DIMENSIONAL PROPERTIES OF A RANDOM DISTRIBUTION FUNCTION ON THE SQUARE

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The notion of a random distribution function on the line associated with a probability measure μ on the unit square S was introduced and investigated by Dubins and Freedman in [3] and [4]. Certain of its properties were discussed by us in [5]. We now introduce a random distribution function F_ω whose values are probability distributions on the square which has properties expressible in formulae from information theory, as was the case in [5].

The construction of F_ω will be given formally below but it may be helpful to describe it briefly first. F_ω is normalized by setting $F_\omega(S) = 1$. A point (x_ω, y_ω) in S is chosen according to the distribution μ. F_ω is now defined on the dyadic rectangles

$$S_{00} = [(x, y) \mid 0 \leqq x < \tfrac{1}{2}, 0 \leqq y < \tfrac{1}{2}],$$

$$S_{01} = [(x, y) \mid 0 \leqq x < \tfrac{1}{2}, \tfrac{1}{2} \leqq y \leqq 1],$$

$$S_{10} = [(x, y) \mid \tfrac{1}{2} \leqq x \leqq 1, 0 \leqq y < \tfrac{1}{2}]$$

and

$$S_{11} = [(x, y) \mid \tfrac{1}{2} \leqq x \leqq 1, \tfrac{1}{2} \leqq y \leqq 1]$$

by

$$F_\omega(S_{00}) = x_\omega y_\omega$$

$$F_\omega(S_{01}) = x_\omega(1 - y_\omega)$$

$$F_\omega(S_{10}) = (1 - x_\omega)y_\omega$$

$$F_\omega(S_{11}) = (1 - x_\omega)(1 - y_\omega).$$

At the next step the measure in each of the four rectangles is partitioned among its dyadic subrectangles in the same way but independently of each other and of the previous choice (x_ω, y_ω). This process continues and in the limit defines F_ω. Our main result, Theorem 2, gives the dimensions of the supports of F_ω and its margins in terms of various 'average entropies' of μ.

We let $x(n, j) = j/2^n$, $I(n, j) = [x(n, j), x(n, j + 1))$, and $S(n, j, k) = I(n, j) \times I(n, k)$. We also let $I(n, x)$ be that $I(n, j)$ containing x and $S(n, x, y)$ be that $S(n, j, k)$ containing (x, y).

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For any function k on S we denote the expectation and variance of k by

$$E_\mu(k) = \int_S k(x, y)\, d\mu(x, y)$$
$$\sigma_\mu^2(k) = E_\mu([k - E_\mu(k)]^2).$$

We define constants G and H by

$$G = \int_S - [x \log x + (1 - x) \log (1 - x)]\, d\mu(x, y)$$
$$H = \int_S - [y \log y + (1 - y) \log (1 - y)]\, d\mu(x, y).$$

All logs, here and in the remainder of the paper, are taken to the base 2. Thus the integrands above lie between 0 and 1 for all x and y so that the integrals exist and G and H lie between 0 and 1.

The probability space Ω is the set of all $(P_{n,j,k})$ for $n = 0, 1, \cdots$, $j = 0, 1, \cdots 2^n - 1$ and $k = 0, 1, \cdots 2^n - 1$ where each $P_{n,j,k}$ is a point of S. A probability measure is obtained by requiring that the coordinate variables be independent and have the distribution μ. Expectation with respect to this measure will be written E_ω. The (n, j, k) coordinate of an ω in Ω will be written $(x_\omega(n, j, k), y_\omega(n, j, k))$.

We set

$$A_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2) = \begin{cases} x_\omega(n, j, k) & \text{if } \epsilon_1 = 0 \\ 1 - x_\omega(n, j, k) & \text{if } \epsilon_1 = 1 \end{cases}$$

$$B_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2) = \begin{cases} y_\omega(n, j, k) & \text{if } \epsilon_2 = 0 \\ 1 - y_\omega(n, j, k) & \text{if } \epsilon_2 = 1 \end{cases}$$

and let $A_\omega(n, x, y)$ be $A_\omega(n, p, m)$ for that (p, m) with $(x, y) \in S(n + 1, p, m)$.

F_ω is defined inductively on dyadic rectangles by

$$F_\omega(S(0, 0, 0)) = 1$$

$$F_\omega(S(n + 1, 2j + \epsilon_1, 2k + \epsilon_2)) = F_\omega(S(n, j, k))A_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2)B_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2)$$

for $\epsilon_1, \epsilon_2 = 0, 1$. This uniquely specifies a probability measure F_ω on S for each ω. We will write G_ω and H_ω for the marginal distributions of F_ω on the x and y axes respectively.

THEOREM 1. *For almost all* ω

(a) $\lim_{n\to\infty} - n^{-1} \log F_\omega(S(n, x, y)) = G + H$ *for* F_ω*-almost all* (x, y).

(b) $\lim_{n\to\infty} - n^{-1} \log G_\omega(I(n, x)) = G$ *for* G_ω*-almost all* x.

(c) $\lim_{n\to\infty} - n^{-1} \log H_\omega(I(n, y)) = H$ *for* H_ω*-almost all* y.

We will need the following well known result.

LEMMA 1. *If* f_n *is* F_n *measurable, where* $\{F_n\}$ *is an increasing sequence of* σ-*fields,* $E(f_n^2) = \sigma_n^2$ *with* $\sum_{n=1}^\infty \sigma_n^2/n^2 < \infty$, *and if* $E(f_n/F_{n-1}) = 0$ *for all* n, *then*

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^n f_j = 0$$

almost everywhere.

PROOF OF THEOREM 1. Let F_n be the σ-field in $S \times \Omega$ generated by the sets $S(n+1, j, k)$ and the random variables $x_\omega(m, j, k)$ and $y_\omega(m, j, k)$ for $m \leqq n$. We define a probability measure on $S \times \Omega$ by setting

$$E(f(x, y, \omega)) = E_\omega(\textstyle\int_S f(x, y, \omega)\, dF_\omega)$$

whenever the right hand side exists.

Since $n^{-1} \log F_\omega(S(n, x, y)) = n^{-1} \sum_{k=0}^{n-1} \log A_\omega(k, x, y) + \log B_\omega(k, x, y)$ (a) will follow if we can show that

$$g_n(x, y, \omega) = \log A_\omega(n, x, y) + G$$

and

$$h_n(x, y, \omega) = \log B_\omega(n, x, y) + H$$

satisfy the conditions of the lemma for F_n and E as defined above.

Since g_n is constant on each $S(n+1, j, k)$

$$\begin{aligned} E(g_n^2) &= E_\omega(\textstyle\int_S g_n^2(x, y, \omega)\, dF_\omega) \\ &= E_\omega(\textstyle\sum_{j=0}^{2^{n+1}-1} \sum_{h=0}^{2^{n+1}-1} F_\omega(S(n+1, j, k))[\log A_\omega(n, j, k) + G]^2) \\ &= E_\omega(\textstyle\sum_{p=0}^{2^n-1} \sum_{m=0}^{2^n-1} \sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} F_\omega(S(n+1, 2p+\epsilon_1, 2m+\epsilon_2)) \\ &\qquad \cdot [\log A_\omega(n, 2p+\epsilon_1, 2m+\epsilon_2) + G]^2). \end{aligned}$$

But

$$\begin{aligned} &\textstyle\sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} F_\omega(S(n+1, 2p+\epsilon_1, 2m+\epsilon_2))[\log A_\omega(n, 2p+\epsilon_1, 2m+\epsilon_2) + G]^2 \\ &\quad = F_\omega(S(n, j, k)) \textstyle\sum_{\epsilon_1=0}^{1} A_\omega(n, 2p+\epsilon_1, 2m)[\log A_\omega(n, 2p+\epsilon_1, 2m) + G]^2 \\ &\quad \leqq 2F_\omega(S(n, j, k)) \max_{0 \leqq x \leqq 1} x[\log x + G]^2 \\ &\quad \leqq 2F_\omega(S(n, j, k)) \max_{0 \leqq x \leqq 1} [x(\log x)^2 + G^2] \\ &\quad = 2F_\omega(S(n, j, k))(1 + G^2) \end{aligned}$$

so

$$E(g_n^2) \leqq 2(1+G^2) E_\omega(\textstyle\sum_{p=0}^{2^n-1} \sum_{m=0}^{2^n-1} F_\omega(S(n, p, m))) = 2(1+G^2).$$

An F_{n-1} measurable function has the form

$$f(x, y, \omega) = \textstyle\sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \alpha_{jk}(\omega)\beta_{jk}(x, y)$$

where $\beta_{jk}(x, y) = 1$ if $(x, y) \in S(n, j, k)$ and vanishes elsewhere and $\alpha_{jk}(\omega)$ depends only on $x_\omega(m, j, k)$ and $y_\omega(m, j, k)$ for $m \leqq n-1$. We have

$$\begin{aligned} &E(g_n\, \alpha_{jk}\, \beta_{jk}) \\ &\quad = E_\omega(\alpha_{jk}(\omega) \textstyle\sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} F_\omega(S(n+1, 2j+\epsilon_1, 2k+\epsilon_2)) \\ &\qquad \cdot [\log A_\omega(n, 2j+\epsilon_1, 2k+\epsilon_2) + G]) \\ &\quad = E_\omega(\alpha_{jk}(\omega) F_\omega(S(n, j, k))) \textstyle\sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} A_\omega(n, 2j+\epsilon_1, 2k+\epsilon_2) \\ &\qquad \cdot B_\omega(n, 2j+\epsilon_1, 2k+\epsilon_2)[\log A_\omega(n, 2j+\epsilon_1, 2k+\epsilon_2) + G]) \end{aligned}$$

$$= E_\omega(\alpha_{jk}(\omega)F_\omega(S(n, j, k)))\sum_{\epsilon_1=0}^{1} A_\omega(n, 2j + \epsilon_1, 2k)$$
$$\cdot[\log A_\omega(n, 2j + \epsilon_1, 2k) + G])$$
$$= E_\omega(\alpha_{jk}(\omega)F_\omega(S(n, j, k)))E_\omega(\sum_{\epsilon_1=0}^{1} A_\omega(n, 2j + \epsilon_1, 2k)$$
$$\cdot[\log A_\omega(n, 2j + \epsilon_1, 2k) + G]) = 0,$$

so $E(g_n/F_{n-1}) = 0$. The verifications for h_n are similar. The proof of (b) and (c) is also similar.

The following lemma, which will be needed in the proof of Theorem 2, is very similar to Satz 1 and Satz 2 of [2]. It is also implicitly contained in the proof of Theorem 2 of [5].

LEMMA 2. *Let σ be a probability measure defined on the Borel sets of the interval* 0, 1]. *If*

$$\lim_{n\to\infty} - n^{-1} \log \sigma(I(n, x)) = \alpha$$

for all x in a set A and if $\sigma(A) > 0$ then $\dim(A) = \alpha$.

PROOF. Choose an $\epsilon > 0$ and an integer N. For each x in A let $J(n, x)$ be the first $I(n, x)$ for $n \geqq N$ with $-n^{-1} \log \sigma(I(n, x)) \leqq \alpha + \epsilon$. The collection (J_j) of such intervals is disjoint. Moreover, writing $|J|$ for the length of J, $|J_j| \leqq 2^{-N}$ and $\sigma(J_j) \geqq |J_j|^{\alpha+\epsilon}$. Thus $1 \geqq \sum_{j=1}^{\infty} \sigma(J_j) \geqq \sum_{j=1}^{\infty} |J_j|^{\alpha+\epsilon}$ and, since N is arbitrary, this proves that the $\alpha + \epsilon$ measure of A is finite and hence that $\dim(A) \leqq \alpha + \epsilon$. ϵ is also arbitrary so $\dim(A) \leqq \alpha$.

If σ has atoms in A then necessarily $\alpha = 0 = \dim(A)$ and the theorem is true. From now on we assume that σ has no atoms in A.

Now choose an integer M and an $\epsilon > 0$. For each x in [0, 1] let $J(x)$ be the first $I(n, x)$ for $n \geqq M$ with $-n^{-1} \log \sigma(I(n, x)) \leqq \alpha - \epsilon$, if one exists. Let C_M be the union of all these and let $c(C_M)$ be its complement. As M increases C_M decreases and $\sigma(C_M)$ goes to 0. Set $A_M = A \cap c(C_M)$ where M is taken so large that $\sigma(A_M) > 0$.

Now let (I_j) be any covering of A_M of norm $\leqq 2^{-M}$. Let k_j be the integer such that $2^{-k_j-1} < |I_j| \leqq 2^{-k_j}$. I_j intersects at most two k_jth order dyadic intervals, say $I(k_j, x_1)$ and $I(k_j, x_2)$. For each of these either $\sigma(I(k_j, x_i)) < 2^{-k_j(\alpha-\epsilon)}$ or $I(k_j, x_i) \subset C_M$. In the latter case, say $I(k_j, x_1) \subset C_M$, we can replace I_j by the interval I_j' consisting of the part of I_j outside $I(k_j, x_1)$. If both $I(k_j, x_1)$ and $I(k_j, x_2)$ are contained in C_M so is I_j and it can be discarded. In either case $\sigma(I_j') \leqq 2^{-k_j(\alpha-\epsilon)+1} \leqq 4|I_j'|^{\alpha-\epsilon}$. Hence $0 < \sigma(A_M) \leqq \sum \sigma(I_j') \leqq 4\sum |I_j'|^{\alpha-\epsilon}$ so the $\alpha - \epsilon$ measure of A_M is positive and

$$\dim(A) \geqq \dim(A_M) \geqq \alpha - \epsilon.$$

Since ϵ is arbitrary this completes the proof.

THEOREM 2. *For almost all ω there exist sets $A_\omega \subset S$, $B_\omega \subset [0, 1]$, $C_\omega \subset [0, 1]$ with $F_\omega(A_\omega) = G_\omega(B_\omega) = H_\omega(C_\omega) = 1$ such that for any sets $A \subset A_\omega$, $B \subset B_\omega$, $C \subset C_\omega$ with $F_\omega(A) > 0$, $G_\omega(B) > 0$, $H_\omega(C) > 0$ we have*

(a) $\dim A = G + H$

(b) dim $B = G$
(c) dim $C = H$.

PROOF. Take A_ω, B_ω, and C_ω to be the sets where (a), (b) and (c) of Theorem 1 are satisfied. (b) and (c) are immediate consequences of Lemma 2. Let ϕ be the map of A onto [0, 1] given by

$$\phi(x, y) = \sum_{i=1}^{\infty} z_i 2^{-i}$$

where $x = \sum_{i=1}^{\infty} x_i 2^{-i}$, $y = \sum_{i=1}^{\infty} y_i 2^{-i}$, $z_{2i} = x_i$ and $z_{2i-1} = y_i$. Beyer [1] has shown that under this map $\dim(D) = 2\dim(\phi(D))$. It is easy to see that $\phi(S(n, x, y)) = I(2n, \phi(x, y))$ so for every $z = \phi(x, y)$ in $\phi(A_\omega)$

$$\begin{aligned}\lim_{n\to\infty} - (2n)^{-1} \log (F_\omega \circ \phi^{-1})(I(2n, z)) &\\ = \lim_{n\to\infty} - (2n)^{-1} \log F_\omega(S(2n, x, y)) &\\ = \tfrac{1}{2}(G + H). &\end{aligned}$$

Also

$$\begin{aligned}\tfrac{1}{2}(G + H) &= \lim_{n\to\infty} - (2n)^{-1} \log (F_\omega \circ \phi^{-1})(I(2n, z)) \\ &\leqq \lim_{n\to\infty} - (2n + 1)^{-1} \log (F_\omega \circ \phi^{-1})(I(2n + 1, z)) \\ &\leqq \lim_{n\to\infty} - (2n)^{-1} \log (F_\omega \circ \phi^{-1})(I(2n + 2, z)) \\ &= \tfrac{1}{2}(G + H).\end{aligned}$$

Hence, if $A \subset A_\omega$ and $F_\omega(A) > 0$, $\dim(A) = 2\dim(\phi(A)) = G + H$.

In the above case the dimension of the support of F_ω is the sum of the dimensions of the supports of the marginal distributions. We now vary the construction somewhat to get a case where this is not so.

Let ν be a probability measure on [0, 1] and set

$$K = -\int_0^1 (z \log z + (1 - z) \log (1 - z))\, d\nu(z).$$

Ω now is the set of all $(P_{n,j,k})$ for $n = 0, 1, \cdots, j = 0, 1, \cdots, 2^n - 1$ and $k = 0, 1, \cdots, 2^n - 1$ where each $P_{n,j,k}$ is a point of [0, 1]. The coordinates are independent and have the distribution ν. We will write $z_\omega(n, j, k)$ for the (n, j, k) coordinate of ω.

The random measure K_ω on S is determined by

$$K_\omega(S(0, 0, 0)) = 1$$

$$K_\omega(S(n + 1, 2j + \epsilon_1, 2k + \epsilon_2)) = K_\omega(S(n, j, k))C_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2)$$

where

$$\begin{aligned}C_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2) &= \tfrac{1}{2} z_\omega(n, j, k) \text{ if } \epsilon_1 = \epsilon_2 \\ &= \tfrac{1}{2}(1 - z_\omega(n, j, k)) \text{ if } \epsilon_1 \neq \epsilon_2 .\end{aligned}$$

For $(x, y) \varepsilon S(n + 1, 2j + \epsilon_1, 2k + \epsilon_2)$ we set $C_\omega(n, x, y) = C_\omega(n, 2j + \epsilon_1, 2k + \epsilon_2)$.

Since

$$\sum_{j=0}^{2^{n+1}-1} K_\omega(S(n+1, j, k))$$
$$= \sum_{m=0}^{2^n-1} K_\omega(S(n+1, 2m, k)) + K_\omega(S(n+1, 2m+1, k))$$
$$= \tfrac{1}{2} \sum_{1=0}^{2^n-1} K_\omega(S(n, m, [k/2]))$$
$$= \cdots = 2^{-(n+1)},$$

the margins on the x axis are uniform and a similar calculation shows that the margins on the y axis are also uniform.

THEOREM 3. *For almost all* ω

$$\lim_{n\to\infty} - n^{-1} \log K_\omega(S(n, x, y)) = 1 + K$$

for K_ω*-almost all* (x, y). *If* D_ω *is the subset of* S *on which the above relation holds and* $D \subset D_\omega$ *has* $K_\omega(D) > 0$ *then* $\dim D = 1 + K$.

PROOF. The proof is very similar to the proofs of Theorems 1 and 2. We let G_n be the field generated by the $S(n+1, j, k)$ and the $z_\omega(m, j, k)$ for $m \leqq n$. E is defined as in Theorem 1 with F_ω replaced by K_ω. Calculations similar to those in the proof of Theorem 1 yield

$$E((\log C_\omega(n, x, y) + K + 1)^2) \leqq 2(1 + (1 + K)^2)$$
$$E(\log C_\omega(n, x, y) + K + 1/G_{n-1}) = 0.$$

This enables one to apply Lemma 1 to the functions

$$k_n(x, y, \omega) = \log C_\omega(n, x, y) + K + 1.$$

The rest of the proof is an exact duplicate of the proofs of Theorems 1 and 2.

Since $-x \log x - (1 - x) \log (1 - x) < 1$ except at $x = \frac{1}{2}$, $1 + K < 2$ except in the case where ν is concentrated at $\frac{1}{2}$ which is, of course, just the product measure on the square.

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