ON THE BIVARIATE MOMENTS OF ORDER STATISTICS FROM A LOGISTIC DISTRIBUTION[1]

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1. Introduction. A logistic distribution is defined by $x = \ln \{F/(1 - F)\}$, where F is the probability of a value less than x. This is a symmetric distribution with mean zero and variance $\pi^2/3$. The shape of a logistic distribution is nearly the same as that of a normal distribution except at the tails. Birnbaum [2], Birnbaum and Dudman [3], Plackett [9] and others [8] have given tables of the expected values of the order statistics.

In Section 3 a convenient expression for the moment generating function of the ith and jth order statistic $(j > i)$ in a random sample of size n drawn from a logistic distribution is derived. This expression is useful in deriving the higher product moments of the order statistics. In Section 4, a finite and easily computable expression [11] is developed. Also various recurrence relations are obtained. In Section 5, using digamma and trigamma values tabulated in [5], [6], we give the covariances of all pairs of order statistics up to sample size $n = 10$.

2. Notation and order statistics theory. Let $x_{1,n} \leqq x_{2,n} \leqq \cdots x_{n,n}$ be the order statistics in a sample of size n from any continuous distribution. Let the cumulative distribution function (c.d.f.) be denoted by $F(x)$. It is well known that the distribution of ith order statistics has the probability differential element

$$a_{i,n}(x)\,dx = [B(i, n - i + 1)]^{-1}F^{i-1}(x)[1 - F(x)]^{n-i}\,dF(x), \quad i = 1, 2, \cdots, n, \tag{2.1}$$

where $B(k, m) = \Gamma(k)\Gamma(m)/\Gamma(k + m)$, $k > 0$, $m > 0$. And the joint distribution of ith and jth order statistics is

$$\begin{aligned} a_{i,j,n}(x, y)\,dx\,dy &= [B(i, j - i)B(j, n - j + 1)]^{-1}F^{i-1}(x) \\ &\quad \cdot [F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j}\,dF(x)\,dF(y), \\ &\qquad x < y \quad \text{and} \quad 1 \leqq i < j \leqq n. \end{aligned} \tag{2.2}$$

Let

$$\mu_{i,n}^{(k)} = E(X_{i,n}^k) = \int_{-\infty}^{\infty} x^k a_{i,n}(x)\,dx, \quad 1 \leqq i \leqq n, \quad k = 1, 2, \cdots, \tag{2.3}$$

with $\mu_{i,n}^{(1)} = \mu_{i,n}$ and

Received 2 September 1965; revised 30 March 1966.

[1] Research supported by Contract AF33(657)11737 with the Aerospace Research Laboratories. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$$(2.4) \quad \mu_{i,j,n} = E(X_{i,n}X_{j,n}) = \int\int_{-\infty<x<y<\infty} xya_{i,j,n}(x, y)\,dx\,dy, \quad 1 \leqq i < j \leqq n.$$

The covariance of the ith and jth order statistics is denoted by $\sigma_{i,j,n}$, where $\sigma_{i,j,n} = E(X_{i,n}X_{j,n}) - \mu_{i,n}\mu_{j,n}$.

3. Moment generating function and bivariate moments for order statistics of a logistic distribution. The probability differential element for a logistic distribution is

$$(3.1) \qquad f(x)\,dx = \exp(-x)[1 + \exp(-x)]^{-2}\,dx, \quad -\infty < x < \infty.$$

Throughout this paper we will talk of logistic order statistics unless otherwise stated in the text. Since $x = \ln(F(x)/(1 - F(x)))$, the expression for moment generating function can be written as

$$(3.2) \quad M(t, s) = C\int\int_{0<u<v<1} (u/(1 - u))^t(v/(1 - v))^s u^{i-1}(v - u)^{j-i-1} \cdot (1 - v)^{n-j}\,du\,dv,$$

where $C = [B(i, j - i)B(j, n - j + 1)]^{-1}$. Expanding $(1 - u)^{-t}$ in powers of u and integrating with respect to u first and v next, we have

$$(3.3) \quad M(t, s) = C\sum_{r=0}^{\infty} [(t + r - 1)^{(r)}/r!]B(t + r + i, j - i) \cdot B(t + s + r + j, n - j - s + 1).$$

This is a simpler expression for the moment generating function, involving only one summation, than that of the m.g.f. given by Gupta and Shah ([8], p. 912). From Expression (3.3), one can obtain the bivariate moments as follows:

$$(3.4) \qquad E(X_{i,n}^k X_{j,n}^m) = [D_1^k D_2^m M(t_1, t_2)]_{t_1=t_2=0},$$

where $D_i^k = \partial^k/\partial t_1^k$ and $D_2^m = \partial^m/\partial t_2^m$. The case $k = 1$, $m = 1$ is of practical importance in estimating best unbiased linear estimators of the parameter μ and σ of the logistic distribution. Thus differentiating (3.4) with respect to t_1 and t_2 and then putting $t_1 = t_2 = 0$, we have (details omitted)

$$(3.5) \quad \begin{aligned} E(X_{i,n}X_{j,n}) &= \Psi^{(1)}(j - 1) + [\Psi^{(0)}(j - 1) - \Psi^{(0)}(n - j)] \\ &\cdot [\Psi^{(0)}(i - 1) - \Psi^{(0)}(n)] + \sum_{r=1}^{\infty} B(n - i + 1, i + r) \\ &\cdot [rB(i, n - i + 1)]^{-1}[\Psi^{(0)}(j + r - 1) - \Psi^{(0)}(n - j)], \end{aligned}$$

where $\Psi^{(r-1)}(x) = d^r \ln \Gamma(x)/dx^r$, (see [12]). Expression (3.5) is a convergent infinite series with only one summation. Expected values of higher powers of the product of two order statistics can be obtained from (3.3) using (3.4).

4. Exact results for the bivarate moment. Expression (3.5) is slowly convergent, and it is rather difficult to obtain an answer correct to four or five decimal places. But for higher order product moments of two order statistics, the method explained in Section 3 is quite suitable. In the section we derive more convenient results for the expected values of the product of any two order statistics. First we prove certain important lemmas.

LEMMA 4.1. *If x is a gamma variate with parameter r, then*

$$E(\ln x) = \Psi^{(0)}(r-1). \tag{4.1}$$

Proof is obvious and given by M. S. Bartlett and D. G. Kendall [1].

LEMMA 4.2. *If x is a Beta variable of the first kind with parameters r and s $(r, s > 0)$, then*

$$\text{(a)}\quad E(\ln x) = \Psi^{(0)}(r-1) - \Psi^{(0)}(r+s-1), \tag{4.2}$$

$$\text{(b)}\quad E(\ln x)^2 = \Psi^{(1)}(r-1) - \Psi^{(1)}(r+s-1) + \{\Psi^{(0)}(r-1) - \Psi^{(0)}(r+s-1)\}^2, \tag{4.3}$$

$$\text{(c)}\quad E[\{\ln x\}\{\ln(1-x)\}] = -\Psi^{(1)}(r+s-1) + \{\Psi^{(0)}(r-1) - \Psi^{(0)}(r+s-1\}\{\Psi^{(0)}(s-1) - \Psi^{(0)}(r+s-1)\}. \tag{4.4}$$

PROOF. Consider the identity $B(t_1 + r, t_2 + s) = \int_0^1 x^{t_1+r-1}(1-x)^{t_2+s-1}\,dx$, $t_1 + r > 0$, $t_2 + s > 0$. Differentitating this identity w.r.t. t_1 and then putting $t_1 = 0$ and $t_2 = 0$, we will have (a). Similarly differentiating the identity two times w.r.t. t_1 and setting $t_1 = t_2 = 0$, we have the result (b). Result (c) in the above lemma can be obtained by differentiating the identity partially w.r.t. t_1 and t_2 and then setting $t_1 = t_2 = 0$.

COROLLARY 4.1.

$$E(\ln(1-x)) = \Psi^{(0)}(s-1) - \Psi^{(0)}(r+s-1). \tag{4.5}$$

Proof is similar, with x and $(1-x)$ interchanged.

COROLLARY 4.2.

$$E[\ln(1-x)]^2 = \Psi^{(1)}(s-1) - \Psi^{(1)}(r+s-1) + \{\Psi^{(0)}(s-1) - \Psi^{(0)}(r+s-1)\}^2. \tag{4.6}$$

Proof is similar to Lemma 4.2 (b) with x and $(1-x)$ interchanged.

LEMMA 4.3. *For a logistic distribution,*

$$\textstyle\int_{-\infty}^{y} F^r(x)\,dx = -[\ln[1-F(y)] + \sum_{i=1}^{r-1} F^i(y)/i], \qquad r > 1. \tag{4.7}$$

PROOF. Denoting the L.H.S. by I_r and using $f(x) = F(x)[1-F(x)]$, we have

$$I_r = \textstyle\int_{-\infty}^{y} F^r(x)\,dx = \int_{-\infty}^{y} F^{r-1}(x)\,dx - \int_{-\infty}^{y} F^{r-2}(x)f(x)\,dx$$
$$= I_{r-1} - F^{r-1}(y)/(r-1).$$

Hence

$$I_r = -\textstyle\sum_{i=1}^{r-1} F^i(y)/i - \ln[1-F(y)].$$

Lemma 4.3 is true for $r = 1$, also, provided the summation is interpreted as vacuous when $r = 1$.

RESULT 1. For a logistic distribution,

$$\mu_{k,m,n} = \mu^{(2)}_{m,n} + \sum_{i=k}^{m-1} \sum_{t=1}^{i-1} (-1)^{i+k} \binom{i-1}{k-1} \binom{n}{i} \binom{m-i+t-1}{t}$$
$$(4.8) \quad \cdot B(t, n-i+1) \cdot \mu_{m+t-i,n+t-i} + \binom{n}{k} \sum_{i=0}^{m-k-1} (-1)^i \binom{n-k}{i} (k+i)^{-1}$$
$$\cdot \{-\Psi^{(1)}(n-m) + [\Psi^{(0)}(n-m) - \Psi^{(0)}(n-k-i)]$$
$$\cdot [\Psi^{(0)}(m-k-i-1) - \Psi^{(0)}(n-m)]\}.$$

PROOF. From (2.5) we have

$$(4.9) \quad \mu_{k,m,n} = C \int_{-\infty}^{\infty} y[1 - F(y)]^{n-m} \, dF(y) \int_{-\infty}^{y} xF^{k-1}(x)$$
$$\cdot [F(y) - F(x)]^{m-k-1} \, dF(x),$$

where $C = [B(k, m-k)B(m, n-m+1)]^{-1}$. Using Lemma 4.3 in the last integral w.r.t. x we have

$$\int_{-\infty}^{y} xF^{k-1}(x)[F(y) - F(x)]^{m-k-1} \, dF(x) = \sum_{i=0}^{m-k-1} (-1)^i$$
$$(4.10) \quad \binom{m-k-1}{i} F^{m-k-i-1}(x) \cdot [k+i]^{-1} [yF^{k+i}(y)$$
$$+ \sum_{t=1}^{k+i-1} t^{-1} F^t(y) + \ln\{1 - F(y)\}].$$

NOTE. If $k = 1$ and $i = 0$, then the second term in the second square bracket is understood to have value zero. Substituting (4.10) in (4.9) and after little adjustment we have

$$\mu_{k,m,n} = \sum_{i=0}^{m-k-1} K(i, k, m) \int_{-\infty}^{\infty} y^2 F^{m-1}(y)[1 - F(y)]^{n-m} \, dF(y)$$
$$(4.11) \quad + \sum_{i=0}^{m-k-1} K(i, k, m) \sum_{t=1}^{k+i-1} t^{-1} \int_{-\infty}^{\infty} yF^{m+t-k-i-1}(y)[1 - F(y)]^{n-m} \, dF(y)$$
$$+ \sum_{i=0}^{m-k-1} K(i, k, m) \int_{-\infty}^{\infty} y \ln[1 - F(y)] F^{m-k-i-1}(y)[1 - F(y)]^{n-m} \, dF(y),$$

where $K(i, k, m) = C(-1)^i (k+i)^{-1} \binom{m-k-1}{i}$.

Let $\mu_{k,m,n} = A + B + C_1$, where A, B and C_1 are the three integrals including the summation w.r.t. i. Thus, we have

$$A = [B(k, m-k)]^{-1} \mu^{(2)}_{m,n} \sum_{i=0}^{m-k-1} (-1)^i \binom{m-k-1}{i} (k+i)^{-1}$$
$$(4.12) \quad = [B(k, m-k)]^{-1} \mu^{(2)}_{m,n} \int_0^1 x^{k-1}(1-x)^{m-k-1} \, dx$$
$$= \mu^{(2)}_{m,n},$$

$$B = \sum_{i=k}^{m-1} \sum_{t=1}^{i-1} (-1)^{i+k} \binom{i-1}{k-1} \binom{n}{i} \binom{m-i+t-1}{t}$$
$$(4.13) \quad \cdot B(t, n-i+1) \mu_{m+t-i,\, n+t-i}, \quad \text{and}$$
$$C_1 = \sum_{i=0}^{m-k-1} K(i, k, m) \int_{-\infty}^{\infty} y \ln[1 - F(y)] F^{m-k-i-1}(y)$$
$$\cdot [1 - F(y)]^{n-m} \, dF(y).$$

Now using the previous lemmas, and after simplification, we have,

$$C_1 = \binom{n}{k} \sum_{i=0}^{m-k-1} (-1)^i k(k+i)^{-1}\binom{n-k}{i}$$

$$\cdot\{-\Psi^{(1)}(n-m) + [\Psi^{(0)}(n-m) - \Psi^{(0)}(n-k-i)] \cdot[\Psi^{(0)}(m-k-i-1) - \Psi^{(0)}(n-m)]\}. \tag{4.14}$$

Thus adding (4.12), (4.13) and (4.14) we have the Result 1.

COROLLARY 4.3.

$$\mu_{1,m,n} = \mu_{m,n}^{(2)} + \sum_{i=2}^{m-1} \sum_{t=1}^{i-1} (-1)^{i+1}\binom{n}{i}\binom{m-i+t+1}{t} \cdot B(t, n-i+1)\mu_{m+t-i,n-t-i} + n \sum_{i=0}^{m-2} (-1)^i\binom{n-1}{i}(1+i)^{-1} \cdot\{-\Psi^{(1)}(n-m) + [\Psi^{(0)}(n-m) - \Psi^{(0)}(n-i-1)] \cdot[\Psi^{(0)}(m-i-2) - \Psi^{(0)}(n-m)]\}, \quad 1 < m \leqq n. \tag{4.15}$$

Ruben [10] gave a formula for $\mu_{1,n,n}$ for any distribution only for n even.

COROLLARY 4.4.

$$\mu_{1,2,n} = \mu_{2,n}^{(2)} + n\{-\Psi^{(1)}(n-2) + [\Psi^{(0)}(n-2) - \Psi^{(0)}(n-1)] \cdot[\Psi^{(0)}(0) - \Psi^{(0)}(n-2)]\}. \tag{4.16}$$

COROLLARY 4.5.

$$\mu_{k,k+1,n} = \mu_{k+1,n}^{(2)} + \binom{n}{k} \sum_{t=1}^{k-1} \binom{n}{k} B(t, n-k+1)\mu_{t+1,n+t-k} + \binom{n}{k}\{-\Psi^{(1)}(n-k-1) + [\Psi^{(0)}(n-k-1) - \Psi^{(0)}(n-k)] \cdot[\Psi^{(0)}(0) - \Psi^{(0)}(n-k-1)]\}, \quad 1 \leqq k \leqq n-1. \tag{4.17}$$

RESULT 2. For a logistic distribution for $1 < k < m \leqq n$ and $m - k \geqq 2$, one has

$$(n+1)\mu_{k,n} = (n+1)(n-m+1)\mu_{k,m,n} - (n-k+1)(n-m+1)\mu_{k,m,n+1} - k(n-m+1)\mu_{k+1,m,n+1}. \tag{4.18}$$

PROOF. Consider

$$\mu_{k,n} = E(X_{k,n}) = C\int_{-\infty}^{\infty} xF^{k-1}(x)\, dF(x) \int_x^{\infty} F(y)[1-F(y)]^{n-m+1} [F(y) - F(x)]^{m-k-1}\, dy.$$

Now denote the second integral w.r.t. y by I_x. Writing $F(y)[1 - F(y)]^{n-m+1} = [1 - F(y)]^{n-m+1} - [1 - F(y)]^{n-m+2}$, we have $I_x = \int_x^{\infty} 1\cdot\{[1 - F(y)]^{n-m+1} - [1 - F(y)]^{n-m+2}\}[F(y) - F(x)]^{m-k-1}\, dy$. Integrating I_x by parts and substituting back in $\mu_{k,n}$, we have (omitting details),

$$\mu_{k,n} = C(n-m+1)\int_{-\infty}^{\infty}\int_x^{\infty} xyF^{k-1}(x)[F(y) - F(x)]^{m-k-1} \cdot[1-F(y)]^{n-m}\, dF(x)\, dF(y) - C(n-m+2)\int_{-\infty}^{\infty}\int_x^{\infty} xyF^{k-1}(x) \cdot[F(y)-F(x)]^{m-k-1}[1-F(y)]^{n-m+1}\, dF(x)\, dF(y) - C(m-k-1)\int_{-\infty}^{\infty}\int_x^{\infty} xyF^{k-1}(x)[F(y)-F(x)]^{m-k-2} \cdot F(y)\,[1-F(y)]^{n-m+1}\, dF(x)\, dF(y). \tag{4.19}$$

Using $F(y)[1 - F(y)]^{n-m+1} = [F(y) - F(x)][1 - F(y)]^{n-m+1} + F(x)[1 - F(y)]^{n-m+1}$, in the last integral of Equation (4.20) and after simplifying, we get the Result 2.

COROLLARY 4.6. *For* $m = k + 1$, *we have,*

$$(n - k)(n - k + 1)\mu_{k,k+1,n+1} = (n + 1)(n - k)\mu_{k,k+1,n} - k(n - k)\mu^{(2)}_{k+1,n+1} - (n + 1)\mu_{k,n}, \quad 1 \leqq k \leqq n - 1.$$

RESULT 3. For a logistic distribution, for $1 < k < m \leqq n$ and $m - k \geqq 2$ one has

$$(4.20) \quad (n + 1)\mu_{m,n} = -(n + 1)k\mu_{k,m,n} + mk\mu_{k+1,m+1,n+1} + k(n - m + 1)\mu_{k+1,m,n+1} .$$

PROOF. Proceed in a similar manner to Result 3, by considering the quantity,

$$\mu_{m,n} = E(y_{m,n}) = C \int\int_{-\infty<x<y<\infty} yF^{k-1}(x)[F(y) - F(x)]^{m-k-1} \cdot[1 - F(y)]^{n-m}\, dF(x)\, dF(y).$$

RESULT 4. For a logistic distribution for $1 < k < m \leqq n$ and $m - k \geqq 2$ one has

$$(4.21) \quad mk\mu_{k+1,m+1,n+1} = (n + 1)[\mu_{k,n} + \mu_{m,n}] + (n - k + 1)(n - m + 1)\mu_{k,m,n+1} - (n - m - k + 1)\mu_{k,m,n} .$$

PROOF. Adding (4.18) and (4.20), we have the proof.

RESULT 5. For the logistic distribution, if all $\mu_{i,j,n-1}(i \neq j)$, $\mu_{k,k}$ for $k = 1, 2, \cdots, n - 1$ and $\mu^{(2)}_{i,n}$, $i = 1, 2, \cdots, n$ are known, the number of linear and independent constraints among the distinct $\mu_{i,j,n}(i \neq j)$ and the number of linearly independent $\mu_{i,j,n}(i \neq j)$ are as shown in Table A.

TABLE A

	Number of Independent Constraints	Corollary 4.3 or Corollary 4.4 or Ruben's [10] Result	Corollary 4.5 or Corollary 4.6	Number of linearly independent $\mu_{i,j,n}(i \neq j)$
n even	$n(n - 2)/4$	1	$(n - 2)/2$	0
n odd	$(n - 1)^2/4$	0	$(n - 1)/2$	0

PROOF. n is even: Total number of distinct elements are $n(n + 2)/4$ (see [7], Theorem 4.11, p. 642). The number of distinct and independent constraints among the distinct $\mu_{i,j,n}(i \neq j)$ imposed by the recurrence formula obtained by Govindarajulu ([7], Theorem 4.2, p. 636) are $n(n - 2)/4$. The number of distinct elements of the type $\mu_{i,j,n}(i = j)$ are $n/2$, which are known [3], [8]. One independent constraint is due to Ruben [10]. The number of distinct elements of the type $\mu_{i,i+1,n}$ are $n/2$ from Corollary 4.5 or Corollary 4.6 except one, which is contained in relation proved by Govindarajulu ([7], Theorem 4.2, p.

636), hence the total number of distinct independent elements of the type $\mu_{i,i+1,n}$ are $(n-2)/2$. Thus the number of linearly independent elements of the type $\mu_{i,j,n}(i, j = 1, 2, \cdots, n \text{ and } i \neq j)$ are $n(n+2)/4 - n(n-2)/4 - n/2 - 1 - (n-2)/2$ which is zero. Hence the proof.

n is odd: Proof is similar to above. We simply write a parallel equation as

$$(n+1)^2/4 - (n-1)^2/4 - (n+1)/2 - 0 - (n-1)/2 = 0.$$

Thus one can obtain all distinct elements $\mu_{i,j,n}(i \neq j)$ using Ruben's result, Corollary 4.5 or Corollary 4.6 and the Theorem 4.2 [7].

ILLUSTRATION. $N = 4$: Assume that all the moments $\mu_{i,n}$ and cross moments $\mu_{i,j,n}$ are known for $n = 3$. Also $\mu_{i,n}$ and $\mu_{i,j,n}(i = j)$ are known for $n = 4$. So the distinct elements to be obtained are $\mu_{1,2,4}$, $\mu_{1,3,4}$, $\mu_{1,4,4}$ and $\mu_{2,3,4}$. $\mu_{1,4,4}$ is known using Ruben's formula. $\mu_{1,2,4}$ can be obtained using Corollary 4.5 or Corollary 4.6 and $\mu_{1,3,4}$ and $\mu_{2,3,4}$ can be obtained using Theorem 4.2 [7]. Thus putting $k = 1$ and $n = 3$ in Corollary 4.6, we have

$$3\mu_{1,2,4} = 4\mu_{1,2,3} - \mu_{2,4}^{(2)} - 2\mu_{1,3},$$

and using Theorem 4.2 [7], by putting $i = 2, j = 4, n = 4$ and $i = 2, j = 3, n = 4$, we have

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_{1,3,4} \\ \mu_{2,3,4} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} \mu_{1,3,3} \\ \mu_{1,4,4} \\ \mu_{1,2,3} \\ \mu_{1,2,4} \end{bmatrix}.$$

Thus one can write down the whole matrix $((\mu_{i,j,n}))$.

$N = 5$: The number of distinct elements of the type $\mu_{i,j,n}(i \neq j)$ are 6. Obtain first $\mu_{1,2,5}$ and $\mu_{2,3,5}$ using Corollary 4.5 or Corollary 4.6, and solve $\mu_{1,3,5}$, $\mu_{1,4,5}$, $\mu_{1,5,5}$ and $\mu_{2,4,5}$ from the following equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{1,3,5} \\ \mu_{1,4,5} \\ \mu_{1,5,5} \\ \mu_{2,4,5} \end{bmatrix} = \begin{bmatrix} 5\mu_{1,2,4} - 3\mu_{1,2,3} - \mu_{2,3,5} \\ 5\mu_{1,3,4} \\ 5\mu_{1,4,4} \\ 5\mu_{2,3,4} - 4\mu_{2,3,5} \end{bmatrix}.$$

5. Table of covariances. Table 1 gives all the covariances of a logistic distribution whose mean is zero and variance is unity. The covariances are given for upper half of the sample size n. The remaining values can be obtained from the symmetry relation $\sigma_{i,j,n} = \sigma_{n-j+1,n-i+1,n}$. The values in Table 1 are correct up to eight decimal places. Independent checks using recurrence relations and others revealed no errors. However, the eight decimal may be off by one unit.

The exact covariances given in Table 1 greatly simplify the solution of various problems of determination of best linear combinations of order statistics for

TABLE 1

Covariances of Logistic Order Statistics

n	i	j	$\sigma_{i,j,n}$	n	i	j	$\sigma_{i,j,n}$	n	i	j	$\sigma_{i,j,n}$	n	i	j	$\sigma_{i,j,n}$	n	i	j	$\sigma_{i,j,n}$
1	1	1	1.0000,0000	6	1	4	0.1008,2608	8	1	1	0.5404,6880	9	1	6	0.0613,2806	10	1	6	0.0606,5025
2	1	1	0.6960,3645	6	1	5	0.0789,9471	8	1	2	0.2167,1840	9	1	7	0.0520,0932	10	1	7	0.0514,2578
2	1	2	0.3039,6355	6	1	6	0.0649,2999	8	1	3	0.1339,9059	9	1	8	0.0451,4723	10	1	8	0.0446,3486
3	1	1	0.6200,4556	6	2	2	0.2633,1056	8	1	4	0.0968,0385	9	1	9	0.0398,8400	10	1	9	0.0394,2729
3	1	2	0.2599,0888	6	2	3	0.1660,8954	8	1	5	0.0757,3982	9	2	2	0.2365,0525	10	1	10	0.0353,0738
3	1	3	0.1641,0023	6	2	4	0.1213,8204	8	1	6	0.0621,9468	9	2	3	0.1474,6444	10	2	2	0.2317,5582
3	2	2	0.3920,7290	6	2	5	0.0956,8709	8	1	7	0.0527,5587	9	2	4	0.1070,4290	10	2	3	0.1442,0196
4	1	1	0.5862,7183	6	3	3	0.2063,1739	8	1	8	0.0458,0303	9	2	5	0.0840,0503	10	2	4	0.1045,4997
4	1	2	0.2411,8450	6	3	4	0.1526,8026	8	2	2	0.2427,0859	9	2	6	0.0691,2766	10	2	5	0.0819,8495
4	1	3	0.1508,8840	7	1	1	0.5466,7214	8	2	3	0.1517,4294	9	2	7	0.0587,2820	10	2	6	0.0674,2839
4	1	4	0.1097,6462	7	1	2	0.2199,6718	8	2	4	0.1103,2079	9	2	8	0.0510,4948	10	2	7	0.0572,6118
4	2	2	0.3160,8201	7	1	3	0.1362,0964	8	2	5	0.0866,6614	9	3	3	0.1667,1770	10	2	8	0.0497,5853
4	2	3	0.2037,3574	7	1	4	0.0984,9346	8	2	6	0.0713,6930	9	3	4	0.1217,9342	10	2	9	0.0439,9440
5	1	1	0.5672,7411	7	1	5	0.0771,0561	8	2	7	0.0606,6558	9	3	5	0.0959,8001	10	3	3	0.1605,1436
5	1	2	0.2309,0355	7	1	6	0.0633,4151	8	3	3	0.1751,6113	9	3	6	0.0792,1504	10	3	4	0.1170,1461
5	1	3	0.1437,3579	7	1	7	0.0537,4458	8	3	4	0.1283,2475	9	3	7	0.0674,4643	10	3	5	0.0920,8533
5	1	4	0.1042,5208	7	2	2	0.2511,5202	8	3	5	0.1013,1881	9	4	4	0.1413,8740	10	3	6	0.0759,2473
5	1	5	0.0817,7677	7	2	3	0.1575,9749	8	3	6	0.0837,3560	9	4	5	0.1119,9643	10	3	7	0.0645,9627
5	2	2	0.2823,0828	7	2	4	0.1148,2178	8	4	4	0.1535,4594	9	4	6	0.0927,7672	10	3	8	0.0562,1331
5	2	3	0.1794,9883	7	2	5	0.0903,2930	8	4	5	0.1220,1945	9	5	5	0.1345,4822	10	4	4	0.1329,4397
5	2	4	0.1318,1478	7	2	6	0.0744,6087	9	1	1	0.5357,1937	10	1	1	0.5319,6673	10	4	5	0.1050,6760
5	3	3	0.2400,9112	7	3	3	0.1873,1967	9	1	2	0.2142,4505	10	1	2	0.2122,9941	10	4	6	0.0868,9242
6	1	1	0.5551,1557	7	3	4	0.1377,8270	9	1	3	0.1323,0636	10	1	3	0.1309,8463	10	4	7	0.0740,9678
6	1	2	0.2244,2215	7	3	5	0.1090,8155	9	1	4	0.0955,2396	10	1	4	0.0945,2107	10	5	5	0.1223,8968
6	1	3	0.1392,6504	7	4	4	0.1725,4367	9	1	5	0.0747,0667	10	1	5	0.0738,9797	10	5	6	0.1016,0318

various estimation problems concerning parameters of a logistic distribution. The problem of obtaining the best unbiased linear estimators of the parameters of a logistic distribution is studied separately.

6. Acknowledgments. The author is grateful to Professor F. J. Anscombe and Dr. C. G. Khatri for their help and suggestions.

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