y_{N,N}) = \max_{1 \leqq k \leqq N} H(Y_k)$ has the same distribution as $X_{N,N}$, etc. Consequently (3.1) is equivalent to

$$\begin{aligned} 1 &= E[H^2(Y_{N,N}) - H(Y_{N-1,N})H(Y_{N,N})] \\ &= \int_0^1 \int_0^v [H^2(v) - H(u)H(v)]N(N-1)u^{N-2} du\, dv. \end{aligned}$$

Or $N(N-1) \int_0^1 u^{N-2}[\int_u^1 \{H^2(v) - H(u)H(v) - 1\} dv] du = 0$, $N = 2, 3, \cdots$, implying that, for almost all u in $(0, 1)$, $\int_u^1 [H^2(v) - H(u)H(v) - 1] dv = 0$. Now, since H is continuous except on a countable set, we get

$$\int_u^1 [H^2(v) - 1] dv = H(u) \int_u^1 H(v) dv, \qquad 0 < u < 1. \tag{3.5}$$

By applying Lebesgue's differentiation theorem, we have, for almost all u in $(0, 1)$,

$$H'(u) \int_u^1 H(v) dv = 1. \tag{3.6}$$

Since $H(u)$ is nondecreasing, $H'(u) \geqq 0$ for all u wherever the derivative exists. Consequently, we have, for almost all u in $(0, 1)$,

$$(1-u)^{-1} \int_u^1 H(v) dv > 0. \tag{3.7}$$

The expression in (3.7) is nondecreasing in u since H is, hence, $\int_u^1 H(v) dv > 0$ for all u in $(0, 1)$. Thus, from (3.5) one has

$$H(u) = \int_u^1 [H^2(v) - 1] dv / \int_u^1 H(v) dv \tag{3.8}$$

from which it readily follows that H is continuous on $(0, 1)$ and that H is differentiable on $(0, 1)$. Consequently (3.6) holds for all u in $(0, 1)$ and $H'(u) > 0$ for $0 < u < 1$. Let $A = H(0+)$. From (3.8), it is clear that $H(1-) = \infty$. Then, we have that

$$F(x) = H^{-1}(x), \qquad A < x < \infty, \tag{3.9}$$

and F is differentiable on (A, ∞) and $F'(x)H'(F(x)) = 1$, $A < x < \infty$, so that $F'(x) > 0$ for $A < x < \infty$. Also, for $A < x < \infty$, $F(x) > 0$ and $F(x) < 1$. Hence, (3.6) can be written as

$$H'(F(x)) \int_{F(x)}^1 H(z) dz = 1, \text{ or } \int_x^\infty t F'(t) dt = F'(x), \qquad A < x < \infty. \tag{3.10}$$

Now, from (3.10), it is clear that $F''(x)$ exists for $A < x < \infty$. Thus, we have

$$F''(x) = -F'(x) \tag{3.11}$$

from which it follows that $F'(x) = ce^{-x^2/2}$ for some $c > 0$ and $A < x < \infty$; or $F(x) = [\Phi(x) - \Phi(A)]/[1 - \Phi(A)]$, $A < x < \infty$. This completes the proof of Theorem 3.1.

COROLLARY 3.1.1.

$$E(X_{1,N}^2 - X_{1,N}X_{2,N}) = 1, \qquad N = 2, 3, \cdots, \tag{3.12}$$

if and only if there exists an extended real number B $(-\infty < B \leqq \infty)$ *such that*

$$F(x) = \Phi(x)/\Phi(B), \qquad -\infty < x < B. \tag{3.13}$$

PROOF. Change the random variable X to $-X$ in Theorem 3.1.

THEOREM 3.2. *If* $F(0) = 0$, *then*, $\sum_{j=1}^N E(X_{1,N}X_{j,N}) = 1$, $N = 2, 3, \cdots$, *if and only if*

$$F(x) = 2\Phi(x), \qquad 0 < x < \infty. \tag{3.14}$$

PROOF. Consider

$$\begin{aligned}\sum_{j=1}^N E(X_{1,N}X_{j,N}) &= \sum_{j=2}^N E(X_{1,N}X_{j,N}) + E(X_{1,N}^2) \\ &= N \int_0^\infty [\int_z^\infty w dF(w)][1 - F(z)]^{N-1} dz,\end{aligned} \tag{3.15}$$

after integrating by parts once in the double integral. Now, if F is given by (3.14), it readily follows that $\sum E(X_{1,N}X_{j,N}) = 1$. Next, towards proving the converse, using the relation $E(X_{1,N}X_{j,N}) = E\{H(Y_{1,N})H(Y_{j,N})\}$ and proceeding as above, one can easily obtain that $\sum E(X_{1,N}X_{j,N}) = 1$, $N = 2, 3, \cdots$, implies that $1 = \iint_{0<u\leqq v<1} (1 - u)^{N-1}H'(u)H(v)\, du\, dv$, $N = 2, 3, \cdots$. That is, $0 = \int_0^1 (1 - u)^{N-1}[H'(u) \int_u^1 H(v)\, dv - 1]\, du$, $N = 2, 3, \cdots$, which implies that for almost all u in (0, 1), we have $H'(u) \int_u^1 H(v)\, dv = 1$. Now, from the latter part of the proof of Theorems 3.1, with $A = 0$, it follows that F is given by (3.14).

COROLLARY 3.2.1. *If* $F(0) = 1$, *then* $\sum_{j=1}^N E(X_{j,N}X_{N,N}) = 1$, $N = 2, 3, \cdots$, *if and only if*

$$F(x) = 2\Phi(x), \qquad -\infty < x < 0. \tag{3.16}$$

PROOF. Change the random variable X to $-X$ in Theorem 3.2.

THEOREM 3.3. *If* $\int_{-\infty}^\infty x dF(x) = 0$, *then, for* $i = 1, 2, \cdots, N$,

$$\sum_{j=1}^N \cdot E(X_{i,N}X_{j,N}) = 1, \qquad N = 2, 3, \cdots,$$

if and only if

$$F(x) = \Phi(x), \qquad -\infty < x < \infty. \tag{3.17}$$

PROOF.

CASE 1. $i = N$. Consider

$$\begin{aligned}\sum_{j=1}^N E(X_{j,N}X_{N,N}) &= \sum_{j=1}^{N-1} E(X_{j,N}X_{N,N}) + E(X_{N,N}^2) \\ &= N(N-1) \iint_{-\infty<z\leqq w<\infty} zwF^{N-2}(w)\, dF(z)\, dF(w) \\ &\quad + N \int_{-\infty}^\infty z^2F^{N-1}(z)\, dF(z).\end{aligned} \tag{3.18}$$

One can write

$$E(X_{N,N}^2) = N \int_{-\infty}^\infty z [\int_{-\infty}^z (d/dF(w))\{wF^{N-1}(w)\}\, dF(w)]\, dF(z)$$

$$= N \iint_{-\infty<w\leq z<\infty} zF^{N-1}(w)\, dw\, dF(z)$$
$$+ N(N-1) \iint_{-\infty<w\leq z<\infty} wzF^{N-2}(w)\, dF(z)\, dF(w).$$

Substituting the above for $E(X_{N,N}^2)$ in (3.18) and combining the two symmetrical double integrals and noting that $\int_{-\infty}^{\infty} z dF(z) = 0$, it follows that $\sum_{j=1}^{N} E(X_{j,N}X_{N,N}) = N \iint_{-\infty<w\leq z<\infty} zF^{N-1}(w)\, dw\, dF(z)$. If F is given by (3.17), then it follows that $\sum_{j=1}^{N} E(X_{j,N}X_{N,N}) = 1$. For proving the converse, use the relation: $E(X_{j,N}X_{N,N}) = E\{H(Y_{N,N})H(Y_{j,N})\}$ and proceed as above and obtain: $\sum_{j=1}^{N} E(X_{j,N}X_{N,N}) = 1$, $N = 2, 3, \cdots$, implies that

$$1 = N \iint_{0<u\leq v<1} H'(u)H(v)u^{N-1}\, du\, dv, \qquad N = 2, 3, \cdots.$$

Or $0 = N \int_0^1 u^{N-1}[H'(u) \int_u^1 H(v)\, dv - 1]\, du$, $N = 2, 3, \cdots$, which in turn implies that for almost all u in (0, 1), $H'(u) \int_u^1 H(v)\, dv = 1$. Now, it readily follows from the latter part of the proof of Theorem 3.1 with $A = -\infty$ that F is given by (3.17).

CASE 2. $i = 1$: This case can similarly be covered since $i = 1$ is the mirror image of $i = N$.

CASE 3. $2 \leq i \leq N - 1$. Consider

$$\sum_{j=1}^{N} E(X_{i,N}X_{j,N})$$
$$= \sum_{j=1}^{i-1} + E(X_{i,N}^2) + \sum_{j=i+1}^{N} E(X_{i,N}X_{j,N})$$
$$= [N!/(i-2)!(N-i)!] \iint_{-\infty<z\leq w<\infty} zwF^{i-2}(w)[1 - F(w)]^{N-i}\, dF(z)\, dF(w)$$
$$+ [N!/(i-1)!(N-i)!] \int_{-\infty}^{\infty} z^2F^{i-1}(z)[1 - F(z)]^{N-i}\, dF(z)$$
$$+ [N!/(i-1)!(N-i-1)!] \iint_{-\infty<z\leq w<\infty} zwF^{i-1}(z)$$
$$\cdot[1 - F(z)]^{N-i-1}\, dF(z)\, dF(w)$$
$$= -[N!/(i-1)!(N-i)!] \iint_{-\infty<z\leq w<\infty} zF^{i-1}(w)[1 - F(w)]^{N-i}\, dF(z)\, dw$$
$$+ [N!/(i-1)!(N-i-1)!] \iint_{-\infty<z\leq w<\infty} zw\{F^{i-1}(z)[1 - F(z)]^{N-i-1}$$
$$+ F^{i-1}(w)[1 - F(w)]^{N-i-1}\}\, dF(z)\, dF(w),$$

after performing integration of parts with respect to w once in the first double integral. Further, since the integrand in the last double integral in the second equality is symmetric in z and w, one can write the double integral as

$$[N!/(i-1)!(N-i-1)!][\int_{-\infty}^{\infty} z\, dF(z)][\int_{-\infty}^{\infty} wF^{i-1}(w)[1 - F(w)]^{N-i-1}\, dF(w)]$$

which is zero because of the hypothesis. Hence, for $i = 2, 3, \cdots, N - 1$,

$$(3.19) \quad \sum_{j=1}^{N} E(X_{i,N}X_{j,N}) = -[N!/(i-1)!(N-i)!]$$
$$\cdot \iint_{-\infty<z\leq w<\infty} zF^{i-1}(w)[1 - F(w)]^{N-i}\, dF(z)\, dw, \quad N = 2, 3, \cdots.$$

Now, if F is given by (3.17), it is easily seen that the right hand side of (3.19) simplifies to unity.

Next, in order to prove the converse of the theorem, use the fact: $E(X_{i,N}X_{j,N}) = E\{H(Y_{i,N})H(Y_{j,N})\}$ and proceed as above and obtain for $i = 2, 3, \cdots, N-1$, $\sum_{j=1}^{N} E(X_{i,N}X_{j,N}) = 1$, $N = 2, 3, \cdots$, implies that for $i = 2, 3, \cdots, N-1$,

$$1 = -[N!/(i-1)!(N-i)!] \cdot \iint_{0<u\leqq v<1} H(u)H'(v)v^{i-1}(1-v)^{N-i}\,du\,dv, \quad N = 2, 3, \cdots.$$

That is, for $i = 2, 3, \cdots, N-1$,

$$0 = \int_0^1 v^{i-1}(1-v)^{N-i}\{H'(v)\int_0^v H(u)\,du + 1\}\,dv, \quad N = 2, 3, \cdots.$$

Now, the only continuous function $\psi(v)$ which is orthogonal to $v^{i-1}(1-v)^{N-i}$, (a linear combination of $v^0, v, \cdots, v^{N-1}$) for all N is $\psi(v) \equiv 0$. Hence, for almost all v in $(0, 1)$, we have $H'(v)\int_0^v H(u)\,du = -1$. Now, using an argument analogous to the one used in the proof of Theorem 1, one can establish that $F = \Phi(x)$, $-\infty < x < \infty$. This completes the proof of Theorem 3.3.

COROLLARY 3.3.1. *If* $\int_{-\infty}^{\infty} z\,dF(z) = 0$, *then for* $i = 1, 2, \cdots, N$, $\sum_{j=1}^{N}$ Cov $(X_{i,N}; X_{j,N}) = 1$, $N = 2, 3, \cdots$, *if and only if* $F(x) = \Phi(x)$, $-\infty < x < \infty$.

PROOF. Since the population mean is zero, it follows that $\sum_{j=1}^{N} E(X_{j,N}) = 0$. Consequently, $\sum_{j=1}^{N}$ Cov $(X_{i,N}; X_{j,N}) = \sum_{j=1}^{N} E(X_{i,N}X_{j,N})$ and Corollary 3.3.1 readily follows from Theorem 3.3.

COROLLARY 3.3.2. (Bennett [1]). *If* $F(z)$ *is symmetric about zero then, for* $i = 1, \cdots, N$, $\sum_{j=1}^{N} E(X_{i,N}X_{j,N}) = 1$ $(\sum_{j=1}^{N}$ Cov $(X_{i,N}; X_{j,N}) = 1)$, $N = 2, \cdots$, *if and only if* $F(x) = \Phi(x)$, $-\infty < x < \infty$.

PROOF. Symmetry of F about zero, implies that the population mean is zero and consequently the corollary follows from Theorem 3.3.

REMARK 3.3.1. Bennett [1] gives a direct proof of Corollary 3.3.3 exploiting the symmetry of the density function which he assumes to exist and be positive everywhere.

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REFERENCES

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